Revision

ODE $\rightarrow$ S.S.

$$x^{(n)} + a_{n-1} \cdot x^{(n-1)} + \dots + a_0 x = b_1 u_1 + b_2 u_2 + \dots$$

1) $x^{(n)} = -a_{n-1} x^{(n-1)} \dots - a_0 x + b_1 u_1 + b_2 u_2 + \dots$

2) $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$, $\vdots$, $x_n = x^{(n-1)}$ $\overset{d/dt}{\Rightarrow}$ $\dot{x}_1 = \dot{x} = x_2 \Rightarrow \dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = x_3 \Rightarrow \dot{x}_2 = x_3$

$\vdots$

$\dot{x}_n = x^{(n)} =$

$-a_{n-1} \cdot x^{(n-1)} \dots - a_0 x + b_1 u_1 + \dots$

$-a_{n-1} x_n - \dots - a_0 x_1 + b_1 u_1 + \dots$

$$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ & & n\times n & & \\ -a_0 & -a_1 & \dots & & -a_{n-1} \end{bmatrix}}_{} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} + \underbrace{\begin{bmatrix} 0\,0\,0\,0 \\ 0\,0 \dots 0 \\ \dots \\ b_1 \; b_2 \dots \end{bmatrix}}_{n\times m} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \end{bmatrix}}_{m\times 1}$$

$y_1 = c_{11} \cdot x_1 + c_{12} x_2 + \dots + c_{1n} x_n + 0 \dots$

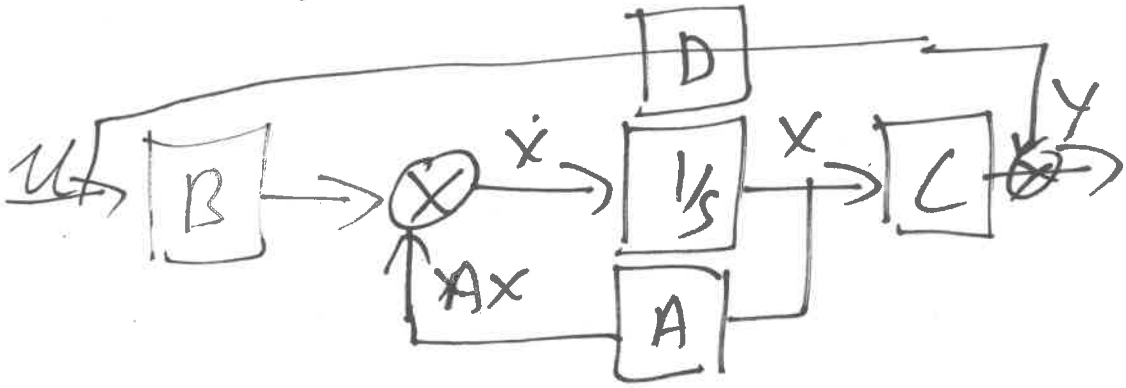
$\vdots$

$y_p = c_{p1} x_1 + c_{p2} x_2 + \dots + c_{pn} x_n + 0$

$$y = \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}}_{p\times 1} = \underbrace{\begin{bmatrix} c_{11} & \dots & c_{1n} \\ \dots & & \dots \\ \dots & p\times n & c_{pn} \end{bmatrix}}_{} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rightarrow n\times 1$$

$\dot{x} = Ax + Bu$

$y = C \cdot x + D \cdot u.$



• ODE $\quad x^{(n)} + \ldots\ldots + = 0 \quad u \to 0$

L.T. ↓ TF ↑ I.L.T. $\quad \hookrightarrow r^n + \ldots\ldots + \cancel{a} = 0$

C.E.

$$\frac{Out(s)}{In(s)} = \frac{\boxed{P(s)}}{\boxed{Q(s)}} \xrightarrow{\text{roots}} s^n + \ldots$$

S.S. ↗ ?

