Chapter #5

EEE8013

State Space Analysis and Controller Design

- **Preliminaries**
- **Geometry of 2D systems**
- **Geometry of 3D systems**

The geometry of the state space

1. Preliminaries

- A set of vectors that satisfy some specific rules (we do not worry about these rules) is called a Vector Space (VS).

- Two vectors c_1, c_2 in that VS are Linear Independent (LI) if the only 2 numbers k_1, k_2 that satisfy the equation $k_1 c_1 + k_2 c_2 = 0$ are $k_1 = k_2 = 0$. If the two vectors are not LI then we can write $c_1 = -\dfrac{k_2}{k_1} c_2$ (assuming that $k_1 \neq 0$). This implies that c_1, c_2 are on the same line or simply speaking c_2 is the scaled version of c_1.

- Finally, the sum $k_1 c_1 + k_2 c_2$ is called the linear combination of c_1, c_2.

Now, let's see what we can do with 2 LI vectors:

A vector $a = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$ can be written as $a = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-4)\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, i.e. expressed using the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, & $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. These two vectors (which are the x, y axes in a Cartesian plane) are orthogonal (as $\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle = 1 \cdot 0 + 0 \cdot 1 = 0$) and have unit length as $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 0^2} = 1$, i.e. they are orthonormal.

But our original vector can easily be expressed in the case where the two basis vectors do not have unit length, for example $\mathbf{a} = 0.5\begin{bmatrix} 4 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ -0.5 \end{bmatrix}$.

The original vector can also be written as a combination of non-orthogonal vectors, for example $\mathbf{a} = -0.5\begin{bmatrix} 4 \\ 6 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -0.5 \end{bmatrix}$.

It is obvious from the above that the first analysis is much easier (and hence is more frequently used) but there are many cases where the other 2 can also be used. In general the coordinates were found by solving the systems.

Case 1: $a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

Case 2: $a\begin{bmatrix} 4 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

Case 3: $a\begin{bmatrix} 4 \\ 6 \end{bmatrix} + b\begin{bmatrix} 2 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$

Note: If the 2 new basis vectors are Linear Dependant (LD) then we cannot find values of a and b that satisfy the above equations:

$$a\begin{bmatrix} 4 \\ 6 \end{bmatrix} + b\begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} \Leftrightarrow \begin{cases} 4a + 8b = 2 \\ 6a + 12b = -4 \end{cases} \Rightarrow \begin{cases} a + 2b = 0.5 \\ a + 2b = -2/3 \end{cases}$$

- So in the Cartesian plane I need two LI vectors to express all the other vectors. Thus we say that the Cartesian plane is a 2 dimensional vector space.

- These 2 LI vectors are called the basis of the vector space.

- We also say that these 2 vectors span the vector space (this means that ALL other vectors in that space can be written as a linear combination of these 2 vectors).

- A final point about this 2 dimensional vector space is that if we take one vector then we can express all the other vectors with the same direction (i.e. on the same line) simply by multiplying that vector with a constant. We say that this new set of vectors (that is smaller than the Cartesian plane) is a vector subspace.

Obviously we can expand this to higher dimensions. For example in the Cartesian space (3D vector space):

- We need 3 LI vectors to express any other vector (i.e. to create a basis). The easiest choice for this vector space is the classical x,y,z axes: $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$. But, as before, I may choose vectors that are not orthogonal and have length other than 1.

- Again, if we take 2 LI vectors in that 3D vector space then by taking their linear combination I can express a smaller set of vectors. Note that I cannot take all possible vectors in that vector space but I can still get a subspace that is spanned by these 2 vectors (i.e. all the possible linear combinations of these 2 vectors).

In n-dimensional vector spaces:

- We need n LI vectors to form a basis.

- Any set of k (where $k<n$) LI vectors define a k-dimensional vector subspace.

2. State Space Analysis

We have seen that the state vector can be written as:

$$\mathbf{x}(t) = x_1(t)\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2(t)\begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots x_n(t)\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

I.e. the state vector can be plotted in a Cartesian space. For example for 2 and 3 dimensions:
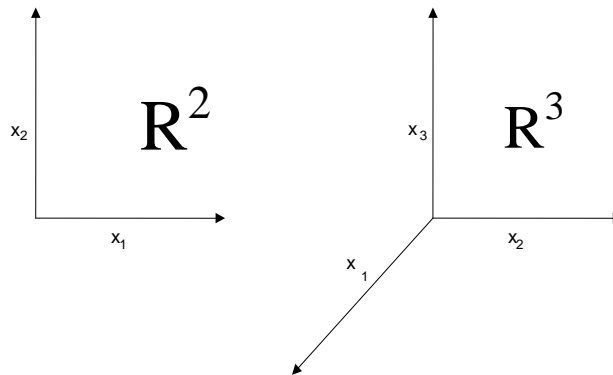
As we have seen in a vector space there are many sets of vectors that can form a basis. Since the state space is another vector space this implies that in the state space we can define other bases. It is of particular importance to find a basis that consists of vectors that are ***invariant*** under the solution of the state equation. The invariance property means that if the solution at some point in time is on one of these vectors then it will remain on that vector for all subsequent times. This will greatly simplify our analysis.

The purpose of this chapter is to explain how these bases can be found for second and third order systems. By doing that we can see how the state space is partitioned, and this will give us an easier geometric approach to study/control the system.

2.1. 2D linear systems

Case A: Distinct and real eigenvalues

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ which has a solution:

$$\mathbf{x}(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

Now if we carefully study this expression we will see that it can be written

as: $\mathbf{x}(t) = a(t) \begin{bmatrix} 2 \\ 1 \end{bmatrix} + b(t) \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, where $a(t) = C_1 e^{-t}, b(t) = C_2 e^{-6t}$. Hence we

express our state vector into 2 other LI vectors (i.e. we found a new basis,

the *eigenbasis*). Note that we can easily scale our two vectors:

$$\mathbf{X}(t) = \frac{a(t)}{2} \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 3b(t) \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix} = c(t) \begin{bmatrix} 4 \\ 2 \end{bmatrix} + d(t) \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix},$$

where $c(t) = \dfrac{a(t)}{2}, d(t) = 3b(t)$.

$$\mathbf{e}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\mathbf{e}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
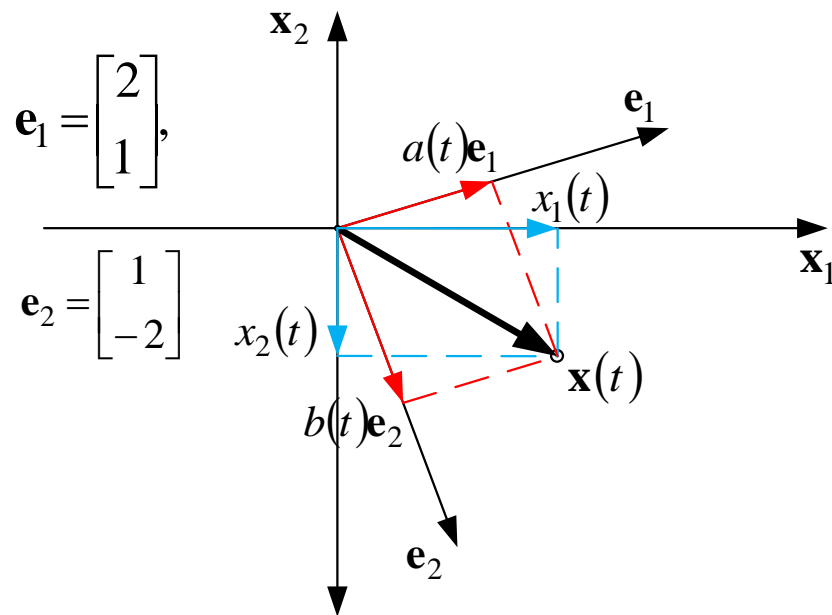
As the eigenvectors can be scaled:
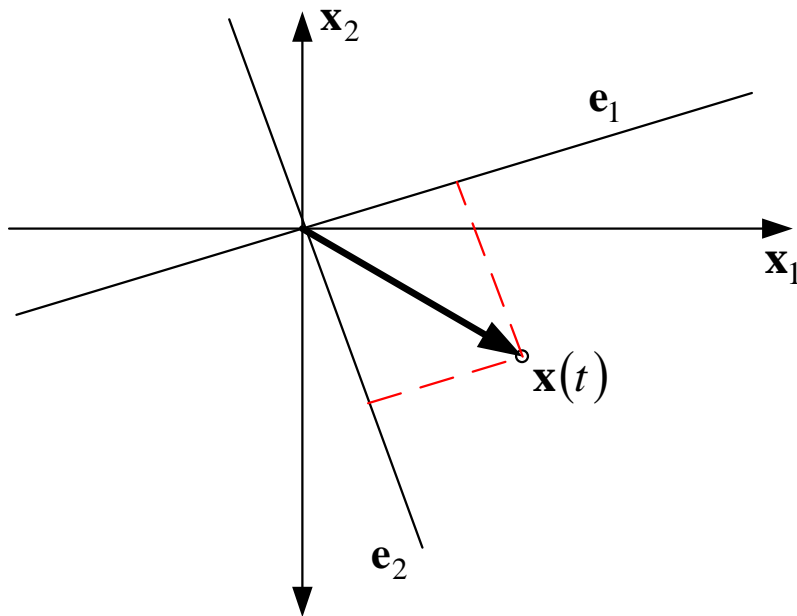
If we also want to state the stability of the corresponding eigenvalue:
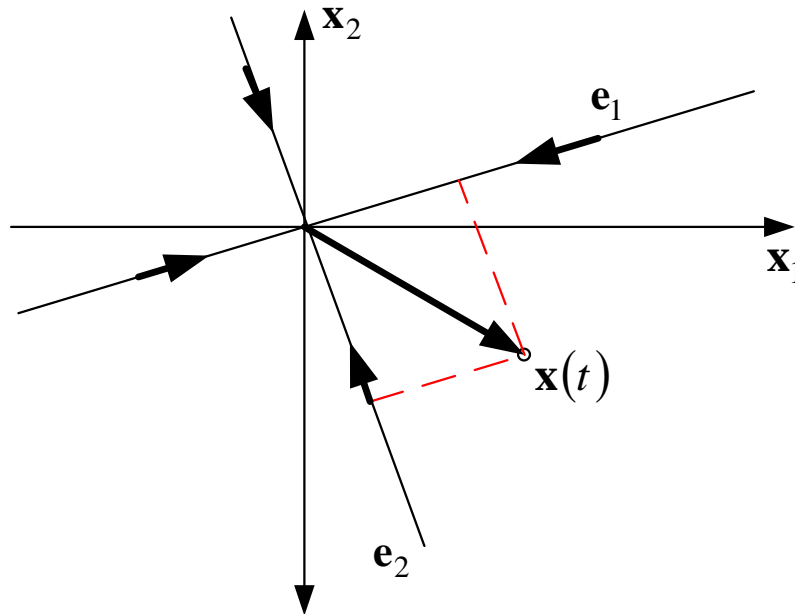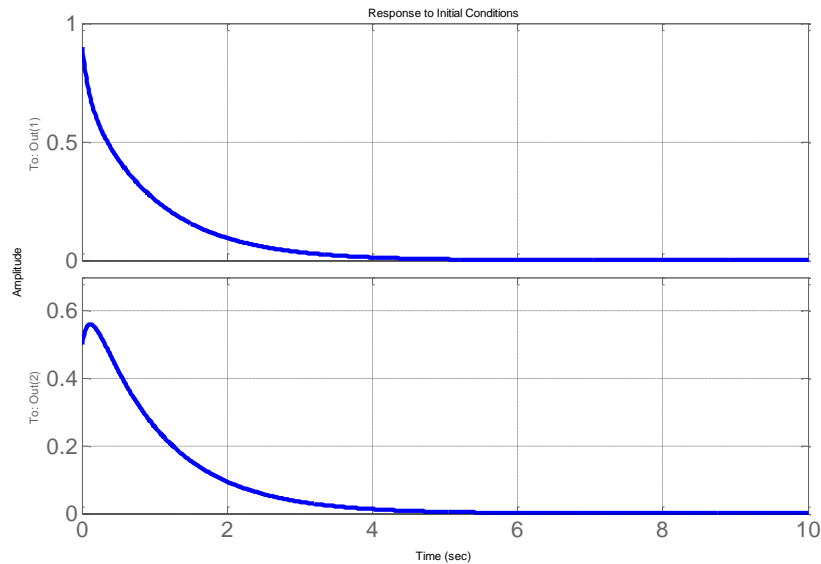

Example:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -5.5 & 4.5 \\ 4.5 & -5.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = -10 \\ \lambda_2 = -1 \end{cases} \Rightarrow \begin{cases} e_1 = \begin{bmatrix} -1 & 1 \end{bmatrix} \\ e_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \end{cases}$$

Thus $\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-10t}$

Giving the initial condition $\mathbf{x}(0) = \begin{bmatrix} 0.9 & 0.5 \end{bmatrix}^T$ will give us $C_1 = 0.7$ and $C_2 = -0.2$

The response from Matlab is:

