Chapter #2

EEE 8007

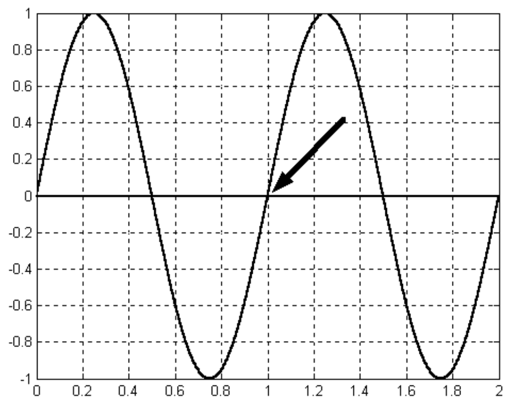
Digital Control

- **Random signals**

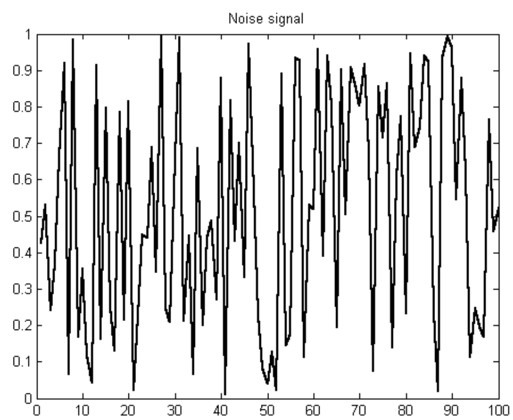
Random Signals

Deterministic signal: $x(t) = A\sin(2\pi f t)$.

If A=1 and f=1 Hz => @ t=1s $x(1) = 1\sin(2\pi) = 0$:



A random signal would be like:



To describe a random signal we have to define some of its properties:

Mean Value: $\mu_x = \frac{1}{N}\sum_{k=1}^{N} x(k)$ (1)

Expectation of x = Mean value: $E[x] = \mu$

DC power of the signal

Mean of the square of x(n) => the **total** (or average) **power**.

```
>> x=rand(1,100);
>> m=mean(x)
m =
    0.5126
```

Variance and **Standard Deviation**: range of values

Standard deviation: $\sigma_x = \sqrt{\frac{1}{N}\sum_{k=1}^{N}(x(k)-\mu)^2}$ (2)

The variance is the square of the standard deviation:

$$Var = \sigma_x{}^2 = \frac{1}{N}\sum_{k=1}^{N}(x(k)-\mu)^2 \qquad (3)$$

Or:

$$Var = \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k) + \mu^2 - 2x(k)\mu\right) = \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k)\right) + \frac{1}{N}\sum_{k=1}^{N}\left(\mu^2\right) - \frac{2}{N}\sum_{k=1}^{N}\left(x(k)\mu\right) =$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k)\right) + \frac{1}{N}\sum_{k=1}^{N}\left(\mu^2\right) - \frac{2}{N}\mu\sum_{k=1}^{N}\left(x(k)\right) = \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k)\right) + \frac{1}{N}\sum_{k=1}^{N}\left(\mu^2\right) - 2\mu\frac{1}{N}\sum_{k=1}^{N}\left(x(k)\right) =$$

$$= \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k)\right) + \mu^2 - 2\mu\mu = \frac{1}{N}\sum_{k=1}^{N}\left(x^2(k)\right) - \mu^2 \qquad (4)$$

And in terms of expectation:

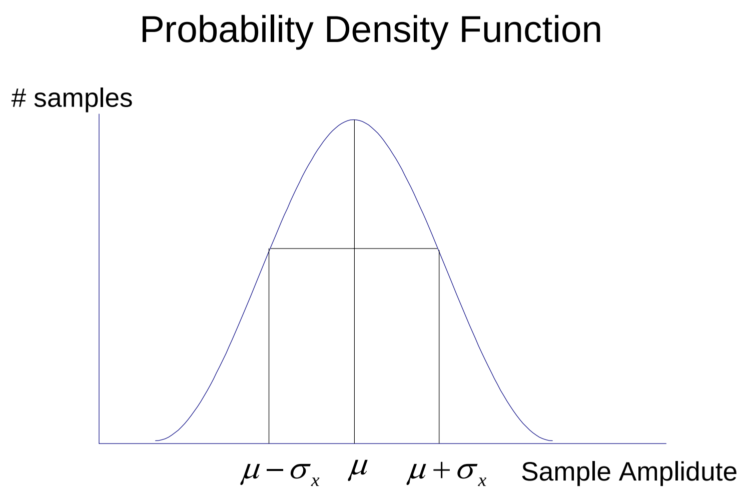
$$Var = E\left[\left(x(k) - \mu\right)^2\right] = E\left[\left(x(k)\right)^2\right] - \mu^2 = E\left[\left(x(k)\right)^2\right] - \left(E\left[x(k)\right]\right)^2 \qquad (3.5)$$

