Revision

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$\dot{x} = A \cdot x$, $x \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$

$\underbrace{x}_{n\times 1} = \underbrace{e^{\lambda t}}_{1\times 1} \cdot \underbrace{e}_{n\times 1}$

$\lambda \to$ eigenvalue

$e \to$ eigenvector

$\lambda \cdot \cancel{e^{\lambda t}} \cdot e = A \cdot \cancel{e^{\lambda t}} \cdot e$

$\lambda e = A \cdot e \Leftrightarrow \dots\dots$ $(\lambda I - A) \cdot e = 0$

or

$(A - \lambda I) \cdot e = 0$ $\quad n+1 \times n$ N.L. SYS

$\lambda \to$ GIVEN

$\downarrow_{n\times n}$ L.H.S.

$\underbrace{|A - \lambda I|}_{\downarrow} = 0 \Longrightarrow \lambda = \dots$

$n = 2$ • 2nd order $\quad \lambda_1 = \dots \quad \lambda_2 = \dots$

I want x_1, x_2 L.I.

Any other sol^n $x = c_1 \cdot x_1 + c_2 \cdot x_2$

- $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

$\lambda_1 \to e_1$, $\lambda_2 \to e_2$ $\quad$ e_1, e_2 are L.I.

$$x = \boxed{c_1}\, e_1 \boxed{e^{\lambda_1 t}} + \boxed{c_2}\, e_2 \boxed{e^{\lambda_2 t}}$$

vectors

scalars

- $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

$e \to$ eigenvector

$b \to$ gen eigenvector.

$$e = (A - \lambda I) \cdot b$$

$$x = c_1 \cdot (e t + b) \cdot e^{\lambda t} + c_2 \cdot e^{\lambda t} \cdot e$$

- $\lambda = a \pm bi$, $a, b \in \mathbb{R}$

$e \in \mathbb{C}^{2\times 1}$ $\quad x_1 = e^{\lambda t} \cdot e \quad , \quad x_2 = e^{\lambda t}\, \bar{e}$

$$x_1 = Re(e^{\lambda t} \cdot e) \quad , \quad x_2 = Im(e^{\lambda t} \cdot e)$$

e.g. $A=\begin{bmatrix}0 & 1\\ -6 & -7\end{bmatrix}$, $X_0=\begin{bmatrix}1\\0\end{bmatrix}$

$$|A-\lambda I|=0$$

$$\begin{vmatrix}-\lambda & 1\\ -6 & -7-\lambda\end{vmatrix}=0 \qquad -\lambda\cdot(-7-\lambda)-(-6)\cdot 1=0$$

$$\lambda^2+7\cdot\lambda+6=0$$

$$\lambda_1=-1,\ \lambda_2=-6$$

$$(A-\lambda I)\cdot e=0$$

$$\begin{bmatrix}-\lambda & 1\\ -6 & -7-\lambda\end{bmatrix}\cdot\begin{bmatrix}V_1\\V_2\end{bmatrix}=0$$

~~we~~ are Vs

- $\lambda_1=-1$ $\quad\begin{bmatrix}+1 & 1\\ -6 & -6\end{bmatrix}\cdot\begin{bmatrix}V_1\\V_2\end{bmatrix}=0 \Rightarrow$

$$\cancel{e_1+e_2=0}$$

$$V_1+V_2=0$$

$$V_1=1 \Rightarrow V_2=-1$$

$$e_1=\begin{bmatrix}1\\-1\end{bmatrix}$$

$\cdot\ \lambda = -6$

$$\begin{bmatrix} 6 & 1 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$6 \cdot v_1 + v_2 = 0$

$\cancel{v_2 = 1 \Rightarrow v_1 = -1/6}$

$v_1 = 1 \Rightarrow v_2 = -6$

$$e_2 = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$$

Gen soln:

$$X(t) = c_1 e_1 e^{\lambda_1 t} + c_2 e_2 e^{\lambda_2 t}$$

$$= c_1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-6t}.$$

spec. soln:

$$X(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$$

..... $c_1 = \ldots$

$c_2 = \ldots$

$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ $\quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ 60

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$(1-\lambda)\cdot(3-\lambda) + 1 = 0$