$$\left.\begin{array}{l} \dot{x} = A \cdot x + Bu \\ y = C \cdot x \end{array}\right\} \xrightarrow{?} \text{T.F.}$$

$x(t) \xrightarrow{L.T.} X(s)$

$u(t) \xrightarrow{L.T.} U(s)$

$y(t) \xrightarrow{L.T.} Y(s)$

$\dot{x}(t) \xrightarrow{L.T.} s \cdot X(s)$

$$sX(s) = A\cdot X(s) + B\cdot U(s)$$

$$sX(s) - A\cdot X(s) = B\cdot U(s)$$

$$\downarrow$$

$$s\cdot I\cdot \underbrace{X(s)} - A\cdot \underbrace{X(s)} = B\cdot U(s)$$

$$(sI - A)\cdot X(s) = B\cdot U(s)$$

$$X(s) = (sI - A)^{-1} B U(s)$$

$$Y(s) = C\cdot X(s)$$

$$Y(s) = \underbrace{C\cdot (sI - A)^{-1}\cdot B}_{\text{something}}\cdot U(s)$$

$Y(s)$ → L.T of out

something → T.F. $= G(s)$

$U(s)$ → L.T. of input

$$\boxed{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$$

$$Y = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

1 in
1 out.

$$\frac{Y(s)}{U(s)} = \dots$$

$$U(s) \longrightarrow \boxed{\quad} \longrightarrow Y(s)$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ +1 & s+0.5 \end{bmatrix}$$

$$|sI - A| = s \cdot (s+0.5) - (-1)$$

$$= s^2 + 0.5s + 1$$

$$(sI-A)^{-1} = \frac{1}{s^2+0.5s+1} \cdot \begin{bmatrix} s+0.5 & +1 \\ -1 & s \end{bmatrix}$$

$$T.F. = G(s) = C \cdot (sI - A)^{-1} \cdot B$$

$$G(s) = \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \cdot \frac{1}{s^2+0.5s+1} \cdot \underset{2\times 2}{\begin{bmatrix} s+0.5 & 1 \\ -1 & s \end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$ 35

$$= \frac{1}{(s^2+0.5s+1)} \cdot \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} s+0.5 & 1 \\ -1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(\quad)} \cdot \begin{bmatrix} \underline{s+0.5}+0 & \underline{1+0} \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(\quad)} \begin{bmatrix} s+0.5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{(\quad)} \begin{bmatrix} 0+1\times 1 \end{bmatrix} =$$

$$= \frac{1}{s^2+0.5s+1} \cdot 1$$

Homework $\ddot{x}+0.5\dot{x}+x=u \xrightarrow{\text{TF}} ?$

$$\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & 1\\ -1 & -0.5\end{bmatrix}}_{A}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B}\cdot \underline{u}$$

$$y = \underbrace{\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}}_{C}\cdot\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

1 input

$y_1 = \dots = x_1$

$y_2 = \dots = 2\cdot x_2$

2 out.

$\frac{y_1}{u_1} = ?$ $\qquad \frac{y_2}{u_1} = ?$

$G_{11} = ?$ $\qquad G_{21} = ?$

$$C\cdot(sI - A)^{-1} B_1 = \cancel{G(s)} = \begin{bmatrix}G_{11}\\ G_{21}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \frac{1}{s^2+0.5s+1}\cdot\begin{bmatrix}s+0.5 & 1\\ -1 & s\end{bmatrix}\cdot\begin{bmatrix}0\\ 1\end{bmatrix}$$

