Revision

$\dot{x} = f(x, t)$ 1st Order ODE

~~Anali~~

An. Soln: $x(t) \Rightarrow 0 = 0$

$x_1(t) = e^{-3t}$ is a soln?

$\dot{x} = -3 \cdot x$

$\dot{x}_1 = -3e^{-3t}$

$-3e^{-3t} = -3 \cdot e^{-3t}$

Another soln: $x_2 = 10e^{-3t}$

$\boxed{x(t) = C \cdot e^{-3t}}$ → Gen. soln.

I.V.P. = ODE + I.C.

$\dot{x} = -3 \cdot x, \quad x(0) = 1$

$x_1(0) = e^{-3 \cdot 0} = 1$

x_1 = spec. soln

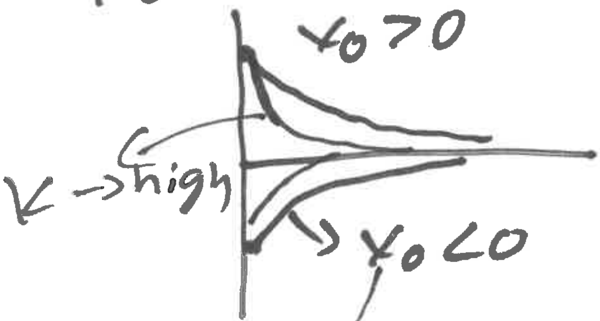
$x_2(0) = 10 \cdot 1 = 10 \neq 1$

$\dot{x} + k \cdot x = u$ $\quad u \rightarrow$ SYS $\rightarrow x$

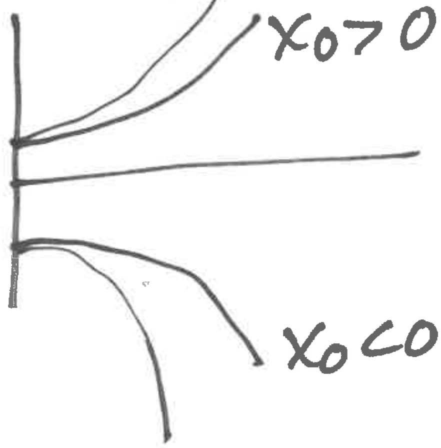
$\hookrightarrow x(t) = \underbrace{e^{-kt} \cdot x_0}_{= \text{ALL SOLNS}} + e^{-kt} \cdot \int_0^t e^{kt_1} \cdot u(t_1) \cdot dt_1$

▪ $u = 0 \Rightarrow x = e^{-kt} \cdot x_0$

- $k > 0 \quad x \rightarrow 0$



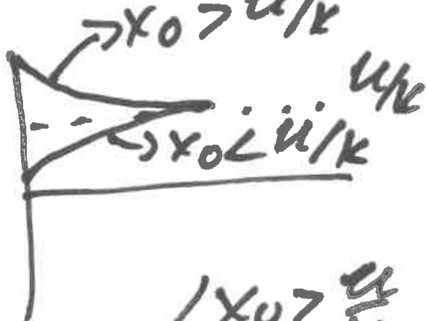
- $k < 0 \quad x \rightarrow \pm\infty$



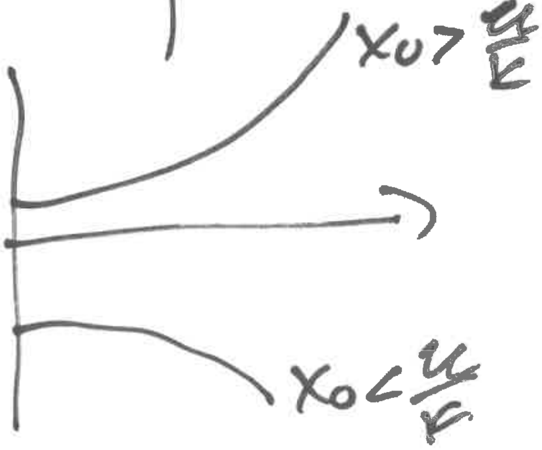
▪ $u \neq 0 \quad (u = \text{const})$

$x(t) = e^{-k \cdot t} \cdot x_0 + \frac{u}{k}\left(1 - e^{-kt}\right)$

- $k > 0 \quad x \rightarrow \frac{u}{k}$



- $k < 0 \quad x \rightarrow \pm\infty$



So in <u>Linear systems</u>

1) x_0 → Not stability
→ Not S.S. $(t = t_0)$
→ The value of x at $t=0$

2) u → Not stability
→ ~~St~~ S.S.
→ Not initial value

3) k → Yes stability/speed
→ Not I.C.
→ Yes S.S (u/k)