The variance is a measure of the AC power of a signal.

In Matlab

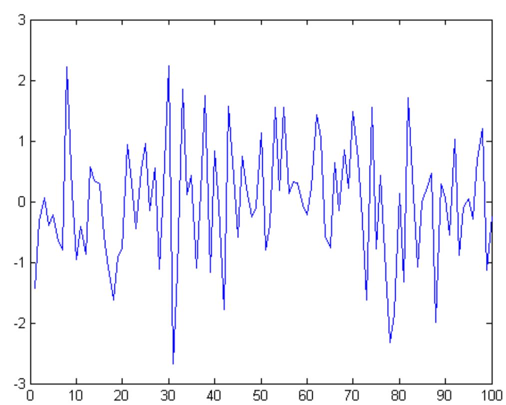
```
>> v=var(x)
v =
    0.0835
```

Normal Distribution: If the values of the random signal follow the next shape then the random signal is called Gaussian (or normal):



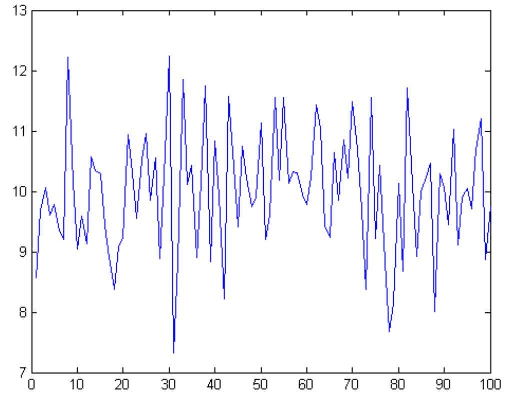
General speaking the Probability Density Function shows how likely is a value to appear in a random signal.

```
>> x=randn(1,100);
```



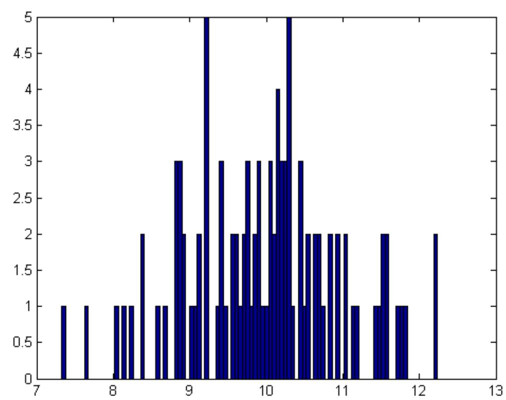
This is normally distribution signal with zero mean. If we want to increase the mean then:

```
>> x=x+10;
>> plot(x)
```



To find the probability density distribution we use:

```
>> plot(x)
>> hist(x,100)
```



White noise: A random signal whose frequency spectrum is flat. Hence all the frequencies will appear.

Correlation between random signals

Similarities among signals (random or not).

$$R = \frac{1}{N}\sum_{k=1}^{N} v(k)w(k) \tag{6}$$

If a signal is similar to another then the product of their samples will be positive and hence the above sum will increase.

Cross correlation

$$R_{vw}(j) = \frac{1}{N}\sum_{k=1}^{N} v(k+j)w(k) = E\{v[n+j]w[n]\} \tag{7}$$

Auto correlation:

$$R_{vv}(j) = \frac{1}{N}\sum_{k=1}^{N} v(k+j)v(k) = E\{v[n+j]v[n]\} \tag{8}$$

The value of the auto correlation for zero shift is the same as the expected square value, and hence is a measure of its **total power**:

$$R_{vv}(0) = \frac{1}{N}\sum_{k=1}^{N} v(k)v(k) = E\{v[n]v[n]\} = \frac{1}{N}\sum_{k=1}^{N} v^2(k) = E\{v^2[n]\} \tag{9}$$

The **cross correlation coefficient** is the normalised cross correlation:

$$R_{Nvw}(j) = \frac{R_{vw}(j)}{\sqrt{[R_{vv}(0)R_{ww}(0)]}} = \frac{Q_{vw}(j)}{\sigma_v \sigma_w} \tag{10}$$

$R_{vv}(0)$ and $R_{ww}(0)$ => represent the total power of each signal. This ratio will give us 1 if the two signals are totally correlated and zero if they are totally uncorrelated.

