Revision

ODEs

$\dot{x} = f(x,t)$ analytical soln $x(t)$

$x(t)$ is a soln $\rightarrow$ $m \cdot x(t)$ is also a soln.

$\rightarrow$ inf. solns

I.V.P.

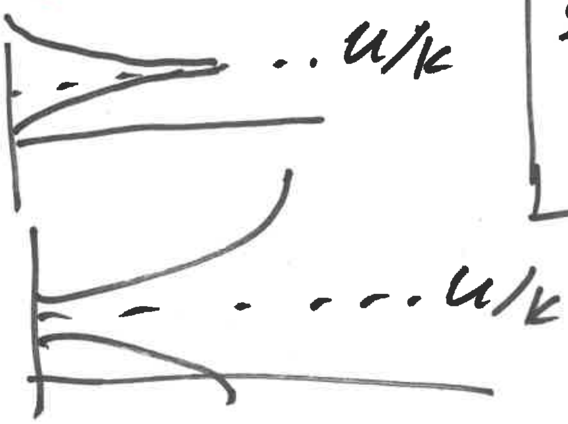
ODE + I.C. $\underline{x_0 = x(0)}$ $\rightarrow$ <u>ONLY</u> one soln

1st order ODEs

$\dot{x} + kx = u$ (MatLab $\dot{x} = kx + u$)

$$x = \underbrace{e^{-kt} \cdot x_0}_{\text{trans.}} + \underbrace{e^{-kt} \cdot \int_0^t e^{kt_1} \cdot u(t_1)\, dt_1}_{\text{Input}}$$

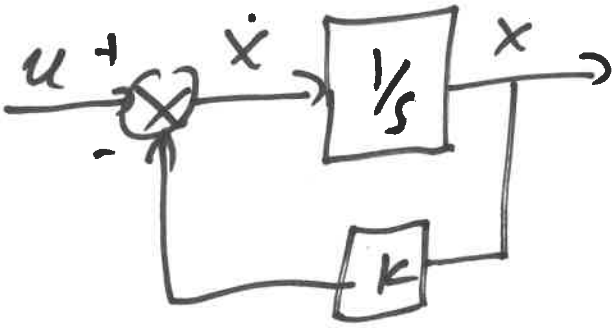
- $k > 0$ $x_{ss} = u/k$
- $k < 0$ $x_{ss} \rightarrow \pm\infty$

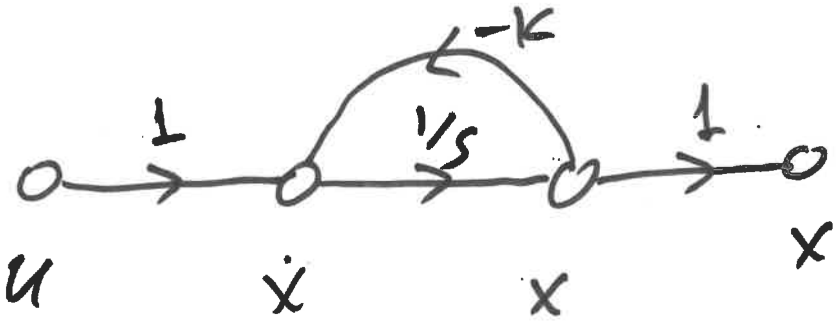


$k \rightarrow$ ~~speed~~ speed

B.D. + S.F.D.

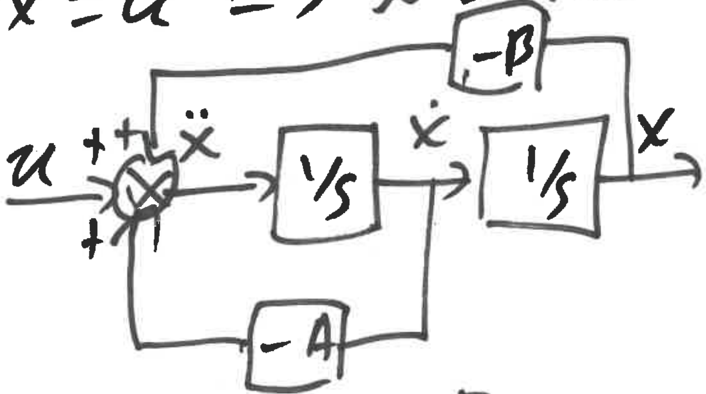
$$\dot{x} + kx = u \Rightarrow \dot{x} = -kx + u$$

