Revision

$\dot{x}+kx=u \rightarrow x(t)=e^{-kt}\cdot x_0+e^{-kt}\int_0^t e^{kt_1}u(t_1)\,dt_1$

- $u=\text{const}$ (bounded)

$x=\frac{u}{k}(1-e^{-kt})$

$x_{ss}=u/k$

$u \rightarrow$ s.s. not stability

$\dot{x}+kx=0 \rightarrow x=e^{-kt}\cdot x_0$
- $\rightarrow k>0 \rightarrow$ stable
- $\rightarrow k<0 \rightarrow$ unstable

$x=e^{rt} \rightarrow r+k=0 \rightarrow$ C.E.

eigenvalue

$r<0 \rightarrow$ stable

$r>0 \rightarrow$ unstable

$\ddot{x}+A\dot{x}+Bx=u$

$\ddot{x}+A\dot{x}+Bx=0 \qquad x=e^{rt}$

$r^2+Ar+B=0$
- $\rightarrow r_1\neq r_2 \in \mathbb{R}$
- $\rightarrow r_1=r_2=r \in \mathbb{R}$
- $\rightarrow r_1=a+bi$, $r_2=\bar{r}_1$

- $x=e^{r_1t}c_1+c_2e^{r_2t}$
- $x=c_1e^{rt}+c_2t\cdot e^{rt}$
- $x=e^{r_1t}c_1+e^{r_2t}c_2$

$u\neq 0$, $u=\text{const}$ sys stable $x \rightarrow x_{ss}$

$\dot{x}_{ss}=0$

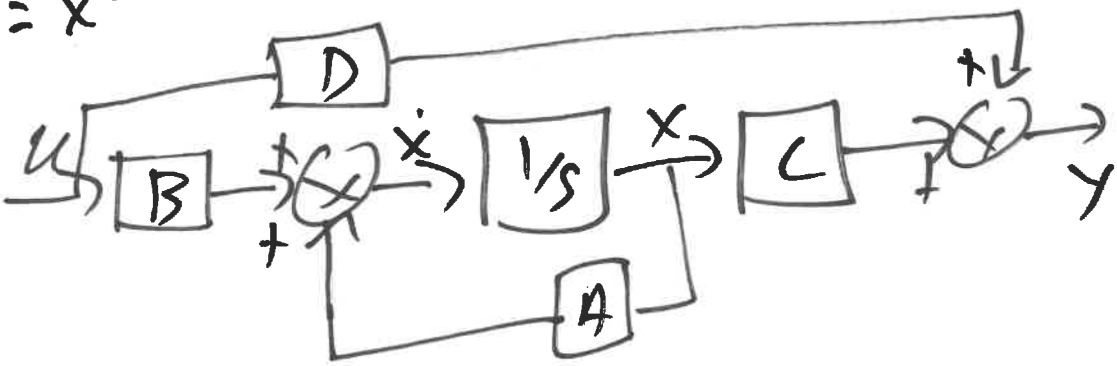
$\ddot{x}_{ss}=0$

$x_{ss}=\frac{u}{B}$

ODE → S.S

$$\left.\begin{aligned} & x^{(n)} = \dots \\ & x_1 = x \\ & x_2 = \dot{x} \\ & \vdots \quad \vdots \\ & x_n = x^{(n-1)} \end{aligned}\right\} \xRightarrow{d/dt} \dot{x} = A \cdot x + B \cdot u$$

$$\left.\begin{aligned} & y_1 = \dots \\ & y_2 = \dots \end{aligned}\right\} \Rightarrow y = C \cdot x + D \cdot u.$$



S.S → T.F.

$$G(s) = C \cdot (sI - A)^{-1} \cdot B + D$$

$\hookrightarrow$ .... $|sI - A|$

C.T.R.B

$u \rightarrow \dots \rightarrow x$

OBSV

$y \rightarrow \dots \rightarrow x$

$u = 0$ $\quad \dot{x} = A \cdot x$ $\quad x = e \cdot e^{rt}$

eigenvector

$|rI - A| = 0$

$|A - rI| = 0 \Rightarrow \ldots \quad (A - \lambda I)e = 0$

2nd. order sys

- $r_1 \neq r_2$, e_1 l.I. e_2 $\Big\}$ $x_1 = e_1 e^{r_1 t}$, $x_2 = e_2 e^{r_2 t}$