$x_0 = u/k$

$$x = \cancel{e^{-kt} \cdot \frac{u}{k}} + \frac{u}{k}\left(1 - \cancel{e^{-kt}}\right) = \frac{u}{k}$$

$$x_{EP} = \frac{u}{k}$$

$k=0$ $\quad \dot{x} + 0 \cdot x = 5$

$$x = \int 5\,dt \Rightarrow x = 5 \cdot t + C$$

unstable

$\ddot{x} = f(\dot{x}, x, t)$

$\downarrow$ Linear

$\ddot{x} + A\dot{x} + Bx = u$
- $\to$ stability / speed
- $\to$ ~~st~~ S.S.
- $\to$ ALL solns

$u = \text{const}$

ODE $\to$ stable

$x \to x_{ss}$

$\dot{x}_{ss} = 0$

$\ddot{x}_{ss} = 0$

$0 + A \cdot 0 + B \cdot x_{ss} = u$

$x_{ss} = u/B$

$u = 0 \to \ddot{x} + A\dot{x} + B \cdot x = 0$

$\ddot{x} - 2\dot{x} - 3x = 0$ $\quad x_1 = e^{3t}$ a soln

$x_2 = 3 \cdot e^{3t}$ –//–

$x_3 = 10 \cdot \sqrt{\pi}\, e^{3t}$ –//–

$x_4 = e^{-t}$ is a soln

$x_5 = 10 \cdot e^{-t}$ –//–

$x_6 = -3\, e^{-t}$ –//–

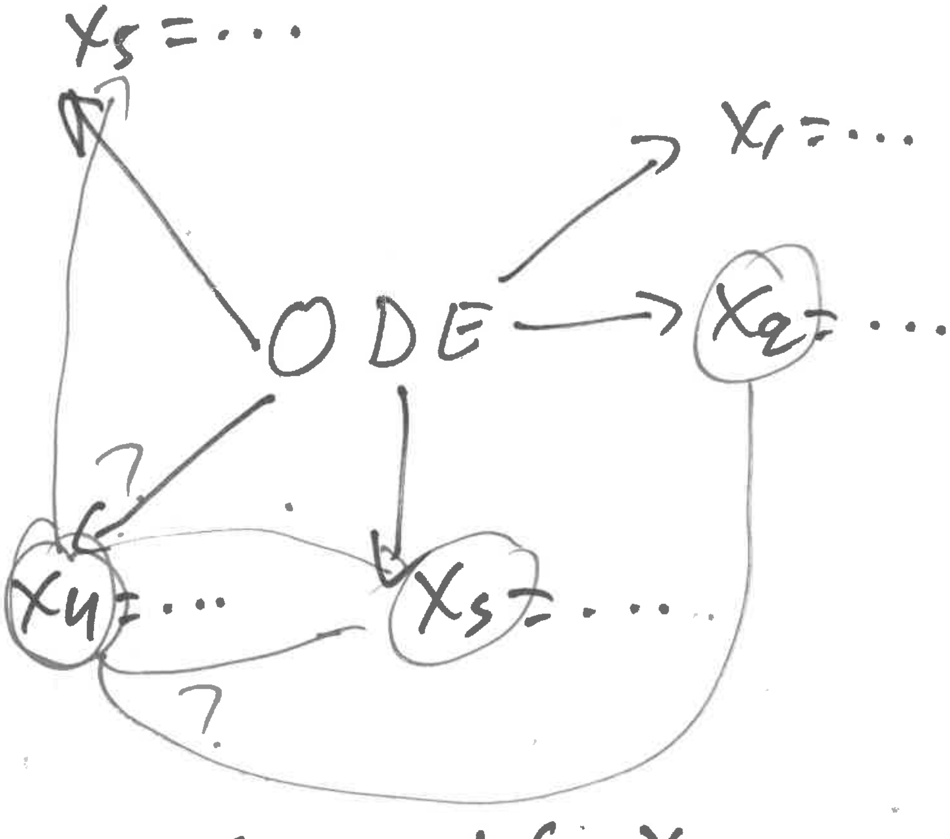
$x_7 = 10 \cdot e^{-t} + \pi \cdot e^{3t}$

Also a soln.

