Revision

$$\left.\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx + Du\end{aligned}\right\} \xrightarrow{L.T.} G(s) = C\cdot(sI-A)^{-1}\cdot B + D$$

Matrix

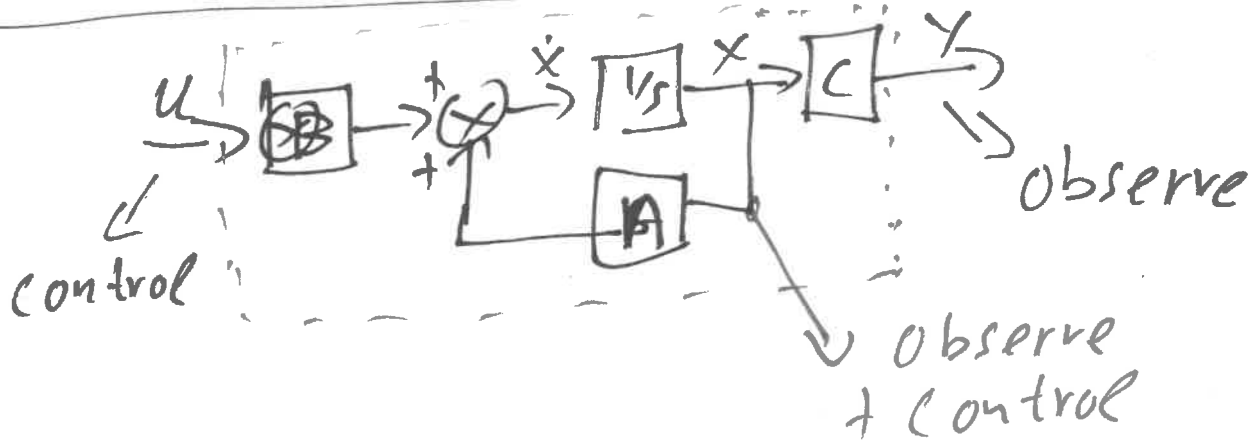
$$G_{i,j}(s) = \frac{\begin{vmatrix} sI-A & -B_i \\ C_j & D \end{vmatrix}}{|sI-A|}$$

scalar

C.E.

$B_i \rightarrow i^{th}$ colmn

$C_j \rightarrow j^{th}$ row



control $u \rightarrow$ control x

observe $y \rightarrow$ how x behaves

$M_c = [B \quad AB \quad A^2B \ \ldots\ldots \ A^{n-1}B]$

$M_o = [C \quad CA \quad CA^2 \ \ldots\ldots \ CA^{n-1}]^T$

$\text{rank}(M_c) = n \Rightarrow$ CTRB

$\text{rank}(M_o) = n \Rightarrow$ OBSV

$\dot{x} = Ax + B\cdot v \rightarrow$ Vector D.E

$\downarrow$ solve ???

$\dot{x} = a\cdot x$, $a, x \in \mathbb{R}$.

$x = e^{at}$

$\ddot{x} + A\dot{x} + B\cdot x = 0$

$x = e^{rt}$ $\downarrow$ r = Not known

$r^2 + Ar + B = 0$

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2}$$

Δ $\rightarrow >0$, <0, $=0$

$\ddot{x} + A\dot{x} + Bx = 0 \rightarrow \dot{x} = A\cdot x$ 2nd.

2nd

$x \in \mathbb{R}$

$A \in \mathbb{R}$

$x = e^{rt}$

$x \in \mathbb{R}^2, A \in \mathbb{R}^{2\times 2}$

$$\underbrace{x}_{\text{vector}} = \underbrace{e^{rt}}_{\text{scalar}} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

r = ? $a_1 = ?$

$a_2 = ?$

Try $x = e^{rt}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\dot{x} = r\cdot e^{rt}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

$$r\cancel{e^{rt}}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = A\,\cancel{e^{rt}}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$\rightarrow$

$$r \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = A \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

unknown

known

say $A = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}$

$$r \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} r a_1 \\ r a_2 \end{bmatrix} = \begin{bmatrix} k_1 \cdot a_1 + k_2 \cdot a_2 \\ k_3 \cdot a_1 + k_4 \cdot a_2 \end{bmatrix} \Rightarrow$$

$$r \cdot a_1 = \boxed{k_1} a_1 + \boxed{k_2} \cdot a_2$$

NLS

$$r \cancel{a_2} = \cancel{k_2 a_2}$$

$$r a_2 = \boxed{k_3} a_1 + \boxed{k_4} \cdot a_2$$

$r \to$ Given $\longrightarrow$ L.S. 2×2

$$(k_1 - r) \cdot a_1 + k_2 \cdot a_2 = 0$$

$$k_3 \cdot a_1 + (k_4 - r) \cdot a_2 = 0$$

$a_1 = 0$

$a_2 = 0$

$\hookrightarrow$ inf $\begin{vmatrix} k_1 - r & k_2 \\ k_3 & k_4 - r \end{vmatrix} = 0 \to$

or $\left| \underbrace{\begin{bmatrix} k_1 & k_2 \\ k_3 & k_4 \end{bmatrix}}_{A} - r \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} \right| = 0$

or $|A - rI| = 0$

or $|rI - A| = 0 \rightarrow$ C.E.

$\hookrightarrow$ 2nd order pol of r.