If the random signal has a mean value other than zero then we can define the **covariance**, **cross covariance** and the **auto covariance (also called covariance function)**:

$$Q_{vw}(j) = \frac{1}{N}\sum_{k=1}^{N}(v(k+j)-\mu_v)(w(k)-\mu_w) = E\{(v(k+j)-\mu_v)(w(k)-\mu_w)\} \tag{11}$$

$$Q_{vv}(j) = \frac{1}{N}\sum_{k=1}^{N}(v(k+j)-\mu_v)(v(k)-\mu_v) = E\{(v(k+j)-\mu_v)(v(k)-\mu_v)\} \tag{12}$$

The auto covariance of random signal is a Dirac pulse when the shift is zero.

$$Q_{vv}(0) = \frac{1}{N}\sum_{k=1}^{N}(v(k)-\mu_v)(v(k)-\mu_v) = E\{(v(k)-\mu_v)(v(k)-\mu_v)\} = \frac{1}{N}\sum_{k=1}^{N}(v(k)-\mu_v)^2 =$$
$$= E\{(v(k)-\mu_v)^2\} = \sigma_v^2 = \text{Variance of } v(k) \tag{13}$$

An important concept that can be defined is that if the two signals are **uncorrelated** (totally independent) then their cross covariance and cross correlation is zero.

Many times it is useful to find the cross covariance and the auto covariance without the shift. Then we will end up to the **covariance**:

$$Q_{vw}(0) = C_{vw} = \frac{1}{N}\sum_{k=1}^{N}(v(k)-\mu_v)(w(k)-\mu_w) = E\{(v(k)-\mu_v)(w(k)-\mu_w)\} = \sigma^2{}_{vw} \quad (14)$$

$$Q_{vv}(0) = C_{vv} = \frac{1}{N}\sum_{k=1}^{N}(v(k)-\mu_v)(v(k)-\mu_v) = E\{(v(k)-\mu_v)(v(k)-\mu_v)\} = \sigma^2{}_{v} \quad (15)$$

As we defined the cross correlation coefficient we can define the cross covariance coefficient as:

$$Q_{Nvw}(j) = \frac{Q_{vw}(j)}{\sqrt{[Q_{vv}(0)Q_{ww}(0)]}} = \frac{Q_{vw}(j)}{\sigma_v \sigma_w} \quad (16)$$

and since we usually do not use the shift in the covariance:

$$Q_{Nvw}(0) = \frac{Q_{vw}(0)}{\sqrt{[Q_{vv}(0)Q_{ww}(0)]}} = \frac{\sigma^2{}_{vw}}{\sigma_v \sigma_w} \quad (17)$$

C**ovariance matrix** of 2 signals:

$$\mathbf{C} = \operatorname{cov}(X(j)) = \begin{bmatrix} Q_{vv}(j) & Q_{vw}(j) \\ Q_{wv}(j) & Q_{ww}(j) \end{bmatrix} =$$

$$= \begin{bmatrix} E((v(k+j)-\mu_v)(v(k)-\mu_v)) & E((v(k+j)-\mu_v)(w(k)-\mu_w)) \\ E(w(k+j)-\mu_w)(v(k)-\mu_v) & E(w(k+j)-\mu_w)(w(k)-\mu_w) \end{bmatrix} =$$

$$\stackrel{j=0}{=} \begin{bmatrix} E((v(k)-\mu_v)(v(k)-\mu_v)) & E((v(k)-\mu_v)(w(k)-\mu_w)) \\ E((w(k)-\mu_w)(v(k)-\mu_v)) & E((w(k)-\mu_w)(w(k)-\mu_w)) \end{bmatrix} = \begin{bmatrix} \sigma_v{}^2 & \sigma_{vw}{}^2 \\ \sigma_{wv}{}^2 & \sigma_w{}^2 \end{bmatrix} \quad (18)$$