Which is not very revealing, but if we see the response using the eigenvectors as basis: $\mathbf{x}(t) = 0.7 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} - 0.2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-10t}$, obviously the term

e^{-10t} is a lot faster than e^{-t} and hence the main response will be on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ as it

is also seen by the next figure.
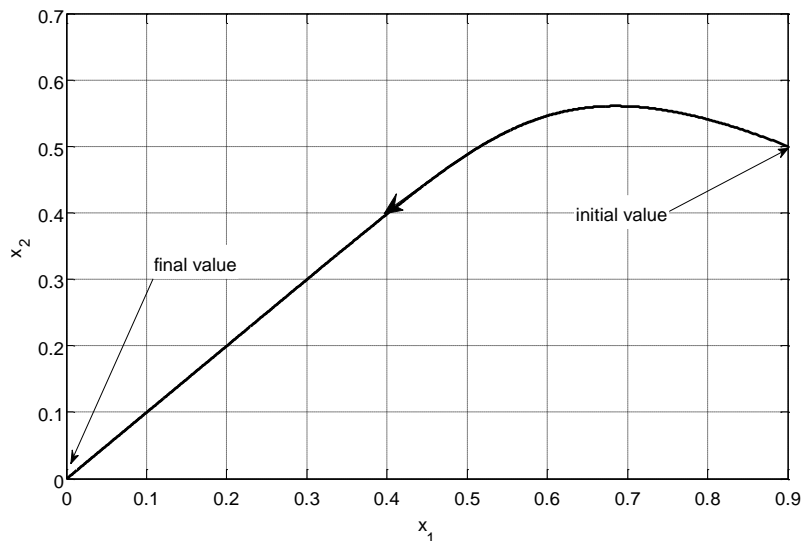
Now if on that we plot the eigenvectors:
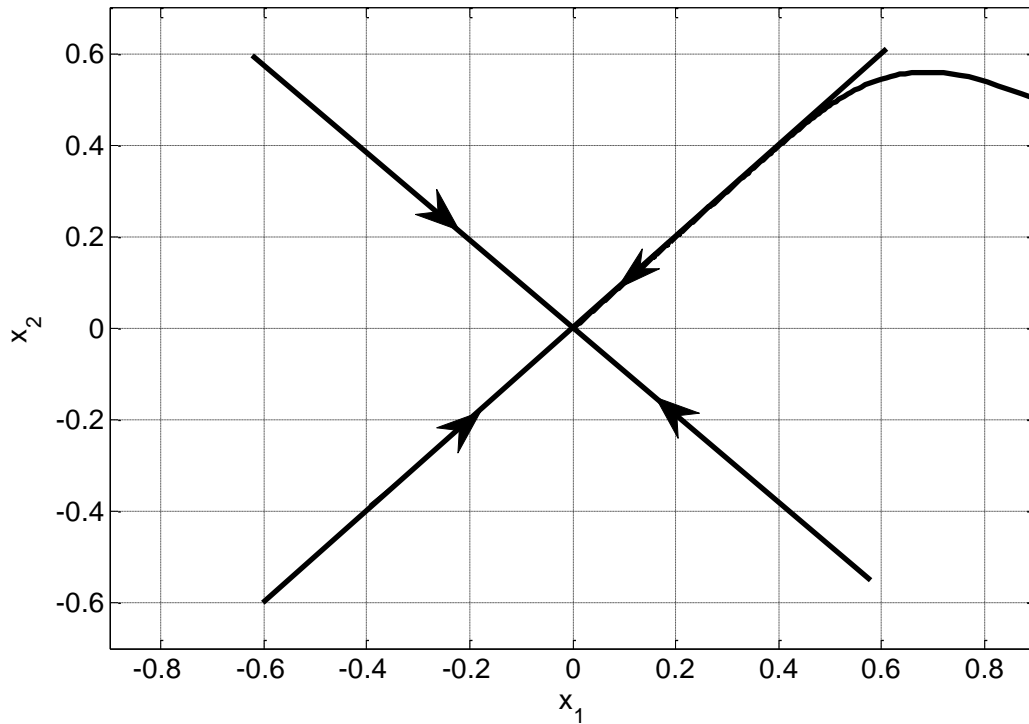
Using Matlab to draw multiple solutions:
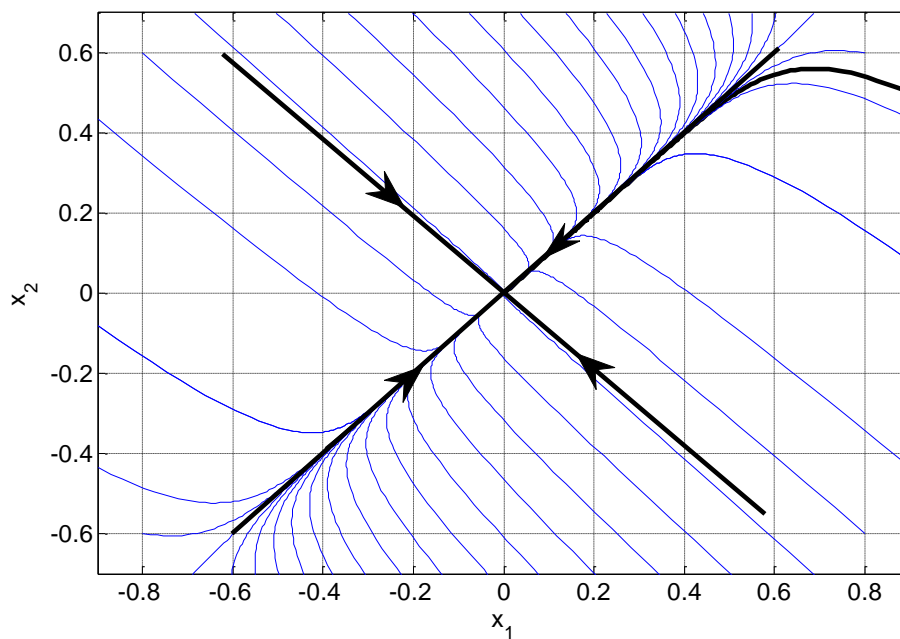
It is interesting to note that:

- If the eigenvalues are negative the solution will converge to zero (**stable**) and if they are positive the solution will diverge to infinity (**unstable**).

- Depending on the absolute value of the eigenvalues we may converge or diverge along one of the two eigenvectors. For example assume that $\lambda_1, \lambda_2 < 0$ and $|\lambda_1| \gg |\lambda_2|$, then $e^{\lambda_1 t}$ will converge to zero faster than $e^{\lambda_2 t}$. This implies that as $t \to \infty$ the solution will effectively coincide with $\mathbf{e}_2$.

- If I have one stable (λ_1) and one unstable eigenvalue (λ_2) (i.e. the system is unstable) then the stable one will converge to zero and hence as $t \to \infty$ the contribution of that eigenvector will be negligible, so the solution for "big" values of t will almost coincide with the unstable eigenvector. In that case it is still possible to converge to zero if $C_2 = 0$, i.e. we start on the stable eigenvector (see next points for a thorough description of this statement).

- But why is this new basis important? After all it is much harder to express the solution in that basis. The importance arises from the fact that in (most of) our cases where $\mathbf{x}(t) = a(t)\mathbf{e}_1 + b(t)\mathbf{e}_2$ the time dependence of the coefficients $a(t), b(t)$ is due to the exponential terms (multiplied by a constant factor, remember that $a(t) = C_1 e^{-t}$, $b(t) = C_2 e^{-6t}$ in our example). As these exponential terms can never be zero this implies that if at any point $a(t)$ (similarly for $b(t)$) is zero this is due to the constant C_1, i.e. the choice of the initial condition is such that $C_1 = 0$. Thus it is clear that $a(t)$ will be zero for all possible values of t. This implies that the solution is going to be $\mathbf{x}(t) = b(t)\mathbf{e}_2$. Geometrically this means that the solution

will be on the second eigenvector. For that reason we say that the 2 vectors are ***invariant***. To connect this with the previous analysis we say that each eigenvector in that case is an ***invariant subspace*** (the eigenspace).

- Each (generalized) eigenvector together with the origin (i.e. the zero vector) define the (generalized) ***eigenspace***, mathematically:

  For each eigenvalue λ (with multiplicity m) the set of vectors that satisfy $(\mathbf{A} - \lambda\mathbf{I})^k \mathbf{e} = 0$, $k = 1,2,..m$ form a basis and hence define a subspace called eigenspace of $\mathbf{A}$ corresponding to λ. Obviously in this case of distinct and real eigenvalues the dimension of each eigenspace is 1.

Exercise 1:

The state matrix of a homogeneous system is given by $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -10 \end{bmatrix}$.

- Find the eigenvalues and eigenvectors.
- Without using Matlab draw in the state space the eigenvectors and indicate the stability of each eigenvalue.
- Choose 4 different initial conditions and without using Matlab sketch in the state space the solution.
- Find the general solution.
- Find the particular solution for the above initial conditions. Discuss your results.

Exercise 2:

Repeat the previous exercise for $\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 10 \end{bmatrix}$.

If we have a diagonal matrix (i.e. a system in a normal form):

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -10 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -10 \end{cases} \Rightarrow \begin{cases} e_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ e_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \end{cases}$$
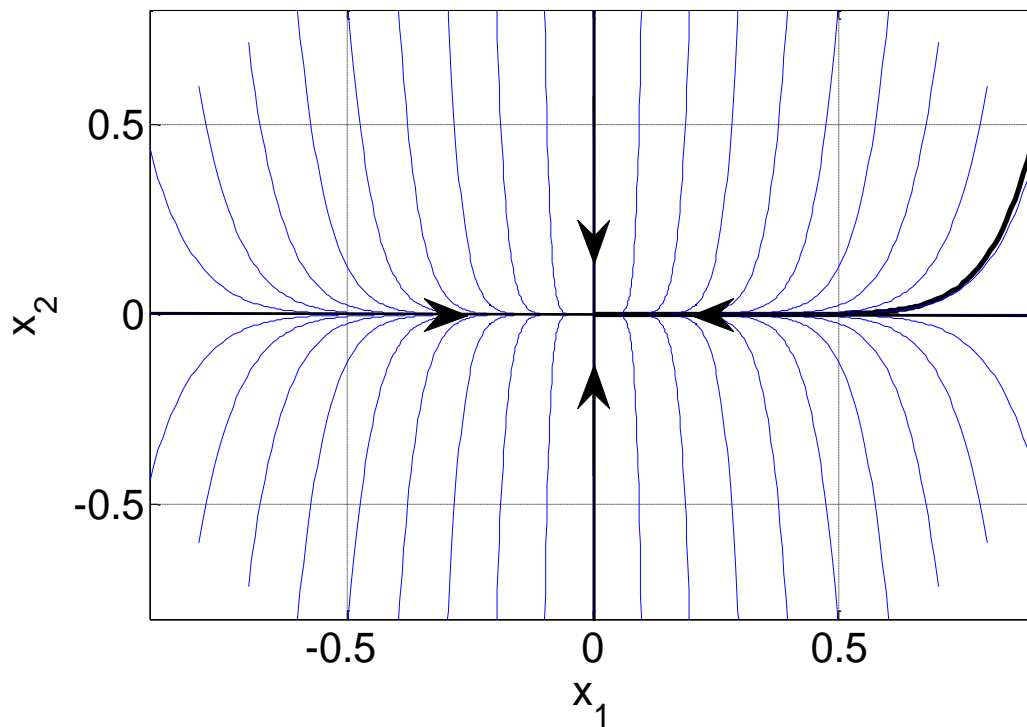
The response to the initial conditions $[1, 0.5]^T$:
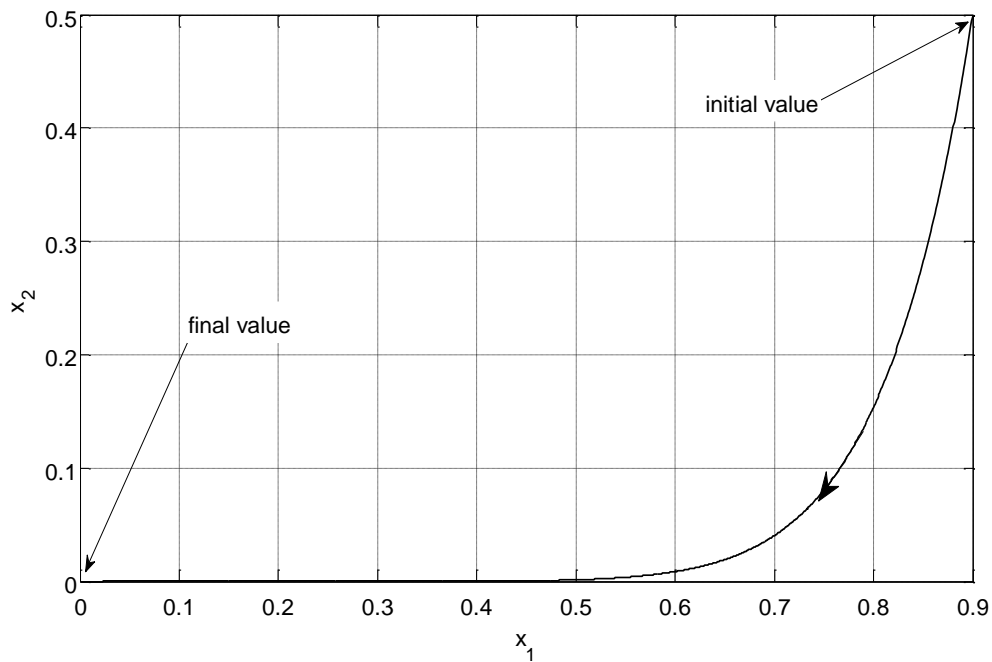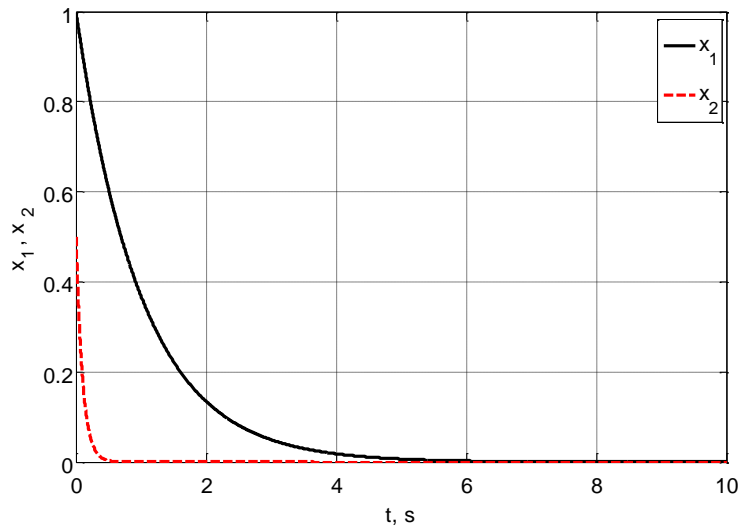
With the eigenvectors:
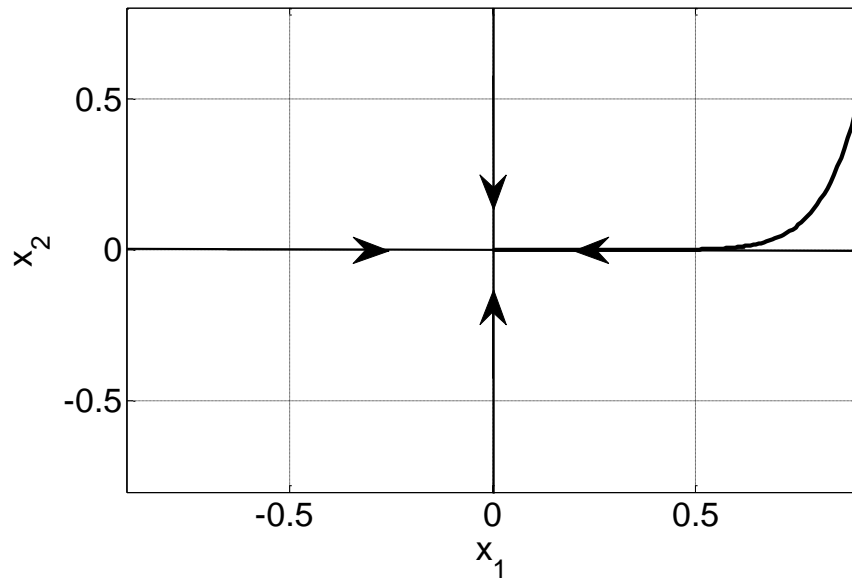
And by plotting many solutions:

Hence the generic diagrams will be (note that the eigenvectors are the x-y axes):
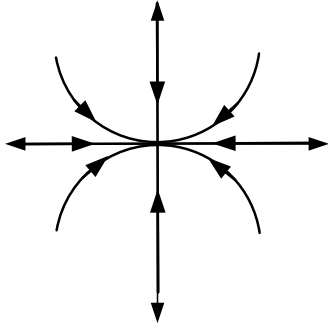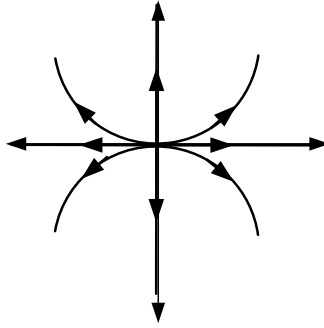
Stable node   Unstable node   Saddle
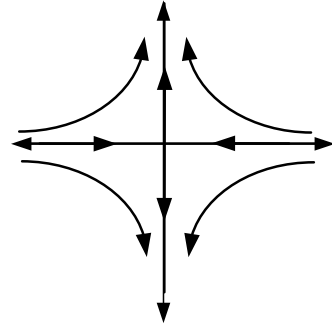
Case B: Repeated eigenvalues.

Now it is possible to have 2 sub-cases:

- I can find 2 LI vectors (a rather artificial case)

$$\mathbf{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2 \Rightarrow \begin{cases} \mathbf{e}_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \end{bmatrix} \end{cases}$$

Hence we have 2 uncoupled 1st order ODEs which can be solved separately:
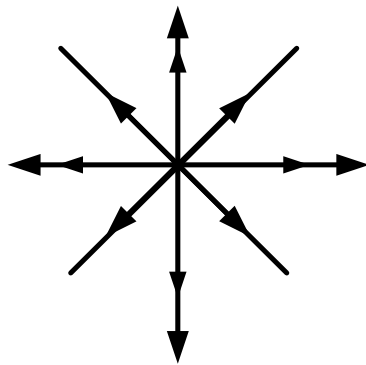
Unstable Proper Node/Star   Stable Proper Node/Star
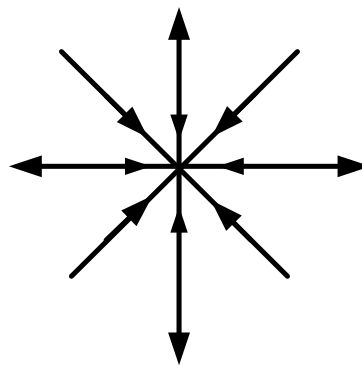
This is a rather trivial case which we will not study.

- I **cannot** find 2 LI vectors

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \Rightarrow \lambda_{1,2} = -2 \Rightarrow \mathbf{v}_{1,2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

In that case I have $\mathbf{x}(t) = C_1(\mathbf{v}t + \mathbf{b})e^{\lambda t} + C_2 \mathbf{v}e^{\lambda t}$, where **b** is our generalized eigenvector. The response in the state space will start from an IC and will converge to zero. The next figure shows the response from many ICs:
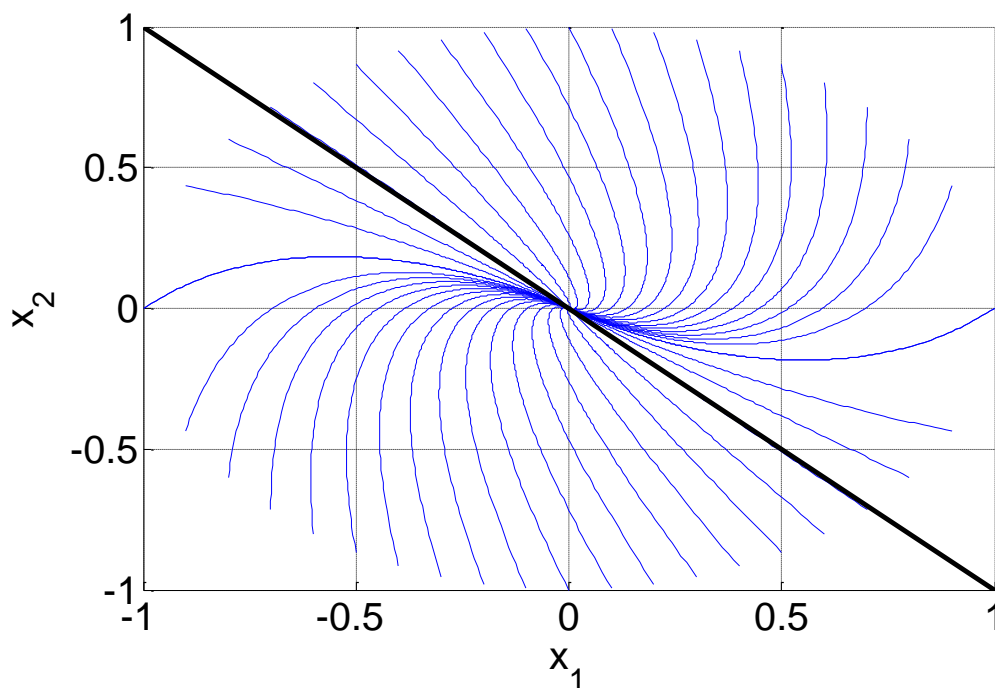
- Notice that since $\mathbf{x}(t) = C_1(\mathbf{v}t + \mathbf{b})e^{\lambda t} + C_2 \mathbf{v}e^{\lambda t}$ the vector **b** is not invariant. This is because even if C_2 and hence at $t=0$ $\mathbf{x}(0) = C_1\mathbf{b}$ and t>0: $\mathbf{x}(t) = C_1(\mathbf{v}t + \mathbf{b})e^{\lambda t}$.

- But the eigenvector **v** is invariant as if $C_1 = 0$ then $\mathbf{x}(t) = C_2\mathbf{v}e^{\lambda t}$ for $t \geq 0$.

- Now we can define the generalised eigenspace using the vector **b** $(\mathbf{A} - \lambda\mathbf{I})^2\,\mathbf{b} = \mathbf{0}$ and obviously the dimension of that space is 2 (it is the subspace spanned by **b** and **v**).

As before we have if we have a system in normal form:
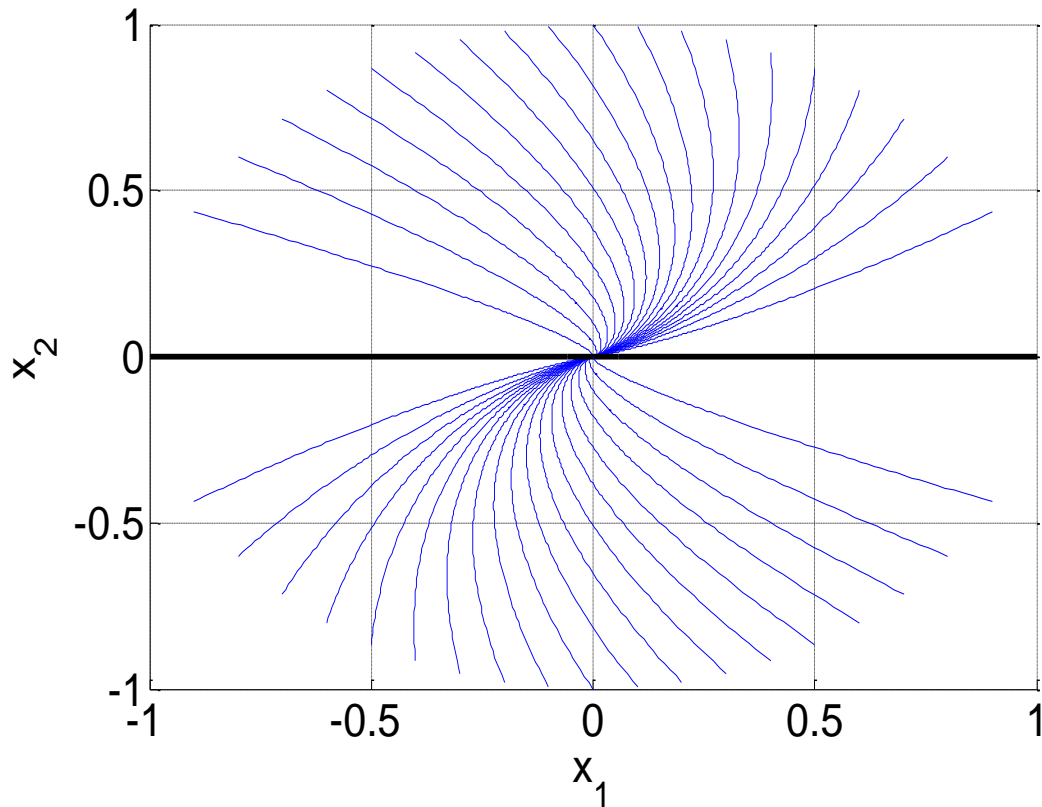
Exercise 4:

The state matrix of a homogeneous system is given by $A = \begin{bmatrix} -1 & -1 \\ 1 & -3 \end{bmatrix}$.

- Find the eigenvalues and eigenvectors.

- Without using Matlab draw in the state space the eigenvectors and indicate the stability of each eigenvalue.

- Choose 4 different initial conditions and without using Matlab sketch in the state space the solution.

- Find the general solution.

- Find the particular solution for the above initial conditions. Discuss your results.