2x2 $\qquad$ 2x2 $\qquad$ 2x1

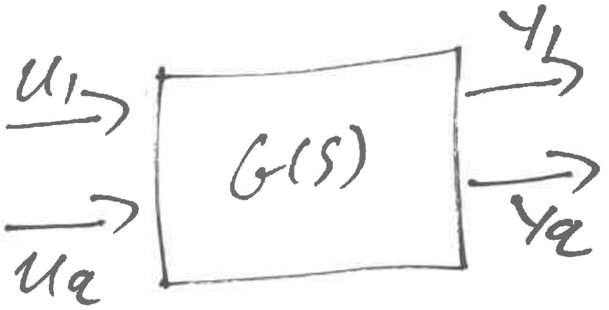
$$\frac{1}{(\ )}\cdot\underbrace{\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}}_{2\times2}\cdot\underbrace{\begin{bmatrix}0+1\\ s\end{bmatrix}}_{2\times1} =$$

$$\cancel{\frac{1}{(\ )}}\cdot\begin{bmatrix}1\\ 2s\end{bmatrix}\frac{1}{s^2+0.5s+1}$$

$$G_{11} = \frac{1}{s^2+0.5s+1}$$

$$G_{21} = \frac{2s}{s^2+0.5s+1}$$

$$A=\begin{bmatrix}0 & 1\\ -1 & -0.5\end{bmatrix},\ B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix},\ C=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$



$\dfrac{y_1}{u_1} = \dots \quad G_{11}$

$\dfrac{y_1}{u_2} = \dots \quad G_{12}$

$\dfrac{y_2}{u_1} = \dots \quad G_{21}$

$\dfrac{y_2}{u_2} = \dots \quad G_{22}$

$$G(s) = C\cdot(sI-A)^{-1}\cdot B$$

$$\underset{2\times2}{\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}} \cdot \frac{1}{s^2+0.5s+1} \cdot \underset{2\times2}{\begin{bmatrix}s+0.5 & 1\\ -1 & s\end{bmatrix}} \cdot \underset{2\times2}{\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}}$$

$$\frac{1}{(\ )} \cdot \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix} \cdot \begin{bmatrix}s+0.5 & 1\\ -1 & s\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} =$$

$$\frac{1}{(\ )} \begin{bmatrix}s+0.5 & 1\\ -2 & 2s\end{bmatrix} \cdot \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$\frac{1}{(\ )}\begin{bmatrix} s+0.5 & s+1.5 \\ -2 & -2+2s \end{bmatrix} \Rightarrow$$

$$G_{11} = \frac{s+0.5}{s^2+0.5s+1} \qquad G_{12} = \frac{s+1.5}{s^2+0.5s+1}$$

$$G_{21} = \frac{-2}{s^2+0.5s+1}, \qquad G_{22} = \frac{2(s-1)}{s^2+0.5s+1}$$

$$G_{i,j}(s) = \frac{\begin{vmatrix} sI-A & -B_i \\ C_j & D \end{vmatrix}}{|sI-A|}$$

$B_i \rightarrow i^{th}$ column of B

$C_j \rightarrow j^{th}$ row of C.

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{matrix} \rightarrow C_1 \\ \rightarrow C_2 \end{matrix}$$

$$\begin{matrix} \downarrow & \downarrow \\ B_1 & B_2 \end{matrix}$$

G_{11}:

$$\begin{vmatrix} \overset{2\times 2}{sI-A} & \overset{2\times 1}{-B_i} \\ \underset{1\times 2}{C_j} & \underset{1\times 1}{D} \end{vmatrix} = \begin{vmatrix} s & -1 & -1 \\ 1 & s+0.5 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

I am minus

$$= s\begin{vmatrix} s+0.5 & 0 \\ 0 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + (-1)\cdot\begin{vmatrix} 1 & s+0.5 \\ 1 & 0 \end{vmatrix}$$