Zero mean

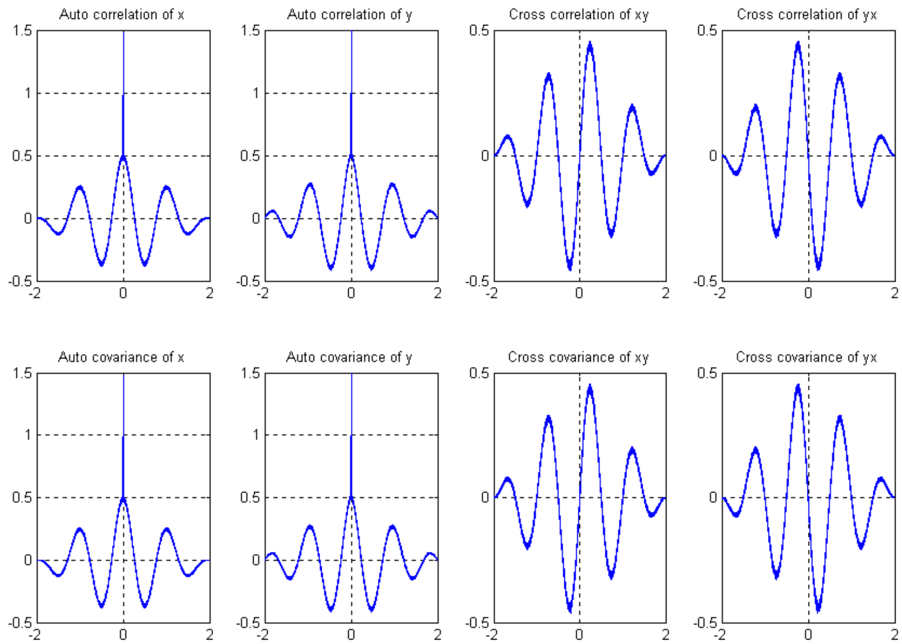
$$\mathbf{C}(X(0)) \stackrel{j=0}{=} \begin{bmatrix} Ev(k)v(k) & Ev(k)w(k) \\ Ew(k)v(k) & Ew(k)w(k) \end{bmatrix} = \begin{bmatrix} \sigma_v{}^2 & \sigma_{vw}{}^2 \\ \sigma_{wv}{}^2 & \sigma_w{}^2 \end{bmatrix} \quad (19)$$

To investigate all these in Matlab initially we will use the above terms in a sine wave and in a cosine wave (with some noise):

```
clc
clear
close all

t=0:0.0001:2;
x1=sin(2*pi*t);
y1=cos(2*pi*t);

y=awgn(y1,0.1);
x=awgn(x1,0.1);

m_x=mean(x)
m_y=mean(y)
```

```
var_x=var(x)
var_y=var(y)

std_x=std(x)
std_y=std(y)

cov_x=cov(x)
cov_y=cov(y)

cov_xy=cov(x,y)
cov_yx=cov(y,x)

corrcoef_xy=corrcoef(x,y)
corrcoef_yx=corrcoef(y,x)

Rxx=xcorr(x,x)/length(t);       % Auto corelation
of x
tshift=-max(t):0.0001:max(t);   % Shift
subplot(2,4,1)
plot(tshift,Rxx), grid
title('Auto correlation of x')

Ryy=xcorr(y,y)/length(t);       % Auto corelation
of y
subplot(2,4,2)
plot(tshift,Ryy), grid
title('Auto correlation of y')

Rxy=xcorr(x,y)/length(t);       % Cross corelation
of xy
subplot(2,4,3)
plot(tshift,Rxy), grid
title('Cross correlation of xy')

Ryx=xcorr(y,x)/length(t);       % Cross corelation
of yx
subplot(2,4,4)
plot(tshift,Ryx), grid
title('Cross correlation of yx')
```

```
Cx=xcov(x,x)/length(t);       % Auto covariance of
x
subplot(2,4,5)
plot(tshift,Cx), grid
title('Auto covariance of x')

Cy=xcov(y,y)/length(t);       % Auto covariance of
y
subplot(2,4,6)
plot(tshift,Cy), grid
title('Auto covariance of y')


Cxy=xcov(x,y)/length(t);       % Cross covariance
of xy
subplot(2,4,7)
plot(tshift,Cxy), grid
title('Cross covariance of xy')

Cyx=xcov(y,x)/length(t);       % Cross covariance
of yx
subplot(2,4,8)
plot(tshift,Cyx), grid
title('Cross covariance of yx')
```




```
m_x =
   -0.0076
m_y =
    0.0067
var_x =
    1.4757
var_y =
    1.4935
std_x =
    1.2148
std_y =
    1.2221
cov_x =
    1.4757
cov_y =
    1.4935
cov_xy =
    1.4757    0.0065
    0.0065    1.4935
cov_yx =
    1.4935    0.0065
```