Case C: Complex eigenvalues

Example:

$$\mathbf{A} = \begin{bmatrix} -3 & 2 \\ -4 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm 2j \Rightarrow \mathbf{e} = \begin{bmatrix} 1 \\ 0.5 \pm 0.5j \end{bmatrix}$$

The response to $[1\ 1]^T$ is:
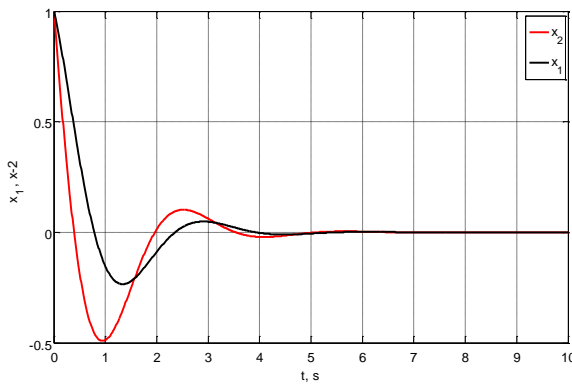
And in the state space:
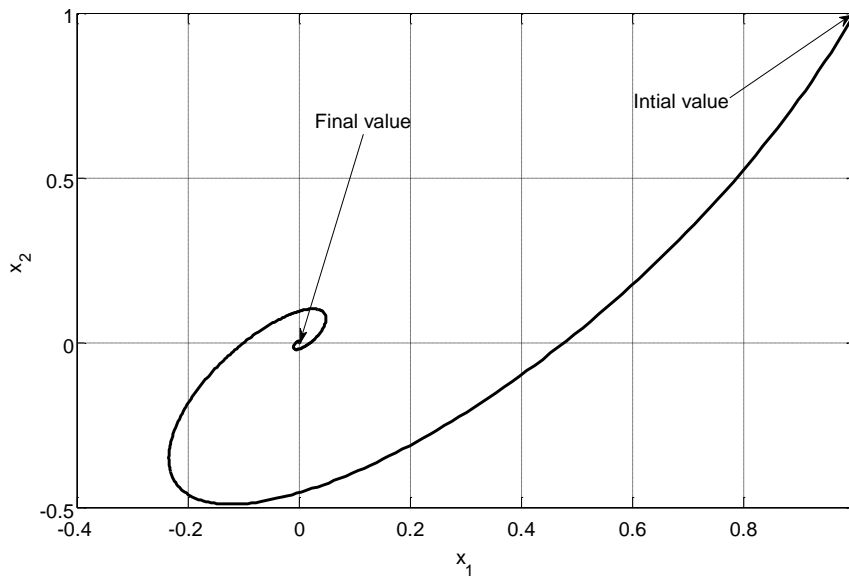
Using many initial conditions:
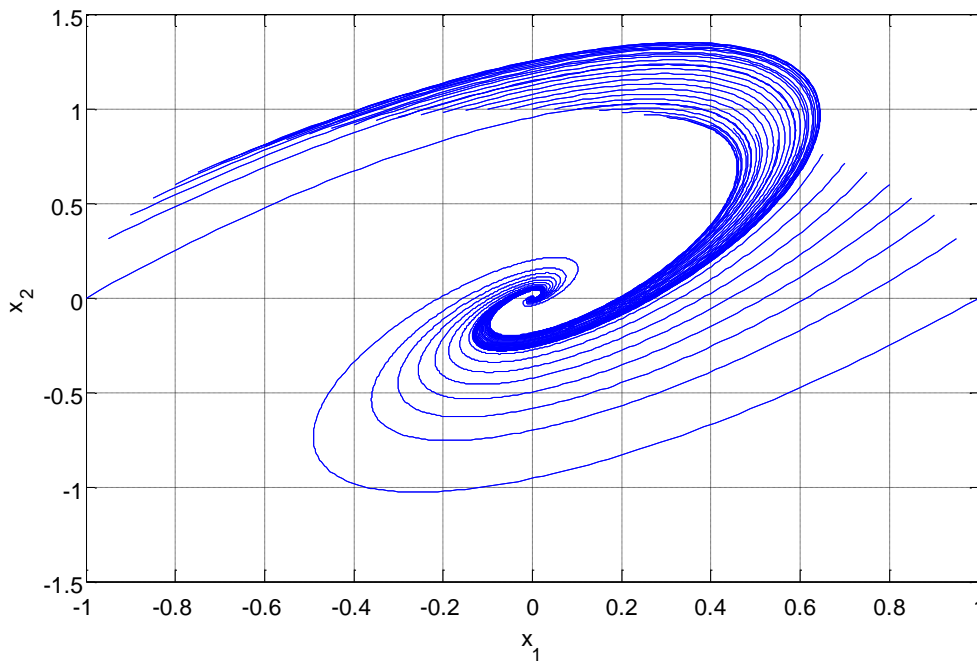
To understand this behavior (which shows no invariant vectors) remember that if I have complex eigenvalues then $\lambda_1 = \overline{\lambda_2} = \mu + vj$ and the corresponding eigenvectors are $\mathbf{e}_1 = \overline{\mathbf{e}_2}$. In that case the general solution is given by: $\mathbf{x}(t) = A_1 \operatorname{Re}\left(\mathbf{e}_1 e^{\lambda_1 t}\right) + A_2 \operatorname{Im}\left(\mathbf{e}_1 e^{\lambda_1 t}\right)$

Now using the Euler formula $e^{njt} = \left(\cos\left(nt\right) + j\sin\left(nt\right)\right)$:

$$\mathbf{x}(t) = e^{\mu t}\left(\mathbf{a}\left(C_1 \cos\left(v\,t\right) + C_2 \sin\left(v\,t\right)\right) + \mathbf{b}\left(C_2 \cos\left(v\,t\right) - C_1 \sin\left(v\,t\right)\right)\right)$$

Hence the solution consists of sinusoidal and exponential terms. The eigenvectors now are not invariant but they define an invariant subspace which is spanned by the real part and imaginary part of one of the eigenvectors. This will be obvious in a 3D case!

Stable Focus
$\mu < 0$

Unstable Focus
$\mu > 0$

Centre
$\mu = 0$
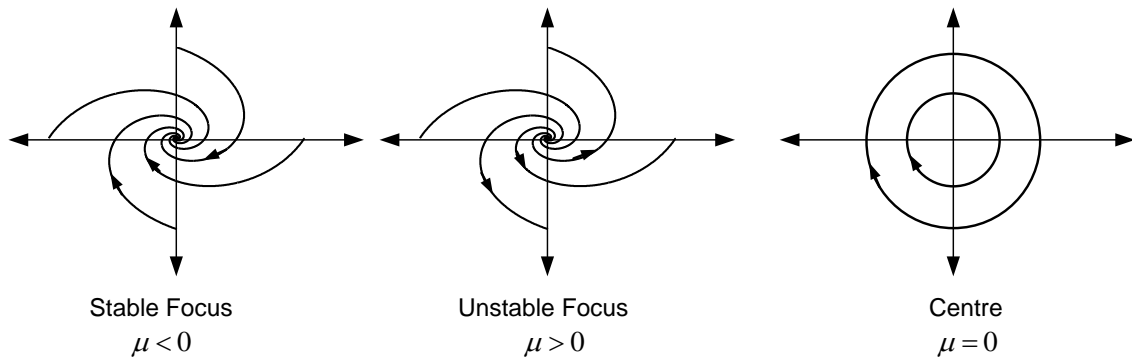
Or in a canonical form:

$$\mathbf{A} = \begin{bmatrix} -1/2 & 1 \\ -1 & -1/2 \end{bmatrix} \Rightarrow \lambda = -0.5 \pm j \Rightarrow \mathbf{e} = \begin{bmatrix} 1 \\ j \end{bmatrix}$$
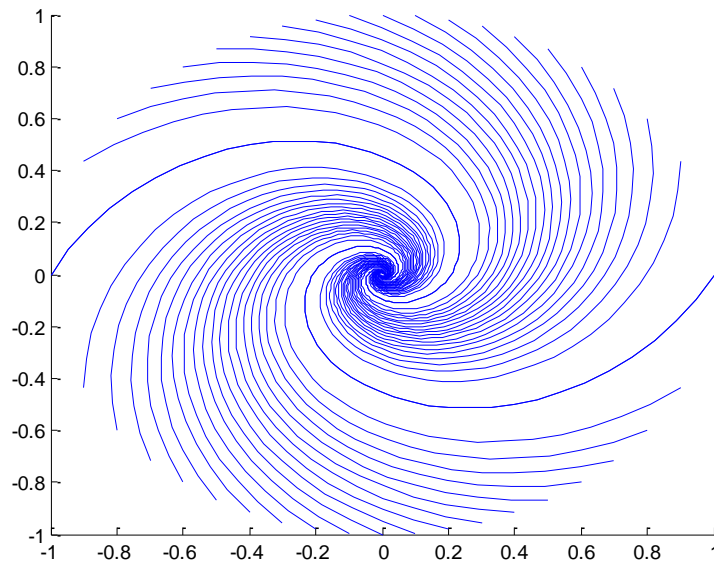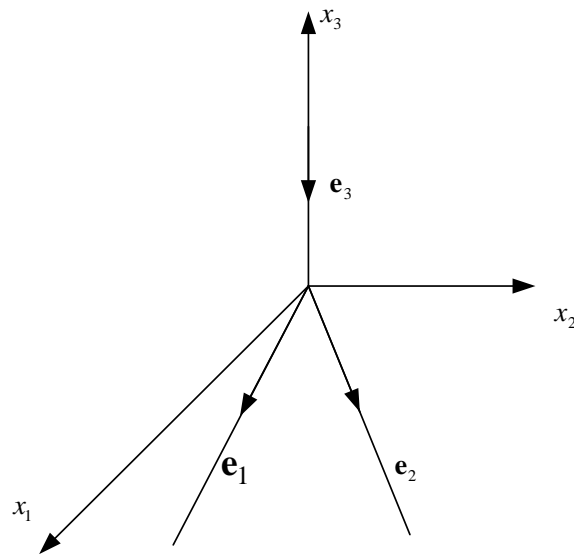
Summary:

Eigenvalues	Eigenvectors	Type	Stability	Eigenspaces	
$\lambda_1 < \lambda_2 < 0$	2 LI	Node	Stable	2	1-dimensional invariant sets
$\lambda_1 > \lambda_2 > 0$	2 LI	Node	Unstable	2	1-dimensional invariant sets
$\lambda_1 < 0 < \lambda_2$	2 LI	Saddle Point	Semi	2	1-dimensional invariant sets
$\lambda_1 = \lambda_2 > 0$	2 LI	Proper Node	Unstable	2	1-dimensional invariant sets
$\lambda_1 = \lambda_2 > 0$	1 => 2LI	Improper Node	Unstable	1	2-dimensional invariant set
$\lambda_1 = \lambda_2 < 0$	2 LI	Proper Node	Stable	2	1-dimensional invariant sets
$\lambda_1 = \lambda_2 < 0$	1 => 2LI	Improper Node	Stable	1	2-dimensional invariant set
$\lambda = \mu + vj$, $\mu < 0$	1 Complex => 2 real LI	Focus	Stable	1	2-dimensional invariant set
$\lambda = \mu + vj$, $\mu > 0$	1 Complex => 2 real LI	Focus	Unstable	1	2-dimensional invariant set
$\lambda = \mu + vj$, $\mu = 0$	1 Complex => 2 real LI	Centre	Neutral	1	2-dimensional invariant set

2.2. 3D linear systems

The same analysis can be expanded to higher dimensions:

Case A – Only real eigenvalues

$$\mathbf{A} = \begin{bmatrix} 2.5 & -1.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = 2 \Rightarrow \begin{bmatrix} 1 & 1/3 & 0 \end{bmatrix}^T \\ \lambda_2 = 1 \Rightarrow \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \\ \lambda_3 = -1 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \end{cases}$$

And the solution is: $\mathbf{x}(t) = C_1 \begin{bmatrix} 1 \\ 1/3 \\ 0 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{t} + C_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$

Using the previous methodology of classifying the eigenspaces, we see that we have 3 1-Dimensional eigenspaces.