  $x = c_1 x_1 + c_2 \cdot x_2$

  $x_0 = \ldots \Rightarrow c_1 = \ldots$, $c_2 = \ldots$

- $r_1 = r_2 = r$, e, b $\Big\} \Rightarrow$ $x_1 = e\, e^{rt}$, $x_2 = (b + te)\, e^{rt}$

  $(A - \lambda I) \cdot e = b$

- $r_1 = a + bi$, $r_2 = \bar{r}_1$, e_1, $e_2 = \bar{e}_1$ $\Big\} \Rightarrow$ $x_1 = e_1 e^{r_1 t}$, $x_2 = \bar{e}_2\, e^{\bar{r}_1 t}$

$x = c_1 \cdot x_1 + c_2 \cdot x_2$

F.S.M.

$x = X \cdot c$

$X \to [x_1 \quad x_2]$

$c \to [c_1 \quad c_2]$

S.T.M.

$x(t) = \underbrace{X(t) \cdot X^{-1}(0)}_{\text{S.T.M.}} \cdot x_0$

$x(t)$: 2x1, $X(t)$: 2x2, $X^{-1}(0)$: 2x2, x_0: 2x1

same as $x(t) = c_1 x_1 + c_2 \cdot x_2$

S.T.M. ? yes L.T.I.

$$\text{S.T.M.} = e^{At} = \frac{I}{0!} + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!}$$

$\dot{x} = Ax \to x(t) = e^{At} \cdot x_0$

$\dot{x} = Ax + B \cdot u \to x(t) = e^{At} \cdot x_0 + \int_0^t e^{A(t-t_1)} \cdot B\, u(t_1)\, dt_1$

- $\lambda_1 \neq \lambda_2$

$e^{At} = T e^{\Lambda t} T^{-1}$

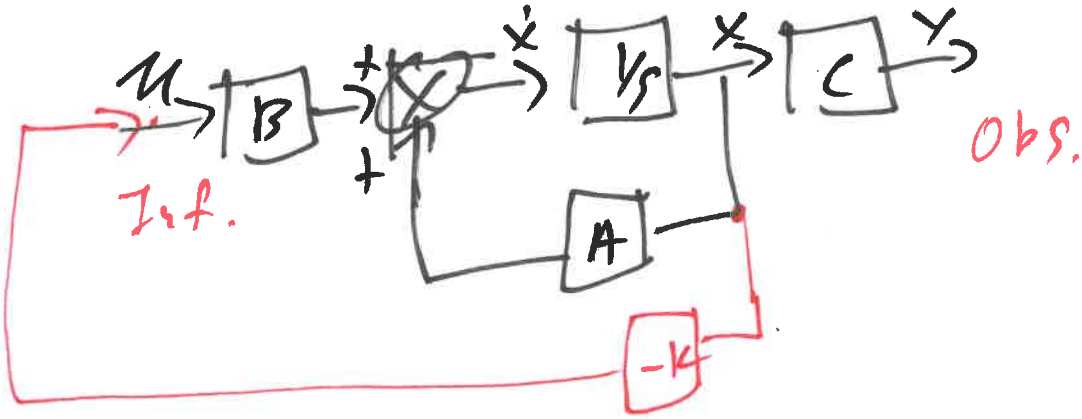
$T = [e_1 \quad e_2]$ eigenmatrix

$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$x = c_1 x_1 + c_2 \cdot x_2 \Longrightarrow x = e^{A \cdot t} \cdot x_0$$

MSc → simulate ALL analytical solns.
+ symbolic toolbox
+ num. solns.



$\dot{x} = Ax + Bu$ | $u = -K \cdot x$

$y = C \cdot x$

$\dot{x} = Ax + B(-Kx)$

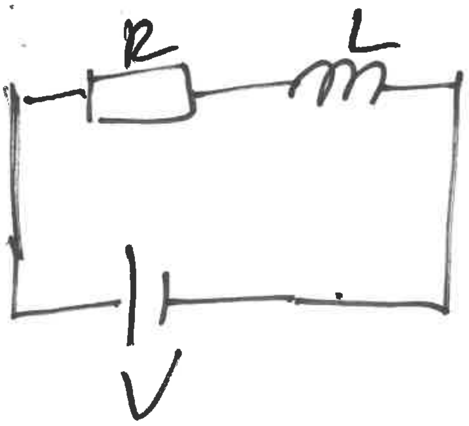
$\boxed{\dot{x} = (A - BK) \cdot x} \rightarrow$ C.L.S.S.