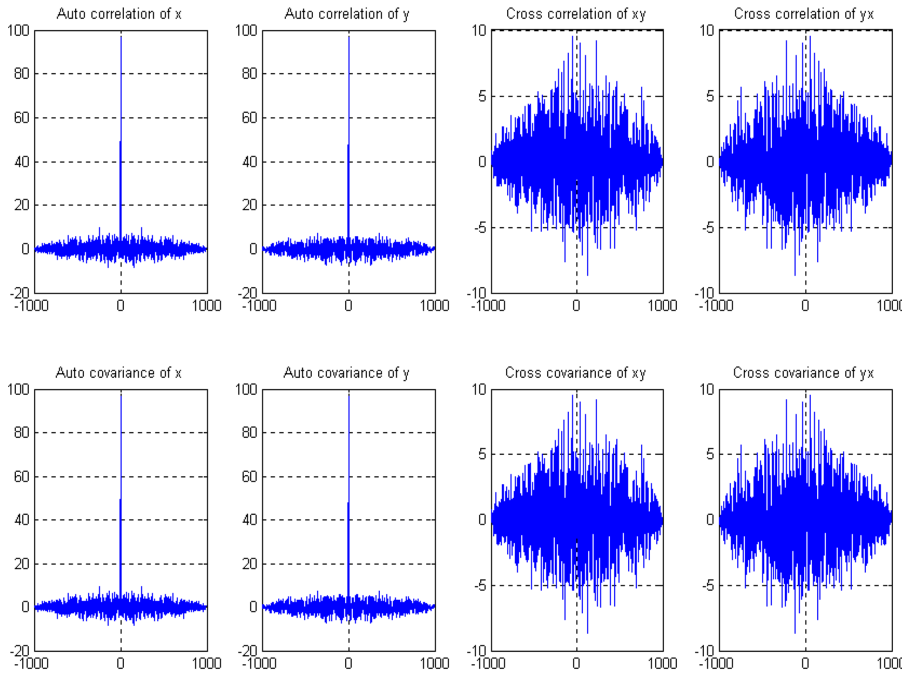
```
    0.0065    1.4757
corrcoef_xy =
    1.0000    0.0043
    0.0043    1.0000
corrcoef_yx =
    1.0000    0.0043
    0.0043    1.0000
```

The same with two random signals with zero mean

```
clc
clear
close all

y=10*randn(1,1000);
x=10*randn(1,1000);

m_x=mean(x)
m_y=mean(y)

var_x=var(x)
var_y=var(y)

std_x=std(x)
std_y=std(y)

cov_x=cov(x)
cov_y=cov(y)

cov_xy=cov(x,y)
cov_yx=cov(y,x)

corrcoef_xy=corrcoef(x,y)
corrcoef_yx=corrcoef(y,x)

Rxx=xcorr(x,x)/length(x);       % Auto corelation
of x
```

```
% tshift=linspace(-length(x),length(x),1000);
% Shift
tshift=-999:999;
subplot(2,4,1)
plot(tshift,Rxx), grid
title('Auto correlation of x')

Ryy=xcorr(y,y)/length(x);       % Auto corelation
of y
subplot(2,4,2)
plot(tshift,Ryy), grid
title('Auto correlation of y')

Rxy=xcorr(x,y)/length(x);       % Cross corelation
of xy
subplot(2,4,3)
plot(tshift,Rxy), grid
title('Cross correlation of xy')

Ryx=xcorr(y,x)/length(x);       % Cross corelation
of yx
subplot(2,4,4)
plot(tshift,Ryx), grid
title('Cross correlation of yx')

Cx=xcov(x,x)/length(x);       % Auto covariance of
x
subplot(2,4,5)
plot(tshift,Cx), grid
title('Auto covariance of x')

Cy=xcov(y,y)/length(x);       % Auto covariance of
y
subplot(2,4,6)
plot(tshift,Cy), grid
title('Auto covariance of y')


Cxy=xcov(x,y)/length(x);       % Cross covariance
of xy
```