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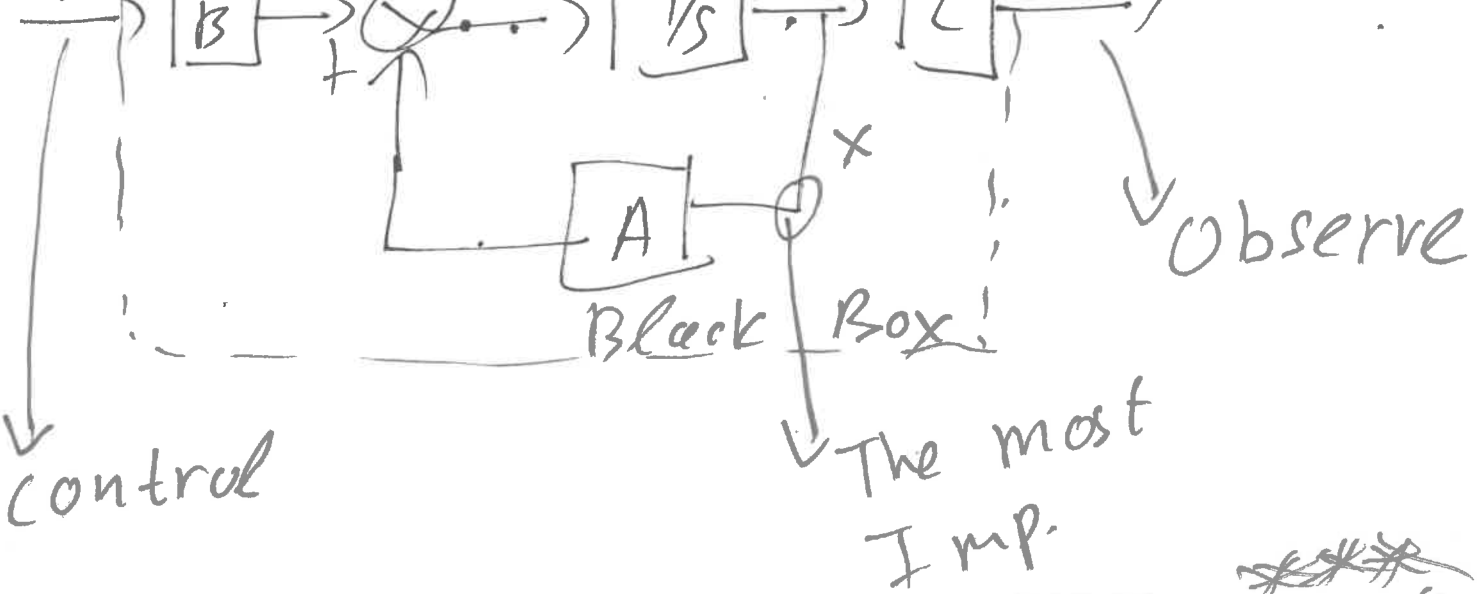
$$G_{12} : \begin{vmatrix} sI-A & -B_1 \\ C_2 & 0 \end{vmatrix} = \cdots$$

$$= \begin{vmatrix} s & -1 & -1 \\ 1 & s+0.5 & 0 \\ 0 & 2 & 0 \end{vmatrix} = \cdots$$

$$G_{21} : \begin{vmatrix} sI-A & -B_2 \\ C_1 & 0 \end{vmatrix} = \cdots$$

$$= \begin{vmatrix} s & -1 & -1 \\ 1 & s+0.5 & -1 \\ 1 & 0 & 0 \end{vmatrix} = \cdots$$

$$G_{22} : \begin{vmatrix} sI-A & -B_2 \\ C_2 & D \end{vmatrix} = \begin{vmatrix} s & -1 & -1 \\ 1 & s+0.5 & -1 \\ 0 & 2 & 0 \end{vmatrix} = \cdots$$



I want to Control x via u
I want to see how x behaves through Y.