If the initial conditions are on the x_3 axis $[0\ 0\ 1]^T$ then we will remain there forever and we will exponentially converge to the origin:
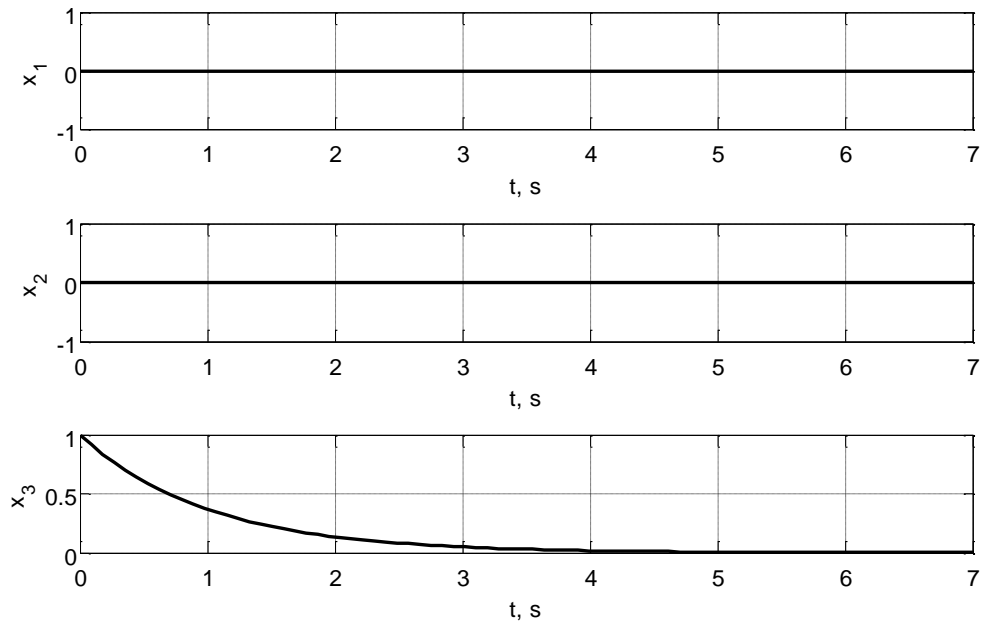
The state space response is:
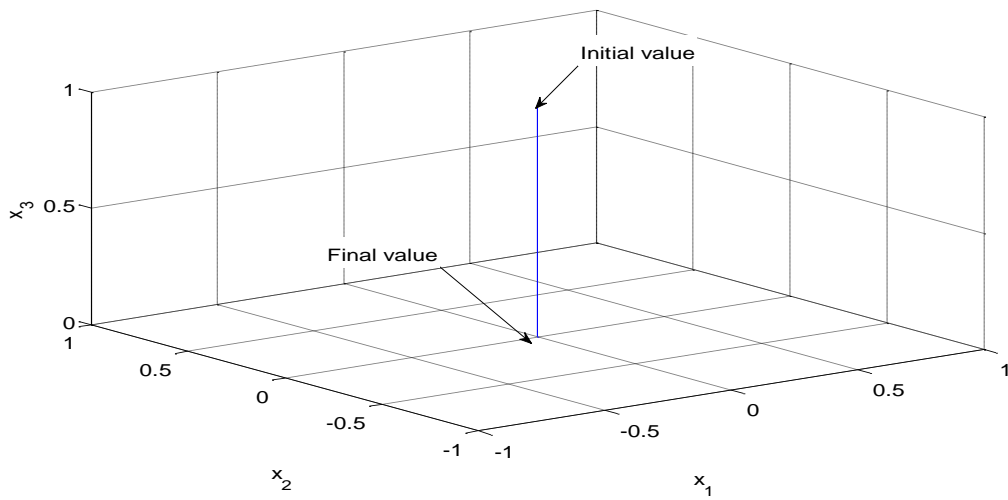
Similarly for the other eigenvectors. Now, if the ICs are such that $c_3 = 0$ then the solution will always be on the x_1-x_2 plane. For example if $\mathbf{x}(0) = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$ then $C_1 = -1.5$, $C_2 = 1.5$ and $C_3 = 0$
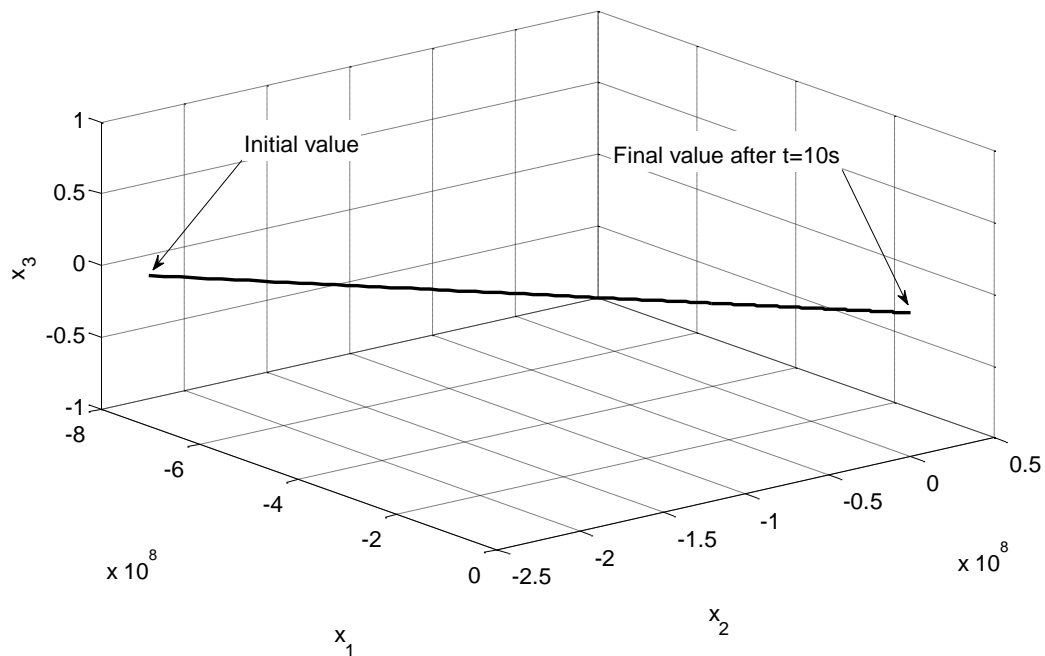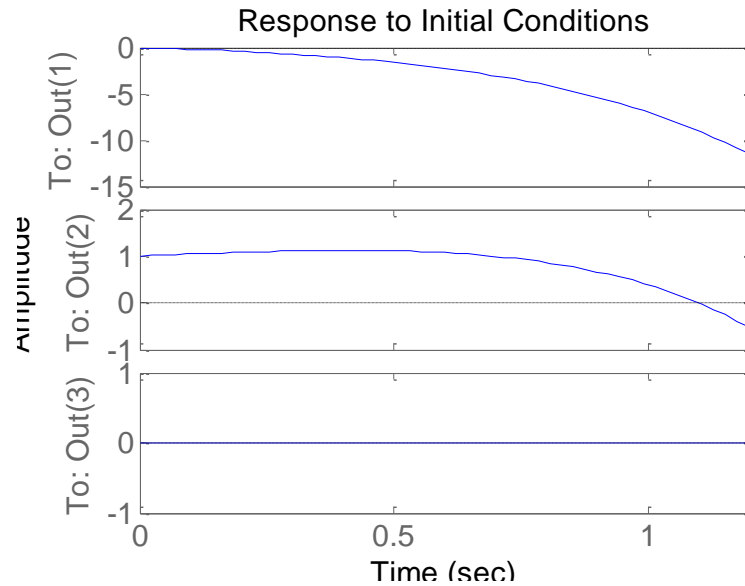
Now, we can see that if the initial conditions are $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ then that the orbit gradually converges to the x_1-x_2 plane and then it diverges along the $\begin{bmatrix} 1 \\ 1/3 \\ 0 \end{bmatrix}$:
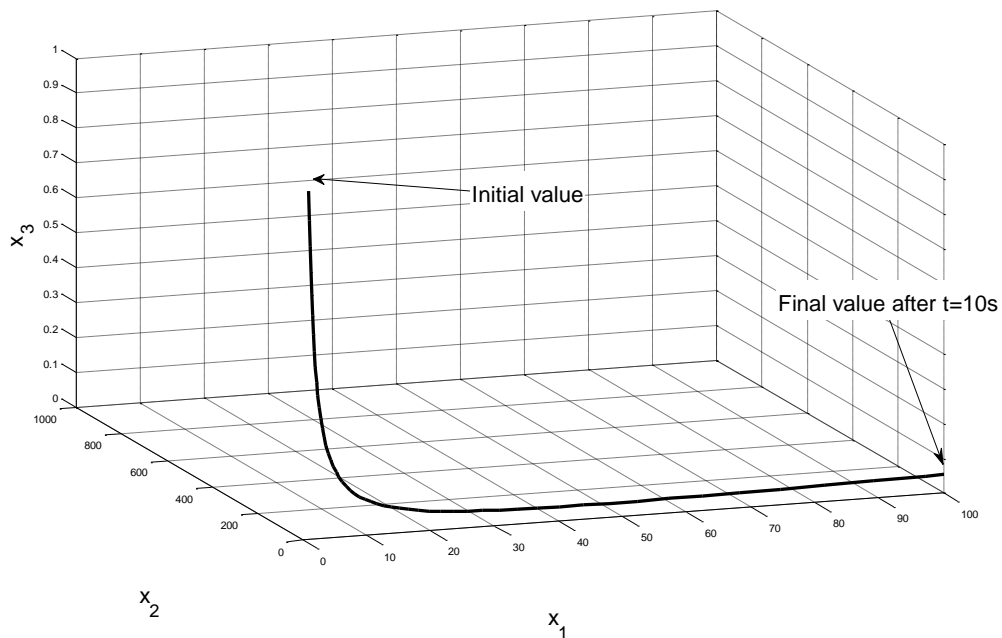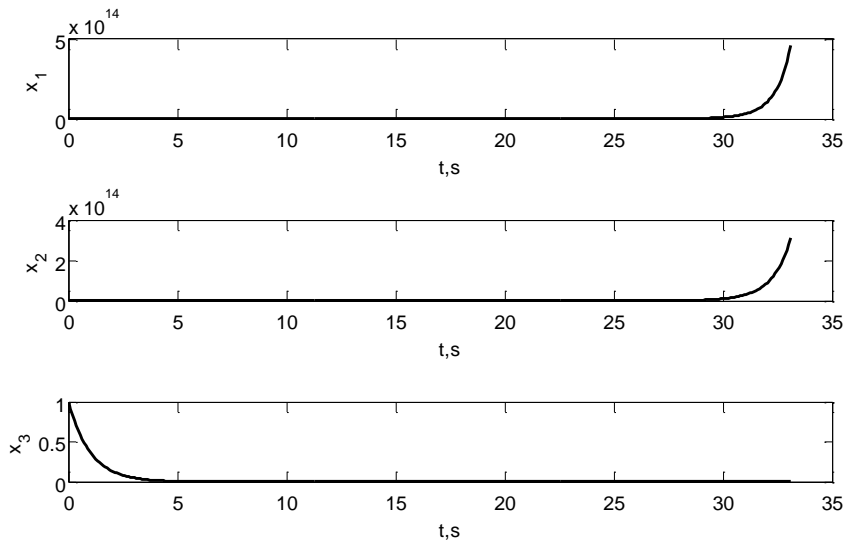
So to summarise, the general case is:
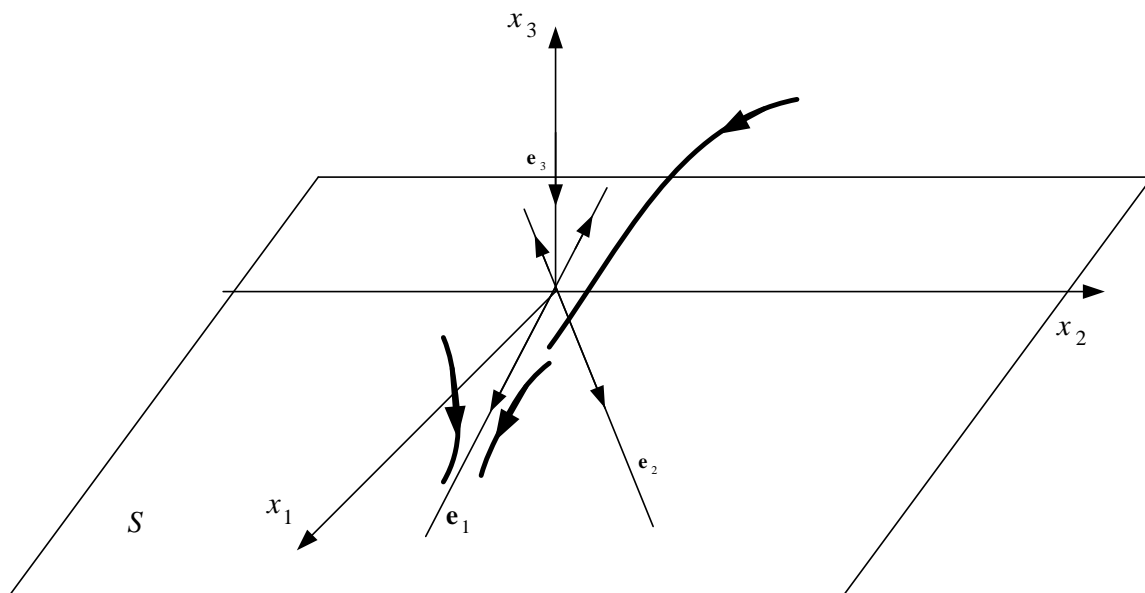
This implies that if we start on the x_1-x_2 plane (shown here as plane S) then we will converge along the eigenvector $\mathbf{e}_1$. Effectively the behaviour in the plane S is exactly the same as the behaviour of a standard unstable node.

Now, if we start on the axis x_3 then we will converge to the origin exponentially. If we start on a general point (not on S or x_3), then we will first converge on the plane S and gradually we will diverge across the eigenvector that corresponds to the eigenvalue with the maximum absolute eigenvalue. This is an example of a **3D saddle**.

So in that case we have 3 LI eigenvectors that create 3 invariant eigenspaces. Also, from the above we see that the x_1-x_2 plane is invariant under the solution of the state space model. This means that if we start in that plane then we will always remain there. Similarly if I start on the x_3 axis (i.e. the third eigenvector) then I will remain on that vector as $t \to \infty$. The x_1-x_2 plane

is invariant because is spanned by the 2 unstable eigenvectors. Similarly the x_3-axis is also invariant because is spanned (coincides to be more precise) by the stable eigenvector. Hence in that case we have a stable 1-dimensional and an unstable 2-dimensional invariant set.