$\hookrightarrow \Delta \rightarrow >0,\ <0,\ =0$

General $\quad x = e \cdot e^{rt}$ — e: eigenvector, r: eigenvalue $\Big\}\ \dot{x} = Ax \Rightarrow$

$e, x \in \mathbb{R}^{n \times 1}$

$A \in \mathbb{R}^{n \times n}$

$\dot{x} = r e e^{rt}$

$re = A \cdot e$

$re - Ae = 0$

$rIe - Ae = 0$ — given

n eqns

$n * 1$ unknowns

$(rI - A) \cdot e = 0 \Rightarrow |rI - A| = 0$

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \cdot x$$

$$x = e \cdot e^{\lambda t} \Rightarrow \ldots \quad (-2-\lambda)\cdot a_1 + 2 \cdot a_2 = 0$$

$$2 \cdot a_1 + (-5-\lambda)\, a_2 = 0$$

$$D = \begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(-2-\lambda)\cdot(-5\lambda) - 4 = 0 \Rightarrow \ldots$$

$$\lambda^2 + 7\lambda + 6 = 0 \qquad \text{C.E.}$$

$$\lambda_1 = -1 \qquad a_1 = ? \qquad e = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$$

$$\lambda_2 = -6 \qquad a_2 = ?$$

$$x_1 = e_1 \, e^{-t}$$

$$x_2 = e_2 \, e^{-6t}$$

???

$(\lambda I - A) \cdot e \rightarrow 2$ unknowns

known.

• $\lambda = -1$

$$\left.\begin{aligned} -a_1 + 2a_2 &= 0 \\ 2a_1 - 4\cdot a_2 &= 0 \end{aligned}\right\} \Rightarrow$$

→ expected

$$-a_1 + 2a_2 = 0$$

Assume $a_2 = 1 \Rightarrow a_1 = 2$

$$\Rightarrow e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow x_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$$

• $\lambda = -6 \Rightarrow \dots \Rightarrow e_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$\rightarrow x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

~~All solns~~

Gen soln $x = c_1 x_1 + c_2 \cdot x_2$

$$x = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

ch 1
$x_0 = \dots$
$\dot{x}_0 = \dots$

$$t = 0: \ x(0) = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot 1 + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot 1$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

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$$\left.\begin{array}{l} C_1 \cdot 2 + C_2 \cdot 1 = 1 \\ C_1 \cdot 1 - 2C_2 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} C_1 = 0.4 \\ C_2 = 0.2 \end{array} \Rightarrow$$

$$x_1 = 0.4 \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 0.2 \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \cdot x \quad , \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

1) stability ?
2) Gen soln ?
3) spec soln ?

C.E.

$$|A - \lambda I| = 0 \quad \Rightarrow |\lambda I - A| = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ -6 & -7-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(-\lambda)(-7-\lambda) - (-6)\cdot 1 = 0 \Rightarrow$$

$$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \begin{array}{l} \lambda_1 = -1 \\ \lambda_2 = -6 \end{array}$$

since both eigs are -ve $\Rightarrow$ sys is stable

Gen soln is

$$x = c_1 \begin{bmatrix} e_1 \end{bmatrix} \cdot e^{-t} + c_2 \begin{bmatrix} e_2 \end{bmatrix} e^{-6t}$$

$$e_1 = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \qquad e_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$(A - \lambda I) \cdot e = 0$$

$$(A - \lambda I) = \begin{bmatrix} -\cancel{\lambda} & 1 \\ -6 & -7-\lambda \end{bmatrix}$$

$\lambda = -1$ $\quad (A+I) = \begin{bmatrix} +1 & 1 \\ -6 & -7-(-1) \end{bmatrix}$

$$(A - \lambda I) \cdot e_1 = 0 \Rightarrow$$

$$\begin{bmatrix} 1 & 1 \\ -6 & -6 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow$$

$$a_1 + a_2 = 0 \rightarrow a_1 = 1$$
$$a_2 = -1$$

$$e_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

ii) $\lambda = -6$

$(A - \lambda I)\cdot e_2 = 0 \Rightarrow \dots$

$$\begin{bmatrix} -6 & -1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow$$

$-6\cdot v_1 - v_2 = 0$

assume $\cancel{v_1 = \sqrt{3}} \Rightarrow v_2 = -6\cdot\sqrt{3}$

$\cancel{v_2 = 1}$ $\quad v_1 = 1 \Rightarrow v_2 = -6$

$$e_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Gen soln

$$X = c_1 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-6t} \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

spec. soln.

$$X_0 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 - 6c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \dots \quad c_1 = 6/5 \qquad c_2 = -1/5$

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$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \cdot x \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$(1-\lambda)(3-\lambda) + 1 = 0 \Rightarrow$$

$$(\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = 2 \Rightarrow$$

~~$x_1 = e^{2t} \cdot e$~~

$$(A - \lambda I) \cdot e = 0$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$v_1 + v_2 = 0 \Rightarrow v_1 = 1$$
$$v_2 = -1$$

$$e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$x_1 = e^{2t} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$b = \dots \Rightarrow x_2 = (e\,t + b)\, e^{\lambda t}$$

$$x = c_1 x_1 + c_2 x_2$$

$b \rightarrow$ Gen. eigenvector

$$e = (A - \lambda I) \cdot b$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$1 = -b_1 - b_2$$

assum $b_1 = 0$

$\Rightarrow b_2 = -1$

$$X = c_1 (et + b) e^{\lambda t} + c_2 e e^{\lambda t}$$

$$= c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$$

$$X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X(0) = c_1 \left(\cancel{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 0} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \cdot 1 + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 1$$

$$= c_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\Rightarrow \dots$