C.L.S.M: A_{CL} = fun of K

K=? : $eig(A_{CL}) = \_\_\_ \quad \_\_\_$

Des. values

pole placement



$$\frac{di}{dt} = \frac{1}{L}(V - iR)$$

$$i_{ss} = V/R.$$

$$x = i, \quad u = V, \quad A = \underbrace{-R/L}_{\text{Time const. eigenvalue}}, \quad B = 1/L.$$

$u = ?$: $i(0) \neq 0$ $\quad i \to 0$ very fast

$u = -K\cdot x = -K\cdot i \quad K = ?$

$$A_{cl} = A - BK = -R/L - \frac{1}{L}K$$

$$\text{eig}(A) = -R/L$$

six faster $\to -6\cdot\frac{R}{L}$

$$\cancel{-}\frac{R+K}{\cancel{L}} = \cancel{-}6\cdot\frac{R}{\cancel{L}}$$

$$K = +5R.$$

$$u = -Kx$$

$$V = -5\cdot R\cdot i$$

$i(0) = 1A.$
$V(0) = -5R$
$R = 1 \quad V(0) = -5V$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

1) stable = ?

2) if not CL. poles at -10, -11

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = 5.37 \\ \lambda_2 = -0.37. \end{cases}$

C.T.R.B *******

$M_c = [B \quad A \cdot B]$

$\begin{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \qquad |M_c| = 1 \cdot 3 - 0 = 3 \neq 0$

CTRB ✓

$u = -Kx$ $\qquad A_{CL} = \underset{2\times 2}{A} - \underset{2\times 1}{B}\,\underset{1\times 2}{K}$

A_{CL}: 2×2

$$K = [k_1 \quad k_2]$$

$$A_{CL} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot [k_1 \quad k_2]$$

$$= \begin{bmatrix} 1-k_1 & 2-k_2 \\ 3 & 4 \end{bmatrix}$$

$$|A_{CL} - \lambda I| = 0$$

$$\begin{vmatrix} 1-k_1-\lambda & 2-k_2 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-k_1-\lambda)\cdot(4-\lambda) - 3(2-k_2) = 0$$

- $\lambda = -10$ $\qquad -14k_1 + 3\cdot k_2 + 148 = 0$
- $\lambda = -11$ $\qquad -15k_1 + 3k_2 + 174 = 0$

$\Rightarrow \ldots$

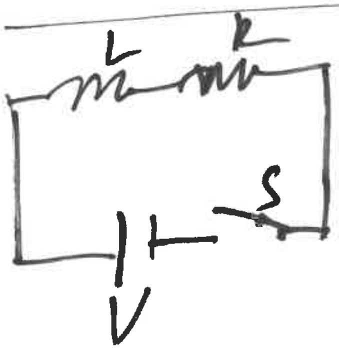
$$k_1 = 26 \qquad k_2 = 72$$

$A_{CL} = A - BK$

$$\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}26 & 72\end{bmatrix} =$$

$$\begin{bmatrix}-25 & -70\\ 3 & 4\end{bmatrix}$$

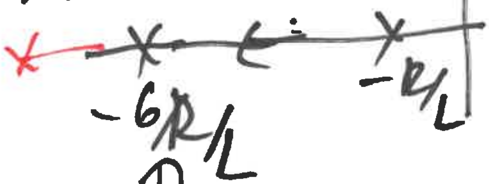
$$\begin{vmatrix}-25-\lambda & -70\\ 3 & 4-\lambda\end{vmatrix} = 0 \Rightarrow \dots \begin{cases}\lambda_1 = -10\\ \lambda_2 = -11\end{cases}$$