Hence we can define 3 more invariant sets:

1. The unstable manifold $E^u(0)$: The invariant space spanned by all the "unstable" eigenvectors. In the previous example $E^u(0) = x_1, x_2$ plane

2. The stable manifold $E^s(0)$: The invariant space spanned by all the "stable" eigenvectors. In the previous example $E^s(0) = x_3$ axis

3. The centre manifold $E^c(0)$: The invariant space spanned by all the "neutral" eigenvectors (eigenvalues with zero real part). In the previous example $E^c(0) = \varnothing$.

Note: the symbols $E^c(0)$, $E^s(0)$, $E^u(0)$ refer to the manifolds of the origin.

Example (another **3D saddle**):

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{cases} \lambda_1 = -1 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T \\ \lambda_2 = -10 \Rightarrow \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \lambda_3 = 1 \Rightarrow \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \end{cases}$$

So the first eigenspace (corresponding to -10) is the x_2-axis, the second (for -1) is the x_1-axis and the third (for 1) is the x_3-axis. To test the invariance of each axis we start on them.
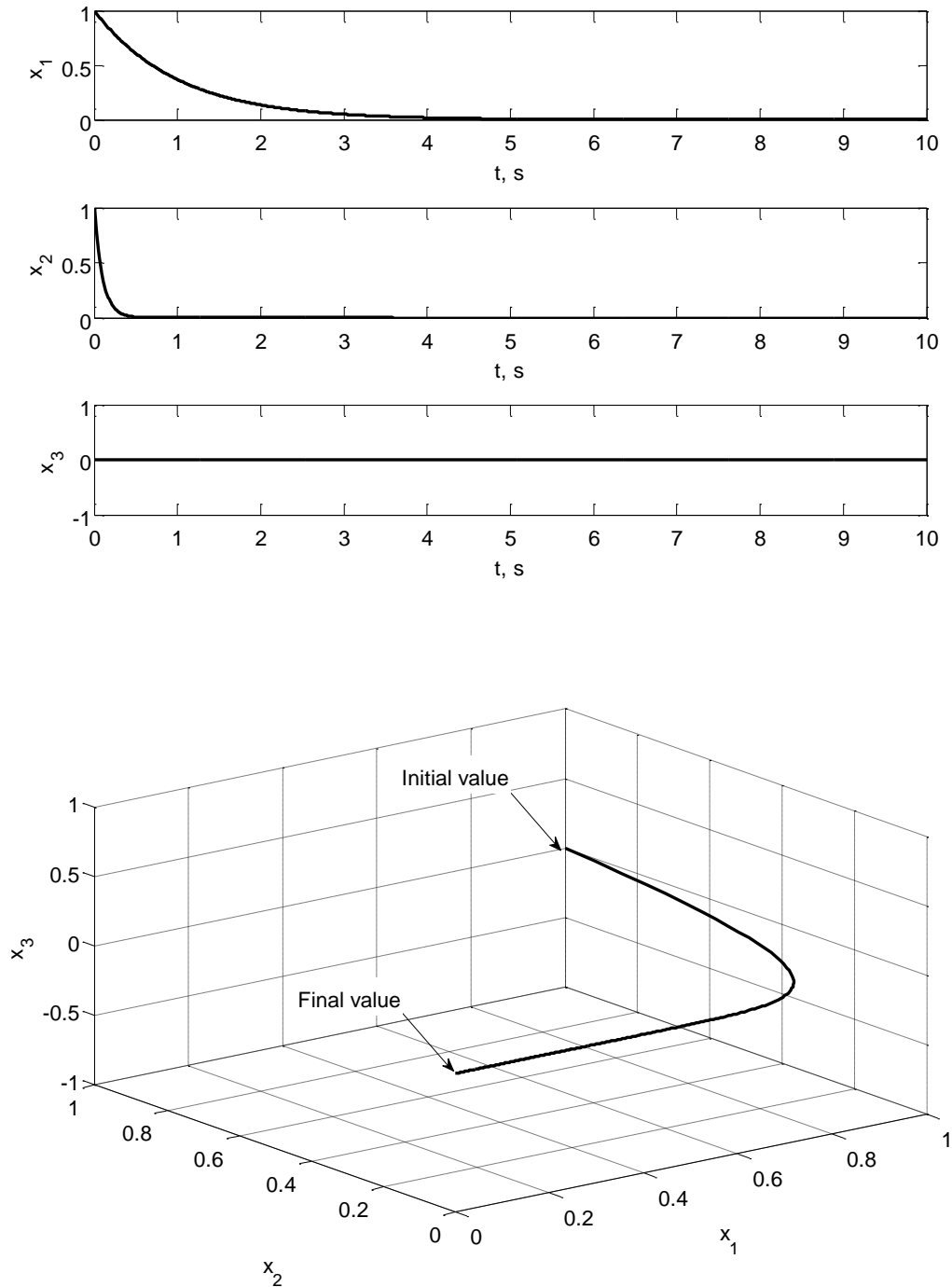
Case 1: Starting at $[0\ 0\ 1]^T$
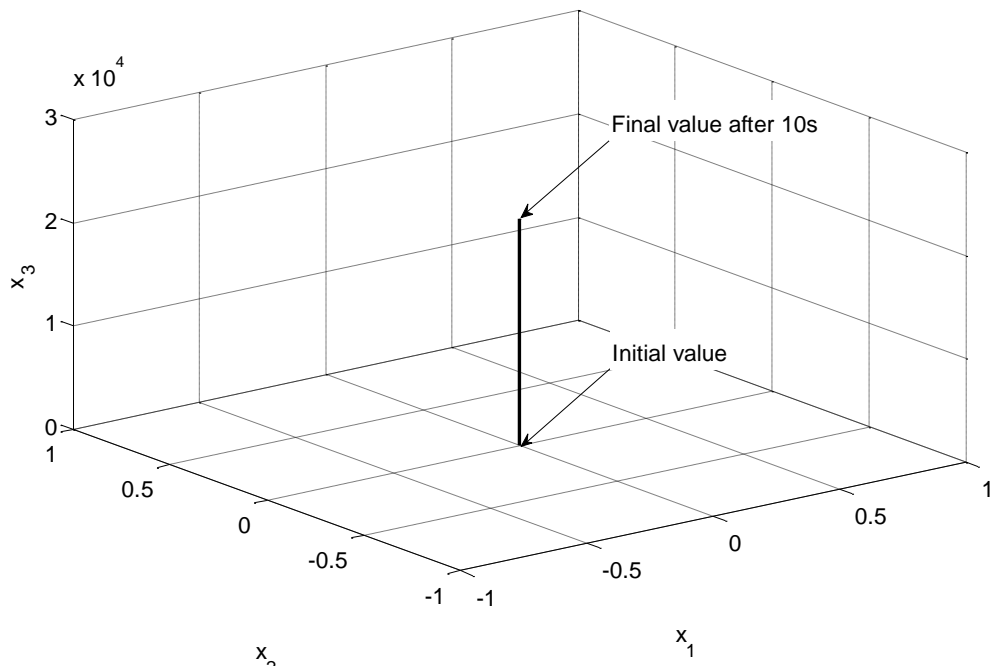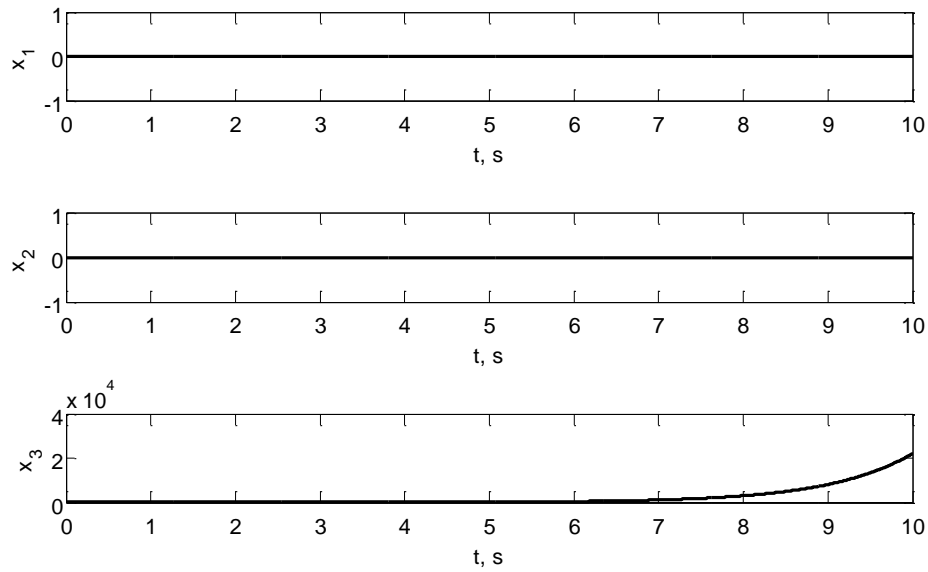
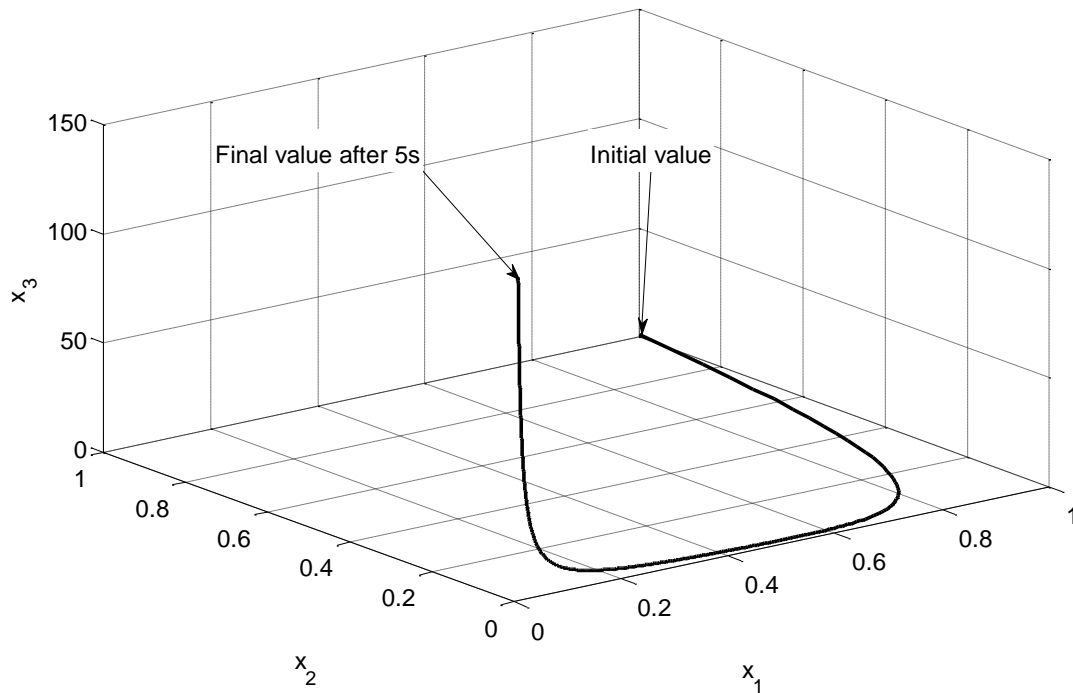Case 2: Starting at $[1\ 0\ 0]^T$
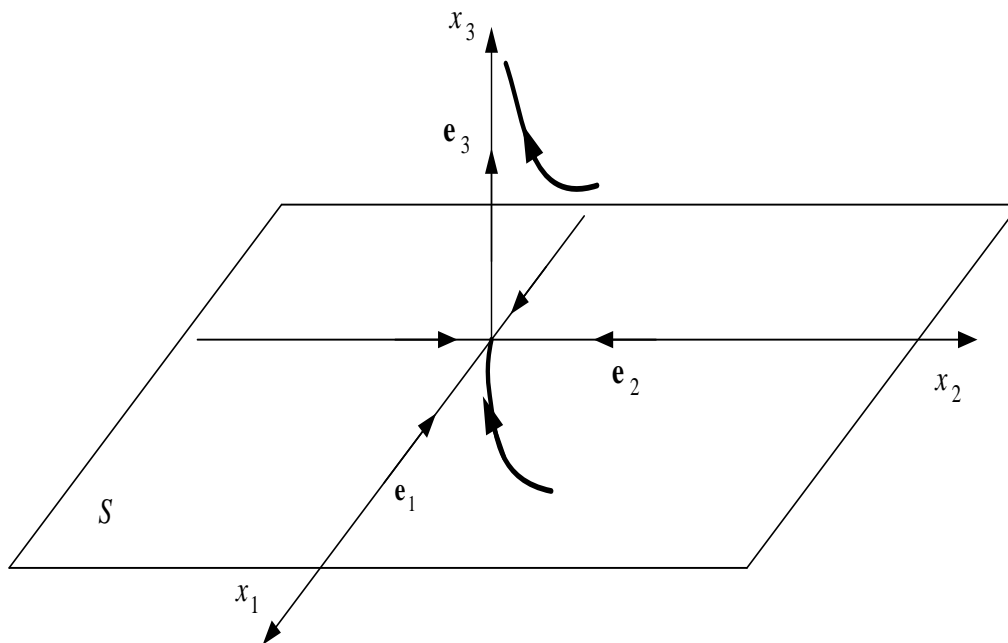
Case 3: Starting at $[0\ 1\ 0]^T$