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & +1 \end{bmatrix} \cdot x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot u. \Rightarrow$$

$\dot{x}_1 = -2 \cdot x_1 + 2 \cdot u$

$\dot{x}_2 = +x_2$

$$\left.\begin{array}{l} x_2 \text{ is unCTRB} \\ x_1 \text{ is CTRB} \end{array}\right\} \Rightarrow$$

sys is unCTRB

$$\dot{x} = \begin{bmatrix} -2 & \cancel{0}\ 1 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot u.$$

$\dot{x}_1 = -2 \cdot x_1 + x_2 + 2 \cdot u$, x_1 is CTRB

$\dot{x}_2 = -x_2 + 0$, x_2 is unCTRB

sys is unCTRB

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ \cancel{0}\ 1 & -1 \end{bmatrix} \cdot x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot u$$

$$\left.\begin{aligned} \dot{x}_1 &= -2x_1 + 2 \cdot u. \\ \dot{x}_2 &= x_1 - x_2 + 0 \end{aligned}\right\} \Rightarrow \text{CTRB}$$

$$M_c = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

$$\mathrm{Rank}(M_c) \leq n$$

if $|M_c| \neq 0 \rightarrow$ CTRB

$$A_1 = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}. \qquad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} B & A \cdot B \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix}$$

$$|M_c| = 0$$

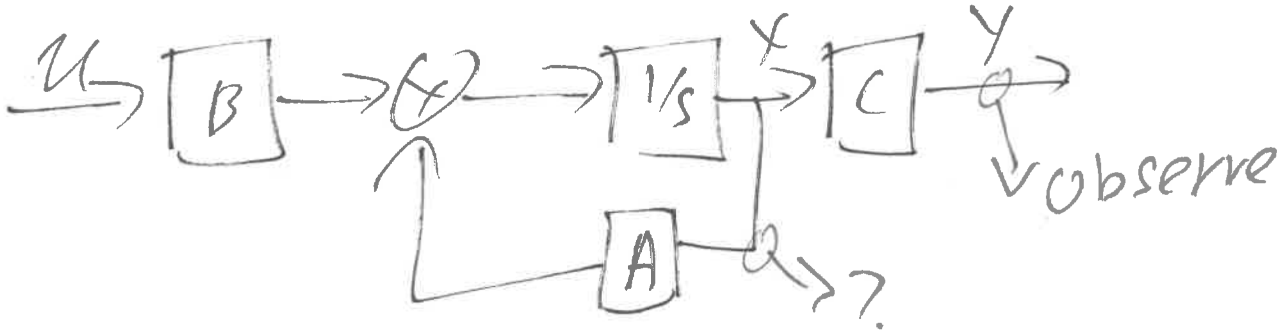
$$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \qquad A_2 \cdot B = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 2 & -4 \\ 0 & 0 \end{bmatrix} \Rightarrow |M_c| = 0$$

$$A_3 = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \qquad A_3 \cdot B = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$M_c = \begin{bmatrix} 2 & -4 \\ 0 & 2 \end{bmatrix} \Rightarrow |M_c| = 4 \neq 0$$



$\cdot A_1 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad C = [1 \quad 0] \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\dot{x}_1 = -2x_1 + u$

$\dot{x}_2 = x_2 + u.$

$\underbrace{\qquad\qquad}_{\downarrow}$

$x_2 = e^{t} \cdot x_2(0)$

$\boxed{y = x_1}$

I cannot observe how x_2 behaves

$\rightarrow x_2$ is unOBSV

$A_2 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad C = [1 \quad 0] \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\dot{x}_1 = -2 \cdot x_1 + x_2 + u$

$\dot{x}_2 = x_2 + u$

$\boxed{y = x_1}$

OBSV.

(obsv)

$A_2 = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \dot{x}_1 = -2x_1 + u$

$\dot{x}_2 = x_1 + x_2 + u.$

$\boxed{y = x_1}$

$A_4 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ $C = [0 \quad 1]$ $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$y = x_2$

Not OBSV

$A_5 = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 3 \\ 5 & 10 \end{bmatrix}$

$y_1 = x_1 + 3 \cdot x_2$

$y_2 = 5 \cdot x_1 + 10 x_2$

$M_o = [C \quad CA \quad CA^2 \ldots \quad CA^{n-1}]^T$

If $\text{rank}(M_o) < n \rightarrow$ Not OBSV

$|M_o| \neq 0 \rightarrow$ OBSV