$\lambda^2 - 4\cdot\lambda + 4 = 0$

$(\lambda - 2)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda = 2$

$\lambda = 2$ $\quad (A - \lambda I)\cdot e = 0$

$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$

$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0 \Rightarrow$

$V_1 + V_2 = 0 \Rightarrow V_1 = 1$
$V_2 = -1$

$e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$(A-\lambda I)\cdot b = e \Rightarrow$

$\cdot\ \lambda = 2 \quad \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$b_1 + b_2 = -1$

$\underline{b_1 = 0}, \quad \underline{b_2 = -1}$

$x = c_1 \cdot (et + b) \cdot e^{\lambda t} + c_2\, e \cdot e^{\lambda t}$

$= c_1 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

$x_0(0) = c_1 \left(\cancel{\begin{bmatrix} 1 \\ -1 \end{bmatrix} \cdot 0} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) \cancel{e^{2t}} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$c_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow$

$\ldots\ldots \quad c_2 = 1 \quad c_1 = -1$

$x = -(et + b)e^{2t} + e\, e^{2t}$

$= -\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) e^{2t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t}$

$= \begin{bmatrix} -t + 0 \\ t + \cancel{t e^{2t}} \end{bmatrix} e^{2t} + \begin{bmatrix} e^{2t} \\ \cancel{-e^{2t}} \end{bmatrix} =$

$= \begin{bmatrix} -te^{2t} + e^{2t} \\ t e^{2t} \end{bmatrix}$

if $b_2=0$ $b_1=-1$

$$c_1\left(\begin{bmatrix}1\\-1\end{bmatrix}\cdot t+\begin{bmatrix}-1\\0\end{bmatrix}\right)e^{2t}+c_2\begin{bmatrix}1\\-1\end{bmatrix}e^{2t}$$

$t=0$

$$c_1\begin{bmatrix}-1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow$$

$c_2=0,\ c_1=-1$

$$x=-\left(\begin{bmatrix}1\\-1\end{bmatrix}t+\begin{bmatrix}-1\\0\end{bmatrix}\right)e^{2t}+0$$

$$=\begin{bmatrix}-te^{2t}+e^{2t}\\ te^{2t}\end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda I) = 0 \quad \ldots\ldots \Rightarrow \lambda_1 = 1 + 2i$$

$$\lambda_2 = 1 - 2i$$

$$(A - \lambda I) \cdot e = 0$$

$$\begin{bmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} 1-(1+2i) & 2 \\ -2 & 1-(1+2i) \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$= \begin{bmatrix} -2i & 2 \\ -2 & -2i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Rightarrow$$

$$-\cancel{2}i v_1 + \cancel{2} v_2 = 0$$

$$-v_1 i + v_2 = 0 \Rightarrow$$

$$v_2 = 1 \Rightarrow v_1 = -i$$

$$1 + 2i \rightarrow \begin{bmatrix} -i \\ 1 \end{bmatrix} = e_1 \qquad e_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$x = c_1 x_1 + c_2 x_2$$

$$= c_1 \cdot e^{(1+2i)t} \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{(1-2i)t} \cdot \begin{bmatrix} i \\ 1 \end{bmatrix}$$

$$x_0 = c_1 \cdot 1 \cdot \begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 \cdot 1 \cdot \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \to \text{Given}$$

$$\left.\begin{aligned} -i \cdot c_1 + c_2 \cdot i &= 1 \\ c_1 + c_2 &= 0 \end{aligned}\right\} \Rightarrow \quad c_1 = -\frac{1}{2i}, \quad c_2 = \frac{1}{2i}$$

Cross check

$$x' = A \cdot x. \qquad n = 2$$

$$\underset{2\times1}{x} = \underset{1\times1}{c_1} \cdot \underset{2\times1}{x_1} + \underset{1\times1}{c_2} \cdot \underset{2\times1}{x_2}$$