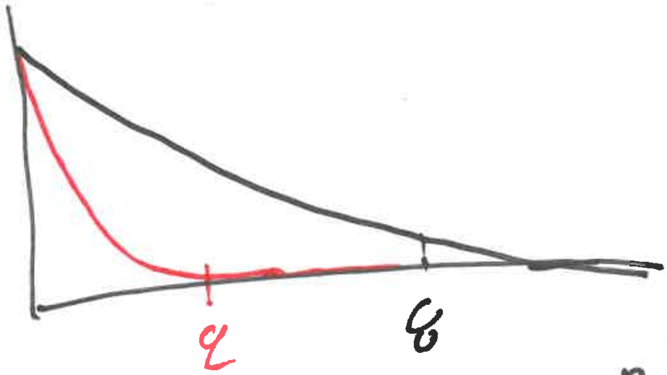
$i(0) \neq 0$ $\quad V = ?$

$i \to 0$ Fast

$V = -S \cdot R \cdot i$



$\uparrow$ C.L. eigs $= -\dfrac{R+K}{L}$



$$-\frac{R+K}{\cancel{L}} = -10\frac{R}{\cancel{L}}$$

$R + K = 10R.$

$K = 9R.$

$C_1: \quad u = -5 \cdot R \cdot i$

$C_2: \quad u = -9 \, R \cdot i$

Assume $\quad i(0) = 1A. \qquad R = 2\Omega.$

$V(0) = -5 \cdot 2 \cdot i(0) = -10\,V.$

$V(0) = -9 \cdot 2 \cdot i(0) = -18\,V$

$C_3: \quad u = -100000 \cdot R \cdot i$

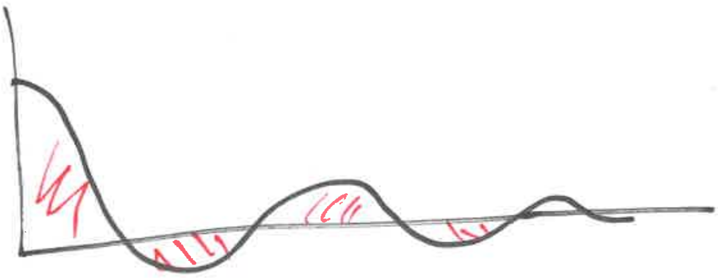
$V(0) = -200000\,V$ → more energy

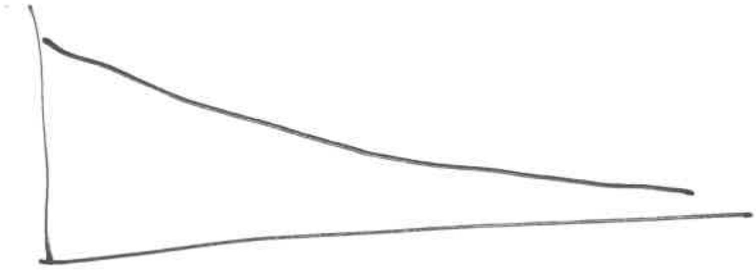
$I_x = \int_0^\infty x(t)\,dt$

$di/dt = -\frac{R}{L} \cdot i \Rightarrow i(t) = e^{-R/L \cdot t}.$

$I_x = i(0) \int_0^\infty e^{-R/L\,t}\,dt = i(0) \cdot \frac{L}{R}$

$I_x = \dots = \frac{L}{R} \cdot \frac{1}{6}$





$$I_x = \cancel{\int |x(t)|}\, dt$$

$$I_x = \int x^2(t)\, dt$$

say $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix}$

$$I_x = \int (x_1^2 + x_2^2 + \dots)\, dt$$

$$I_x = \int (10 \cdot x_1^2 + x_2^2 + \dots)\, dt$$

$$I_x = \int (10 \cdot x_1^2 + 5 \cdot x_2^2 + \dots)\, dt$$

$$I_x = \int X^T \cdot Q \cdot X\, dt$$

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$X^T = [x_1 \quad x_2 \quad \dots \quad]$.

$$Q = \begin{bmatrix} 10 & 0 & 0 & \\ & 5 & & \\ 0 & 0 & \ddots & \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{bmatrix}$$

$$I_x = \int x_1 \cdot 10\, x_1 + x_2 \cdot 5 \cdot x_2 + \dots$$

$K = ?$ $\quad I_x = \int_0^{\infty} X^T . Q X \, dt \rightarrow$ small

$$I_u = \int_0^{\infty} u^T . R . u \, dt \rightarrow \text{small}$$

$K = ?$ $\quad I = \int_0^{\infty} (x^T . Q . x + u^T . R u) \, dt$

$\rightarrow$ small

speed $\quad$ energy

$Q \uparrow$ faster $\quad$ $R \uparrow$ less energy

Last year's exam Q. 94

LQR

$R = \ldots \quad Q = \ldots \quad A = \ldots \quad B = \ldots$

$P = \ldots$

$K = ? \quad K = R^{-1} B^T P$

$A_{CL} = A - BK.$

$\mathrm{eig}(A_{CL})$

speed or energy

unstable A_{CL}

$x \to \pm\infty$

$J_x = \int x^T Q x \, dt$