If x_1, x_2 are solns
of $\ddot{x} + A\dot{x} + B\cdot x = 0$
then so is $y = c_1 \cdot x_1 + c_2 \cdot x_2$

$\Downarrow$

inf. soln.



x_4

$c_1 x_m + c_2 \cdot x_n$

$x_k = c_1 \cdot x_{\boxed{n}} + c_2 x_{\boxed{m}}$

$x_n \neq c \cdot x_m$

$\ddot{x} + A\dot{x} + B \cdot x = 0$ $\quad x_1 \quad x_2$ <u>ARE SOLS</u>

$$y = \boxed{c_1} \cdot x_1 + \boxed{c_2} \cdot x_2$$

Can I find $c_1, c_2 = ?$

$$\dot{y} = c_1 \cdot \dot{x}_1 + c_2 \cdot \dot{x}_2$$

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \underbrace{\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{-1}}_{\text{only if } \det \neq 0} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0$$

$$W(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix} = \text{Wronskian Matrix}$$

$$\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0 \implies$$

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} \neq c \cdot \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix} \implies x_1 \neq c \cdot x_2.$$

$\dot{x} + kx = 0$
$x = e^{-kt} \cdot C$

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$\ddot{x} + A\dot{x} + Bx = 0$

HOPE

$x = e^{rt}$
$\dot{x} = r \cdot e^{rt}$
$\ddot{x} = r^2 \cdot e^{rt}$ $\Rightarrow$

$r^2 \cancel{e^{rt}} + A \cdot r \cancel{e^{rt}} + B \cancel{e^{rt}} = 0$

$r^2 + A \cdot r + B = 0 \Rightarrow$ C. E.
$r =$ C. V.
$r =$ eigenvalues

$r_{1,2} = \dfrac{-A \pm \sqrt{\Delta}}{2}$ $\Delta = A^2 - 4 \cdot B$

$\Delta > 0$ $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

$\Rightarrow x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$

$$|W(x_1, x_2)| = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 \cdot e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = \ldots$$

$$= \ldots = r_2 e^{(r_1 + r_2)t} - r_1 e^{(r_1 + r_2)t}$$

$\Rightarrow$ ALL $x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$x = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

If r_1 **and** $r_2 < 0$ $\quad x \to 0$

If r_1 **or** $r_2 > 0$ $\quad x \to \pm\infty$

e.g. $\ddot{x} + 11 \cdot \dot{x} + 30 \cdot x = 0 \qquad x_0 = 1 \quad \dot{x}_0 = 0$

$\downarrow$

$r^2 + 11 \cdot r + 30 = 0 \longrightarrow$ C. E.

$$r_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2} = \begin{cases} r_1 = -5 \\ r_2 = -6 \end{cases}$$

stable

$$\underbrace{x = c_1 e^{-5t} + c_2 e^{-6t}}_{\text{Gen soln.}}$$

$x(0) = 1 \Leftrightarrow \boxed{c_1 \cdot 1 + c_2 \cdot 1 = 1.}$

$\dot{x}(t) = -5 \cdot c_1 e^{-5t} - 6 \cdot c_2 e^{-6t}$

$\dot{x}(0) = \boxed{-5c_1 - 6c_2 = 0} \Longrightarrow \ldots$

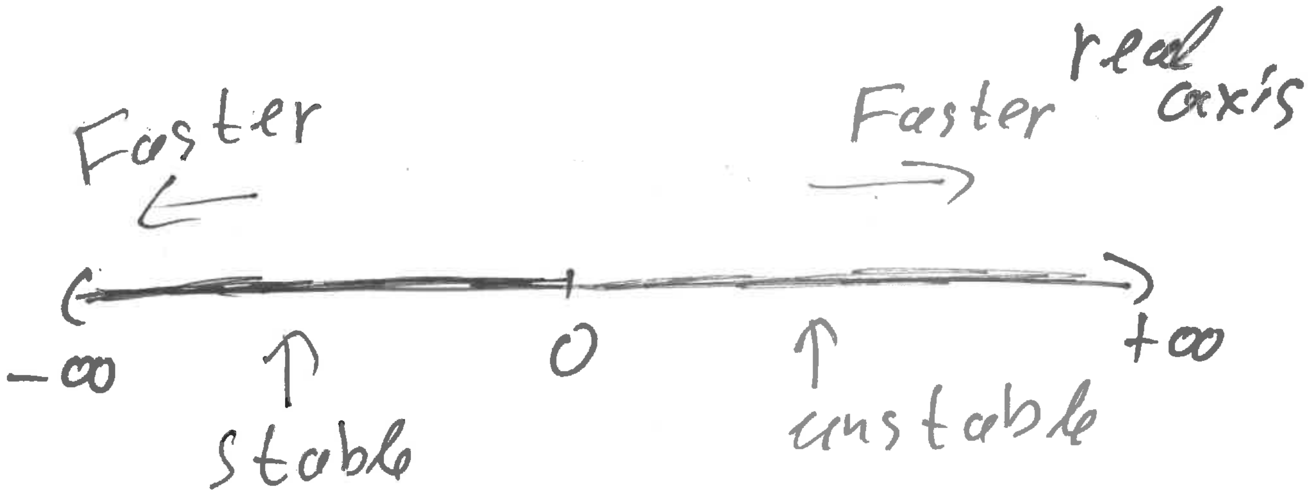
$c_1 = 6 \quad , \quad c_2 = -5$

$\dot{x} + kx = 0$

$\downarrow x = e^{rt}$

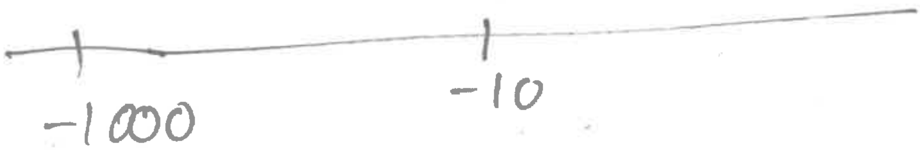
$r\cancel{e^{rt}} + k\cdot\cancel{e^{rt}} = 0$