$x_1 = e_1 e^{\lambda_1 t}$

$x_2 = e_2 e^{\lambda_2 t}$

$x_1 = e \cdot e^{\lambda t}$

$x_2 = (et + b) e^{\lambda t}$

$x_1 = \mathrm{Re}(e^{\lambda t} \cdot e)$

$x_2 = \mathrm{Im}(e^{\lambda t} \cdot e)$

$$\underset{2\times1}{x(t)} = \underset{2\times2}{\begin{bmatrix} \underset{2\times1}{x_1} & \underset{2\times1}{x_2} \end{bmatrix}} \cdot \underset{2\times1}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} \Rightarrow \cancel{x(t)}$$

$$x(t) = X(t) \cdot C.$$

Fundamental sols matrix

$$x(t) = \mathbb{X}(t) \cdot C_1$$

$t=0$ $\quad x(0) = \mathbb{X}(0) \cdot C_1$

$$\boxed{C_1 = \mathbb{X}^{-1}(0) \cdot x(0)}$$

$$\underbrace{x(t)}_{q\times 1} = \underbrace{\boxed{\underbrace{\mathbb{X}(t)}_{q\times q} \cdot \underbrace{\mathbb{X}^{-1}(0)}_{q\times q}}}_{\text{State transition Matrix}} \cdot \underbrace{x(0)}_{q\times 1}$$

State transition Matrix

$e^{At} \rightarrow$ next week

$\ddot{x} + 2\cdot\dot{x} + x = 0$ $\qquad$ $x_0 = 1 = x_1(0) = 1$ $\qquad$ (65)

$\dot{x}_0 = 0 = x_2(0) = 0$

1) Find spec. soln.

2) ODE → S.S. model.

3) solve S.S. using eigs.

4) Find STM.

5) Solve S.S. using ST.M.

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1) $x = e^{rt} \to r^2 + 2\cdot r + 1 = 0$

$\Delta = 4 - 4\cdot 1 = 0 \Rightarrow$

$r_1 = r_2 = r = -1$

$$\left.\begin{aligned} x &= C_1\cdot e^{-t} + C_2\cdot te^{-t} \\ \dot{x} &= -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} \end{aligned}\right\} \xrightarrow{t=0}$$

$$\left.\begin{aligned} x_0 &= C_1 + 0 = 1 \\ \dot{x}_0 &= -C_1 + C_2 - 0 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= 1 \end{aligned}$$

$x = e^{-t} + te^{-t} = e^{-t}\cdot(t+1)$

$\dot{x} = \cancel{-e^{-t}} + \cancel{e^{-t}} - te^{-t} \Rightarrow \dot{x} = -t\cdot e^{-t}$

2) $\ddot{x} = -2\dot{x} - x$

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}\right\} \Rightarrow \begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = -2\dot{x} - x \\ &= -2 \cdot x_2 - x_1 \end{aligned}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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3) $|A - \lambda I| = 0 = \begin{vmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow$

$-\lambda(-2-\lambda) + 1$

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda_1 = \lambda_2 = -1$

$(A - \lambda I) \cdot e = 0$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

• $\lambda = -1$ $\quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$

$\downarrow$

$v_1 + v_2 = 0 \Rightarrow v_1 = 1, v_2 = -1$

$e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$e = (A - \lambda I)\cdot b.$

$$\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1 & 1\\ -1 & -1\end{bmatrix}\cdot\begin{bmatrix}b_1\\b_2\end{bmatrix} \Rightarrow$$

$b_1 + b_2 = 1.$ $\quad b_2 = 0$

$b_1 = 1$

Homework.
$b_1 = 0$ $b_2 =$
find sol's

$$\Rightarrow b = \begin{bmatrix}1\\0\end{bmatrix}$$

$$x = c_1\cdot(et+b)\cdot e^{-t} + c_2 e\, e^{-t}$$

$$x_0 = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Rightarrow x(0) = c_1\cdot(e\cdot 0 + b)\cdot 1 + c_2\cdot e\cdot 1$$

$$= c_1\cdot b + c_2\cdot e \Rightarrow \dots$$

crosscheck

$c_2 = 0$

$c_1 = 1$

$$x(t) = 1\cdot\left(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}1\\0\end{bmatrix}\right)e^{-t}$$

$$= \begin{pmatrix}t+1\\-t\end{pmatrix}e^{-t} = \begin{bmatrix}(t+1)e^{-t}\\ -te^{-t}\end{bmatrix}$$