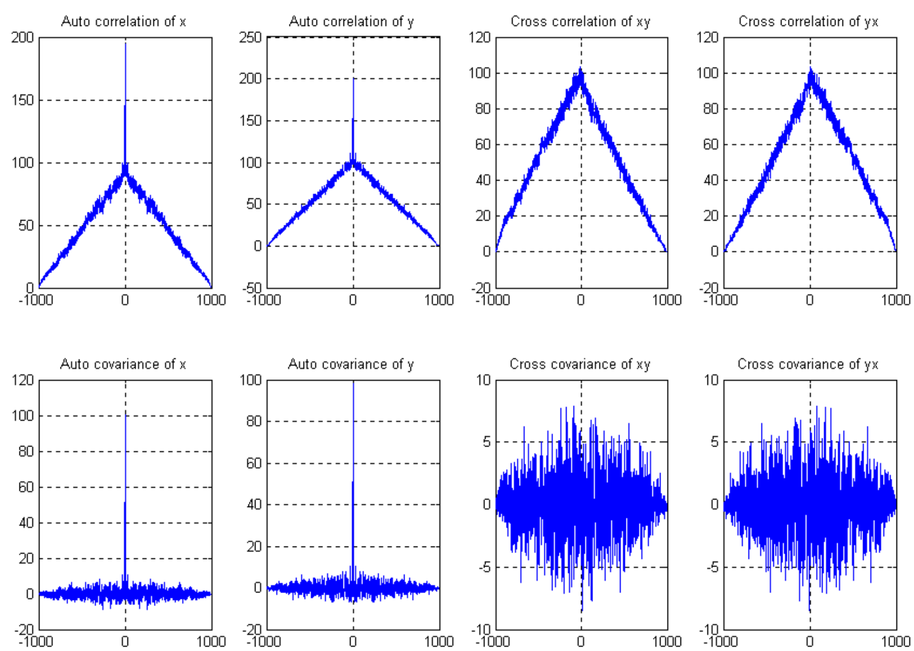
```
subplot(2,4,7)
plot(tshift,Cxy), grid
title('Cross covariance of xy')

Cyx=xcov(y,x)/length(x);        % Cross covariance
of yx
subplot(2,4,8)
plot(tshift,Cyx), grid
title('Cross covariance of yx')
```




```
m_x =
   -0.2104
m_y =
    0.0112
var_x =
   96.6653
var_y =
   96.8008
std_x =
    9.8319
std_y =
```

```
    9.8387
cov_x =
   96.6653
cov_y =
   96.8008
cov_xy =
   96.6653   -0.7819
   -0.7819   96.8008
cov_yx =
   96.8008   -0.7819
   -0.7819   96.6653
corrcoef_xy =
    1.0000   -0.0081
   -0.0081    1.0000
corrcoef_yx =
    1.0000   -0.0081
   -0.0081    1.0000
```

And with a mean +10:



```
m_x =
    9.6748
m_y =
   10.1080
var_x =
  100.6721
var_y =
   98.1436
std_x =
   10.0335
std_y =
    9.9067
cov_x =
  100.6721
cov_y =
   98.1436
cov_xy =
  100.6721    2.7561
    2.7561   98.1436
cov_yx =
   98.1436    2.7561
    2.7561  100.6721
corrcoef_xy =
    1.0000    0.0277
    0.0277    1.0000
corrcoef_yx =
    1.0000    0.0277
1.0000
```

Random Signals and Vectors:

If I have n random signals then we can use vector notation:

$$\mathbf{X} = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_n(k) \end{bmatrix}$$, where $x_1(k)$, x_2, … $x_n(k)$ are random signals.