$r + k = 0 \Rightarrow r = -k.$ $\begin{cases} \to r > 0 \to \text{unst.} \\ \to r < 0 \to \text{stable} \end{cases}$



e^{-10t}

e^{-1000t}



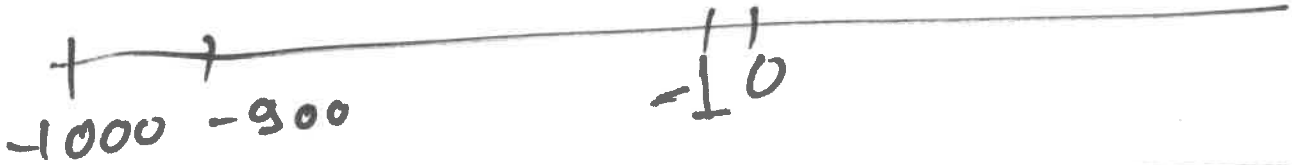
$x = c_1 e^{t} + c_2 e^{-t}$ $\quad x_0 = 1$

$\dot{x} = c_1 e^{t} - c_2 e^{-t}$ $\quad \dot{x}_0 = -1.$

$\left.\begin{array}{l} x_0 = c_1 + c_2 = 1 \\ \dot{x}_0 = c_1 - c_2 = -1 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 0 \\ c_2 = 1 \end{array}$

if $c_1 =$ 0.00000 0.0000 0.00001

$$x = \cancel{c_1}\, \underbrace{e^{-t}}_{\text{Imp. Term.}} + \cancel{c_2}\, e^{-1000000t}$$



- $A^2 - 4B < 0$

$$r_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$$= \frac{-A \pm j\sqrt{4B - A^2}}{2}$$

$$= a \pm bi$$

$$a \rightarrow -A/2 \qquad b \rightarrow \frac{\sqrt{4B - A^2}}{2}$$

$$\Rightarrow \left.\begin{array}{l} r_1 = a + bi \\ r_2 = a - bi \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = e^{(a+bi)t} \\ x_2 = e^{(a-bi)t} \end{array}$$

$$|W| = \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \dots = -2e^{at} bj \neq 0$$

$$x = c_1 x_1 + c_2 \cdot x_2 , \quad c_1, c_2 \in \mathbb{C}$$

$e^{i\theta} = \cos\theta + i\sin\theta$ (20)

$x_1 = e^{rt}$

$x_2 = e^{\bar{r}t}$

$x_1 = e^{(a+bi)t}$

$= e^{at}\cdot e^{bit}$

$= e^{at}\cdot(\cos bt + i\cdot\sin bt)$

$x_2 = e^{at}(\cos bt - i\sin bt)$

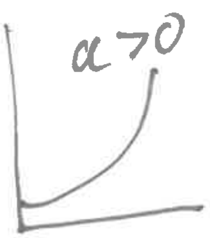
$y_1 = x_1 + x_2 = e^{at}\cdot 2\cos bt \quad \in \mathbb{R}$

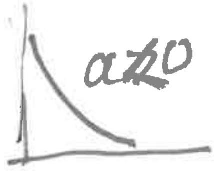
$y_2 = x_1 - x_2 = e^{at}\cdot 2\cdot\sin bt$

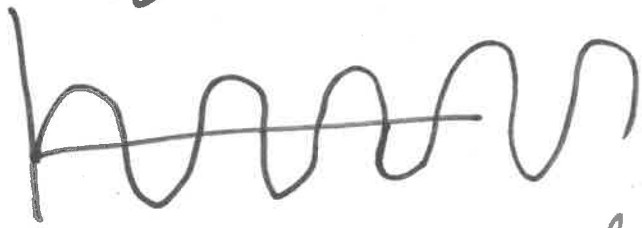
$2i \cdot \sin bt$

$x = c_1\cdot y_1 + c_2\cdot y_2$

$= e^{at}\cdot(\underbrace{c_1\cdot\cos bt + c_2\cdot\sin bt}_{})$







stability

stable	unstable
~~stable~~	~~unst.~~
stable	unstable