Initial value

Final value after 10s
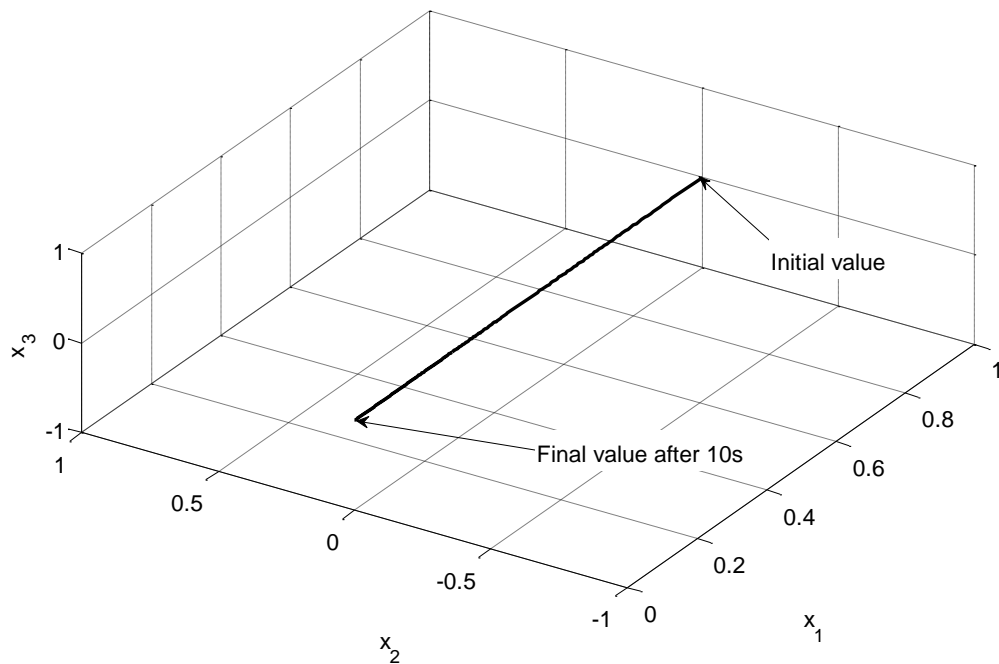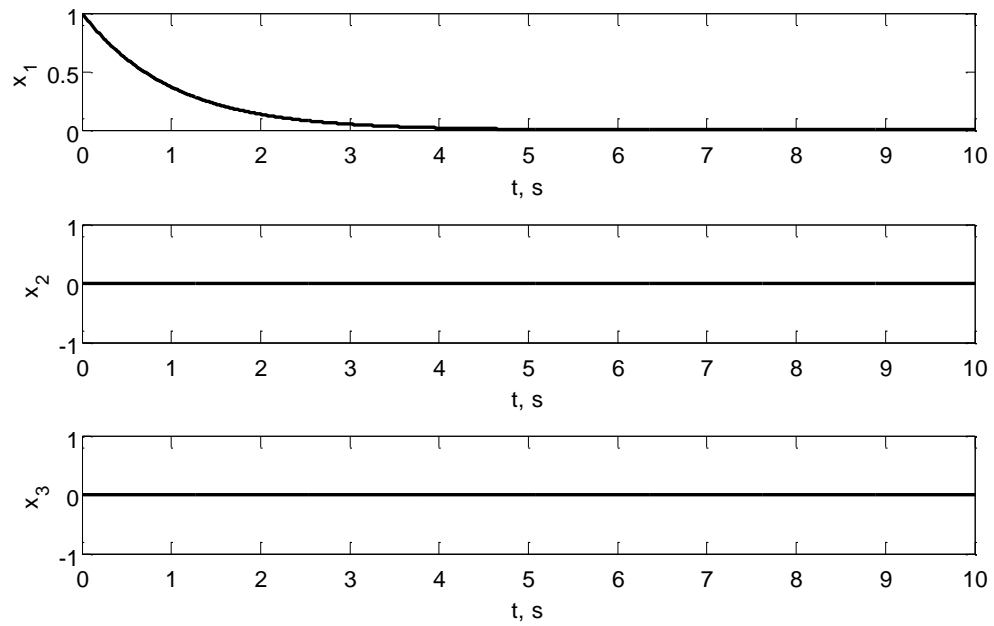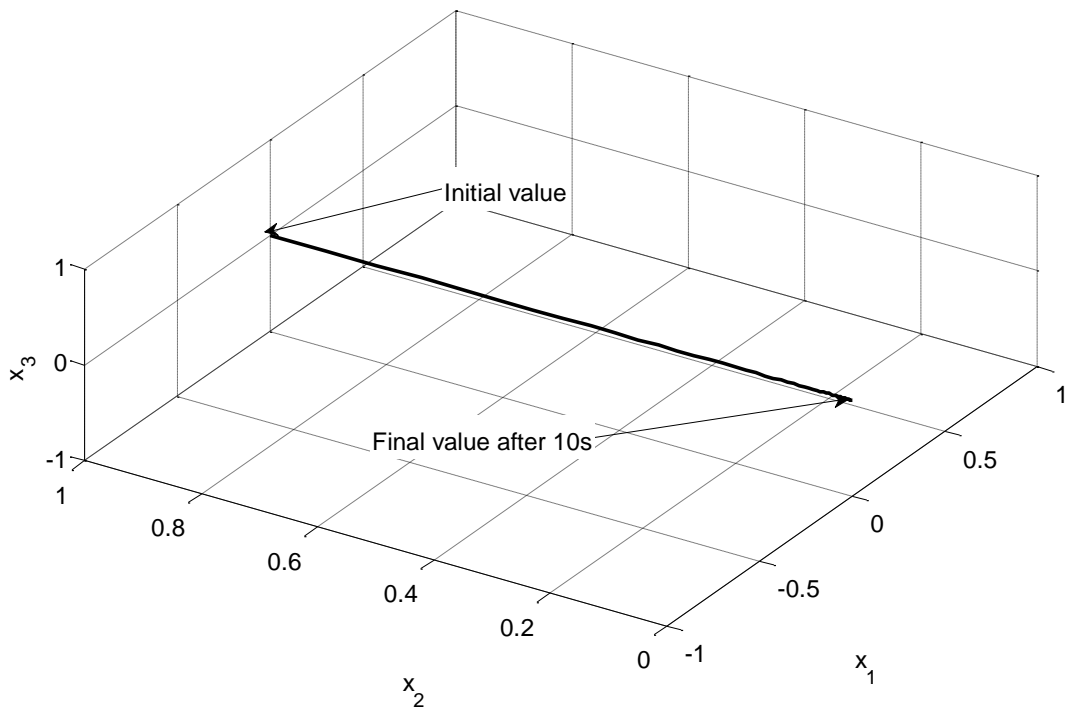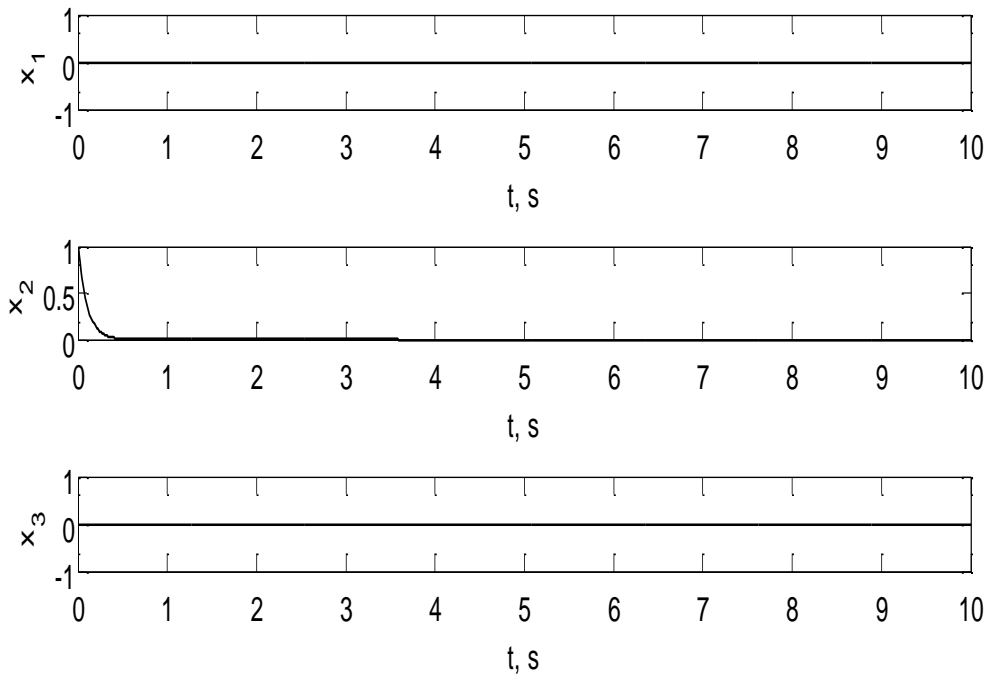
Now, we can see the invariance of the stable manifold, by starting at $[1\ 1\ 0]^T$:

And if we start on a general point $[1\ 1\ 1]^{\mathrm{T}}$:


In general the geometry of our system is:

If we start on the stable plane S we will remain there forever and we will converge to zero exponentially along the slow stable eigenvector $[1\ 0\ 0]^{\mathrm{T}}$. If

we start on the unstable eigenvector $[0\ 0\ 1]^T$ we will exponentially diverge to infinity along this vector. And if we start at another general point (i.e. not on the unstable eigenvector or the plane *S*) we will diverge to infinity along the $\mathbf{e}_3$ vector but as seen in S plane it slides along the slow eigenvector $\mathbf{e}_1$.

The manifolds are:

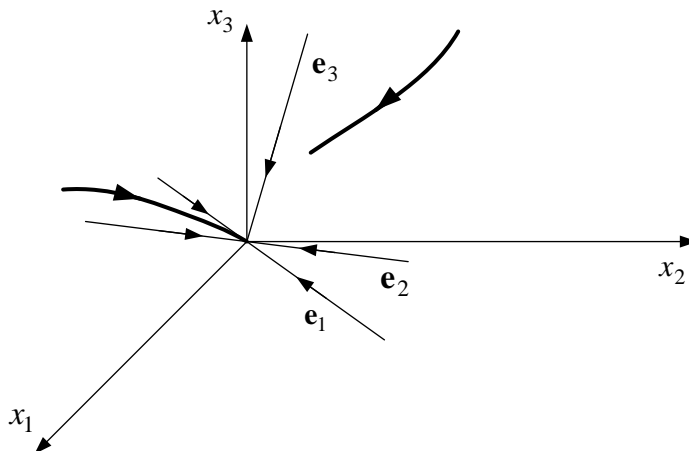1. $E^u(0) = $ The eigenspace created by $\mathbf{e}_3$
2. $E^s(0) = S$
3. $E^c(0) = \varnothing$

Other examples of 3 distinct eigenvalues are:
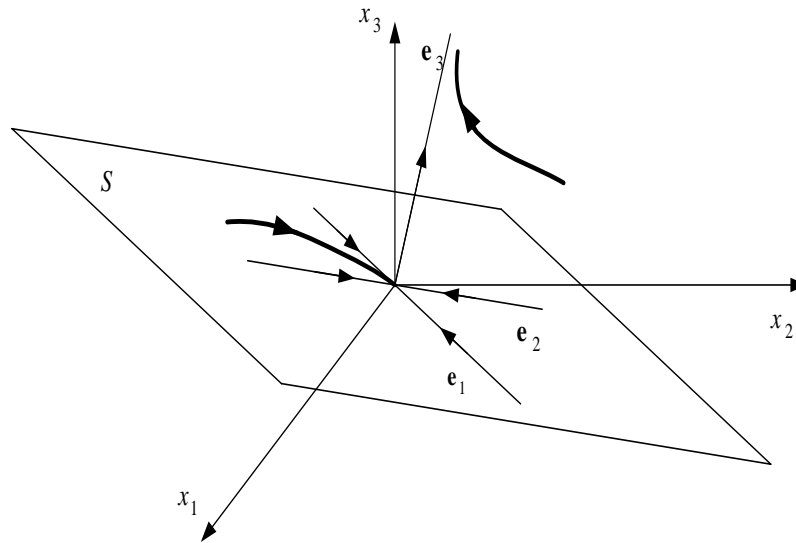
Again 3 invariant eigenspaces, and:

1. $E^u(0) = $ The   eigenspace   created   by $\mathbf{e}_3$
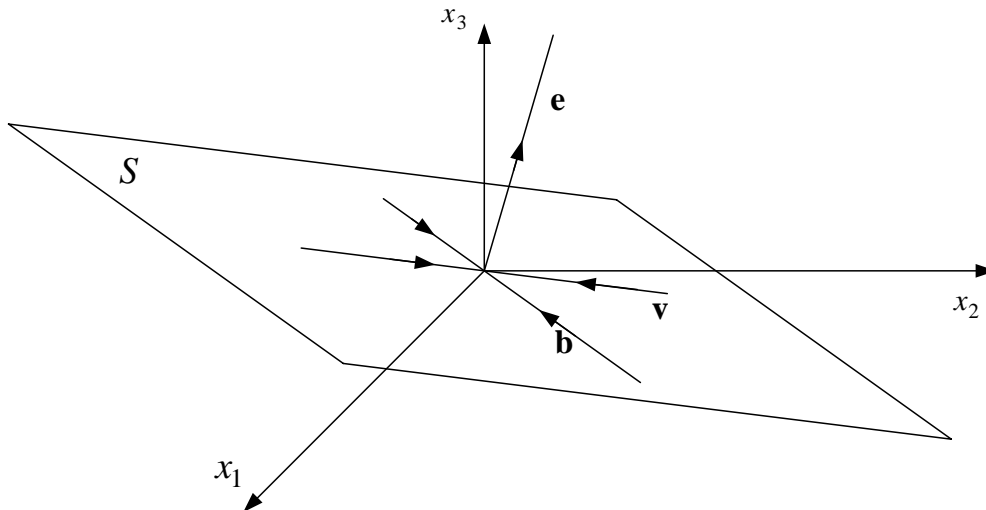2. $E^s(0) = S$
3. $E^c(0) = \varnothing$

Again 3 invariant eigenspaces, and:

1. $E^u(0) = \varnothing$
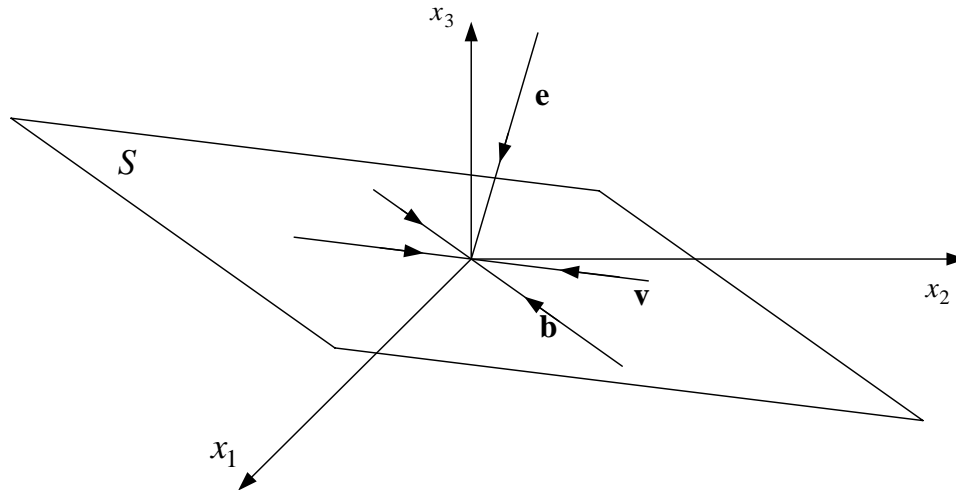2. $E^s(0) = $ The whole state space.
3. $E^c(0) = \varnothing$

Case B – Repeated eigenvalues (Assuming no LI eigenvectors)

Multiplicity 2: $\lambda_1 = \mathbf{v}, \mathbf{b}$ & $\lambda_2 = \mathbf{e}$

2 Invariant eigenspaces (one 2D by $\mathbf{b}$ and $\mathbf{v}$ and is denoted as S in the above figure and one from $\mathbf{e}$).
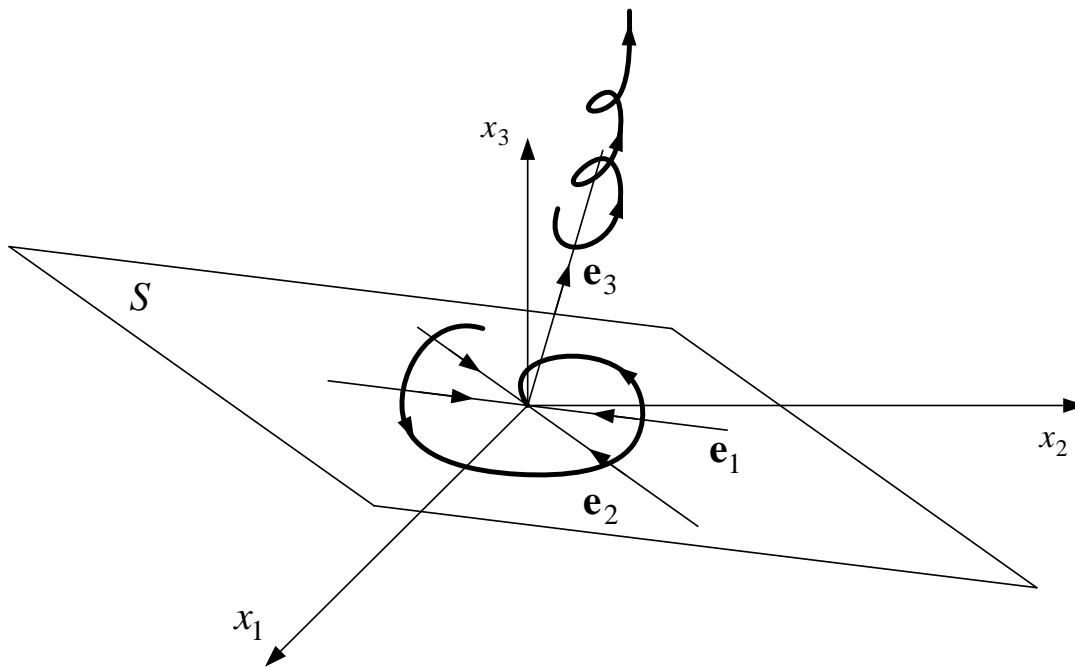
1. $E^u(0) = $ The eigenspace from $\mathbf{e}$
2. $E^s(0) = S$
3. $E^c(0) = \varnothing$

2 Invariant eigenspaces (one 2D by **b** and **v** and is denoted as S in the above figure and one from **e**).

1. $E^u(0) = \varnothing$
2. $E^s(0) = $ The whole state space.
3. $E^c(0) = \varnothing$

Multiplicity 3: $\lambda_1 = \mathbf{v}, \mathbf{b}, \mathbf{w}$

A similar case as before but now one of the $E^u(0)$, $E^s(0)$, $E^c(0)$ will be the whole space as well as the invariant eigenspace of λ_1 while the other two will be empty.

Case C – Complex eigenvalues

Now let's assume that we have a complex eigenvalue (and one real of course). As before I have an eigenvalue λ which is complex and this will give me a complex eigenvector $\mathbf{v}$. Using the same methodology as before we have 2 eigenvectors to express the solution: $\mathbf{e}_1 = \mathfrak{Re}(\mathbf{v})$ and $\mathbf{e}_2 = \mathfrak{Im}(\mathbf{v})$. These 2 vectors will define a 2-dimensional eigenspaces S.
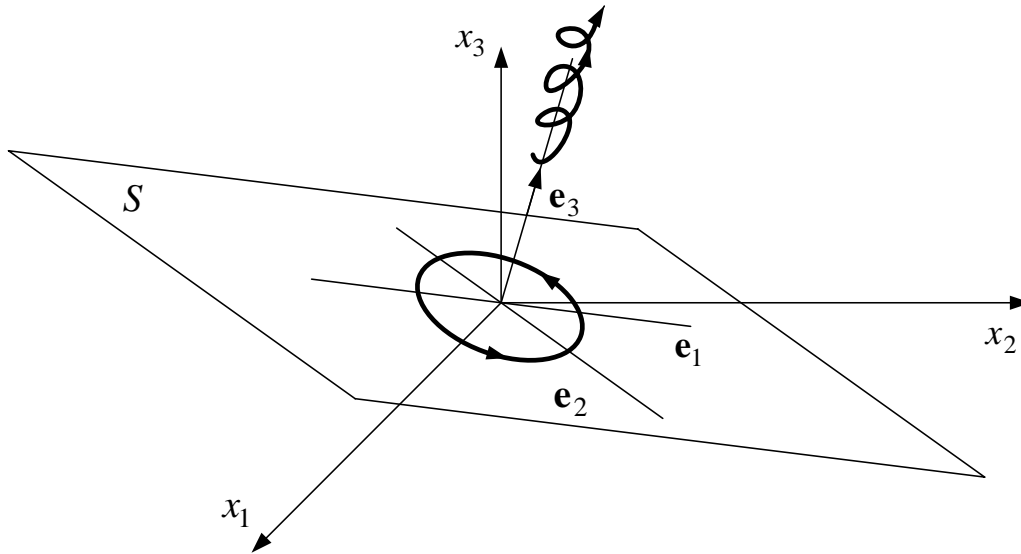

2 invariant eigenspaces, and:

1. $E^u(0) = $ The eigenspace of $\mathbf{e}_3$
2. $E^s(0) = S$
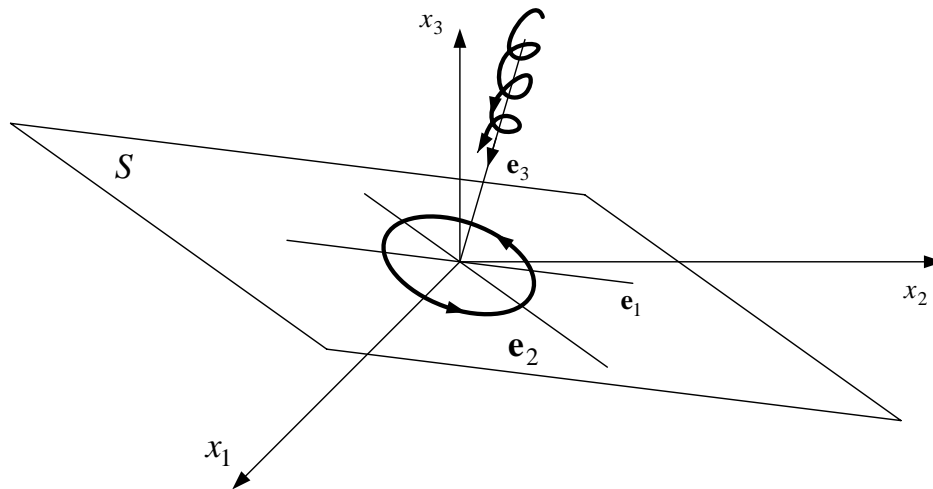3. $E^c(0) = \varnothing$
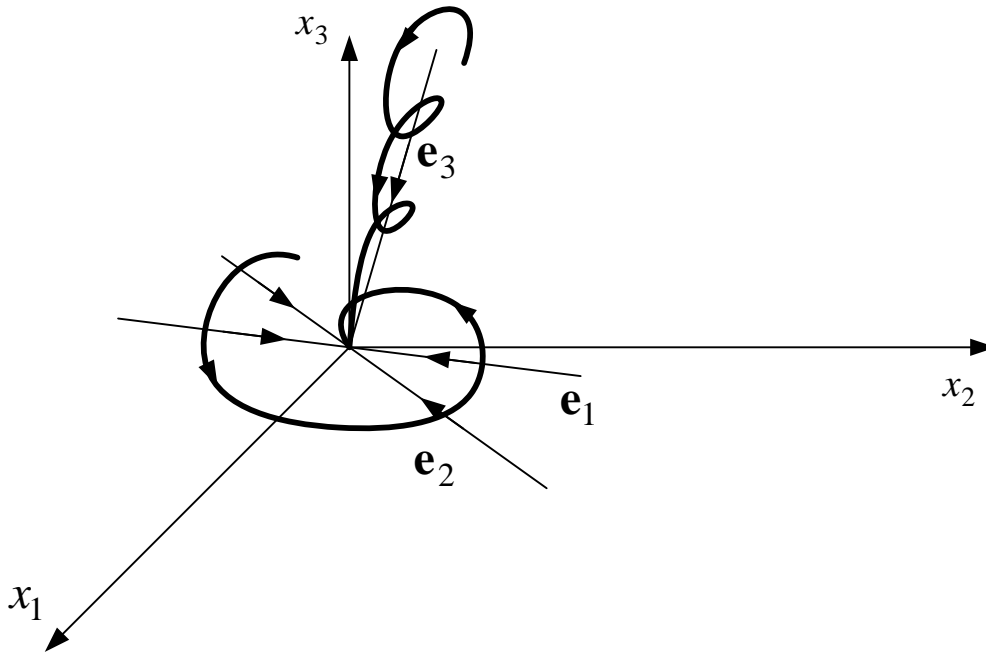
2 invariant eigenspaces, and:

1. $E^u(0) = \varnothing$

2. $E^s(0) =$ The whole state space

3. $E^c(0) = \varnothing$

2 invariant eigenspaces, and:

1. $E^u(0) =$ The eigenspace of $\mathbf{e}_3$

2. $E^s(0) = \varnothing$

3. $E^c(0) = S$

2 invariant eigenspaces, and:

1. $E^u(0) = \varnothing$

2. $E^s(0) =$ The eigenspace of $\mathbf{e}_3$

3. $E^c(0) = S$