Each of these random signals (white or not) has an expected (mean) value:

$$E(\mathbf{X}) = \begin{bmatrix} E(x_1(k)) \\ E(x_2(k)) \\ \vdots \\ E(x_n(k)) \end{bmatrix} \tag{20}$$

The covariance matrix now can be defined as

$$\mathbf{C} = \operatorname{cov}(\mathbf{X}(j)) = \begin{bmatrix} Q_{x_1x_1}(j) & Q_{x_1x_2}(j) & \cdots & Q_{x_1x_n}(j) \\ Q_{x_2x_1}(j) & Q_{x2x_2}(j) & \cdots & Q_{x_2x}(j) \\ \cdots & \cdots & \cdots & \cdots \\ Q_{x_nx_2}(j) & Q_{x_nx_2}(j) & \cdots & Q_{x_nx_n}(j) \end{bmatrix} =$$

$$= \begin{bmatrix} E((x_1(k+j)-\mu_{x_1})(x_1(k)-\mu_{x_1})) & E((x_1(k+j)-\mu_{x_1})(x_2(k)-\mu_{x_2})) & \cdots & E((x_1(k+j)-\mu_{x_1})(x_n(k)-\mu_{x_n})) \\ E((x_2(k+j)-\mu_{x_2})(x_1(k)-\mu_{x_1})) & E((x_2(k+j)-\mu_{x_2})(x_2(k)-\mu_{x_2})) & \cdots & E((x_2(k+j)-\mu_{x_2})(x_n(k)-\mu_{x_n})) \\ \cdots & \cdots & \cdots & \cdots \\ E((x_n(k+j)-\mu_{x_n})(x_1(k)-\mu_{x_1})) & E((x_n(k+j)-\mu_{x_n})(x_2(k)-\mu_{x_2})) & \cdots & E((x_n(k+j)-\mu_{x_n})(x_n(k)-\mu_{x_n})) \end{bmatrix} =$$

$$\overset{j=0}{=}\begin{bmatrix} E\left(\left(x_1(k)-\mu_{x_1}\right)\left(x_1(k)-\mu_{x_1}\right)\right) & E\left(\left(x_1(k)-\mu_{x_1}\right)\left(x_2(k)-\mu_{x_2}\right)\right) & \cdots & E\left(\left(x_1(k)-\mu_{x_1}\right)\left(x_n(k)-\mu_{x_n}\right)\right) \\ E\left(\left(x_2(k)-\mu_{x_2}\right)\left(x_1(k)-\mu_{x_1}\right)\right) & E\left(\left(x_2(k)-\mu_{x_2}\right)\left(x_2(k)-\mu_{x_2}\right)\right) & \cdots & E\left(\left(x_2(k)-\mu_{x_2}\right)\left(x_n(k)-\mu_{x_n}\right)\right) \\ \cdots & \cdots & \cdots & \cdots \\ E\left(\left(x_n(k)-\mu_{x_n}\right)\left(x_1(k)-\mu_{x_1}\right)\right) & E\left(\left(x_n(k)-\mu_{x_n}\right)\left(x_2(k)-\mu_{x_2}\right)\right) & \cdots & E\left(\left(x_n(k)-\mu_{x_n}\right)\left(x_n(k)-\mu_{x_n}\right)\right) \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2}^2 & \cdots & \sigma_{x_1x_n}^2 \\ \sigma_{x_2x_1}^2 & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{x_nx_1}^2 & \sigma_{x_nx_2}^2 & \cdots & \sigma_{x_nx_n}^2 \end{bmatrix} \tag{21}$$

And if there is no mean:

$$\mathbf{C} = \operatorname{cov}(\mathbf{X}(j)) = \begin{bmatrix} E((x_1(k))(x_1(k))) & E((x_1(k))(x_2(k))) & \cdots & E((x_1(k))(x_n(k))) \\ E((x_2(k))(x_1(k))) & E((x_2(k))(x_2(k))) & \cdots & E((x_2(k))(x_n(k))) \\ \cdots & \cdots & \cdots & \cdots \\ E((x_n(k))(x_1(k))) & E((x_n(k))(x_2(k))) & \cdots & E((x_n(k))(x_n(k))) \end{bmatrix} =$$

$$= \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2}^2 & \cdots & \sigma_{x_1x_n}^2 \\ \sigma_{x_2x_1}^2 & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ \sigma_{x_nx_1}^2 & \sigma_{x_nx_2}^2 & \cdots & \sigma_{x_nx_n}^2 \end{bmatrix} \tag{22}$$