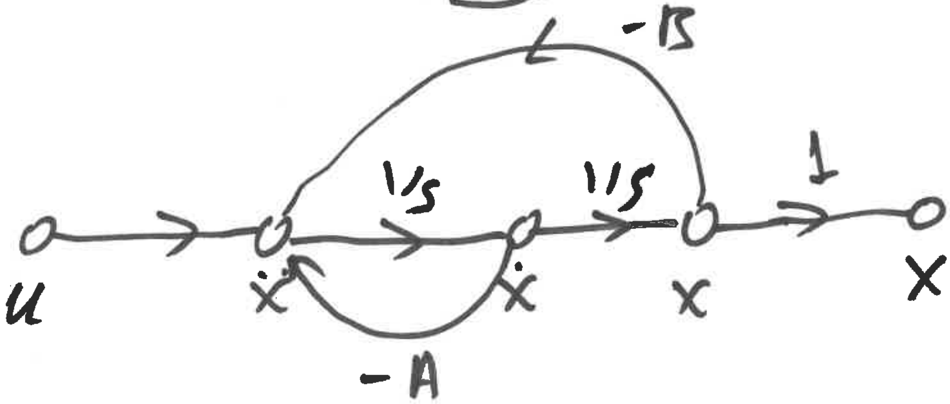




2nd order

$$\ddot{x} + A\dot{x} + Bx = u \Rightarrow \ddot{x} = -A\dot{x} - Bx + u$$





$u=0$

if x_1 is a soln $\to$ $\boxed{m\cdot x_1}$ also a soln.

if x_1, x_2 are soln. $\to$

any L.C. $y=c_1x_1+c_2x_2$

is also a soln.

proof

$$\ddot{x}+A\dot{x}+Bx=0 \qquad x_1, x_2$$

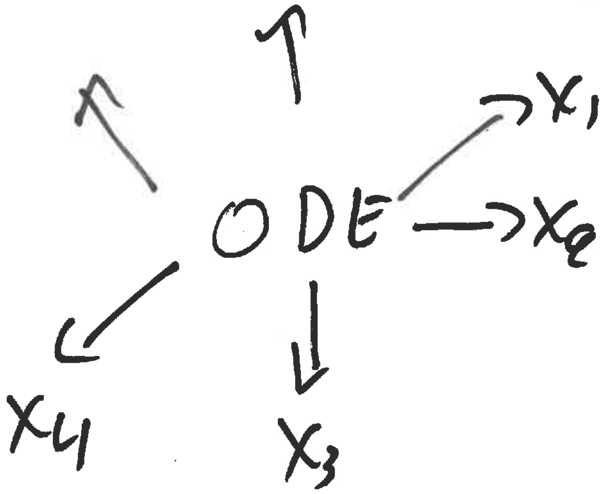
$$\left.\begin{aligned} c_1(\ddot{x}_1+A\dot{x}_1+Bx_1)&=0 \\ c_2(\ddot{x}_2+A\dot{x}_2+Bx_2)&=0 \end{aligned}\right\} \Rightarrow$$

$$\underbrace{c_1\ddot{x}_1+c_2\ddot{x}_2}+A(\underbrace{c_1\dot{x}_1+c_2\dot{x}_2})+B(\underbrace{c_1x_1+c_2x_2})=0$$

$$y=c_1x_1+c_2x_2$$
$$\dot{y}=c_1\dot{x}_1+c_2\dot{x}_2$$
$$\ddot{y}=c_1\ddot{x}_1+c_2\ddot{x}_2$$

$\to$ is a soln.

$$\ddot{y}+A\cdot\dot{y}+By=0$$



Can I express x_4 as a L.C. of x_3 and x_1 =?

$$x_4 = c_1 x_1 + c_2 \cdot x_2$$

$$c_1 = ?$$

$$c_2 = ?$$

$$\ddot{x} - 3\dot{x} - 2x = 0$$

$$x_1 = e^{3t} \qquad x_2 = e^{-t}$$

$$x_3 = e^{3t} + 5e^{-t}$$

$$x_4 = 3e^{3t} + \sqrt{3}\, e^{-t}$$

$$x_5 = e^{-t} + 5e^{-t}$$

$$x_6 = 3e^{-t} + \pi e^{3t}$$

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$$x_6 = c_1 x_1 + c_2 x_2$$

$$3 e^{-t} + \pi e^{3t} = c_1 e^{3t} + c_2 \cdot e^{-t}$$

$$\Rightarrow c_1 = \pi, \quad c_2 = 3$$

H.W.

$$\boxed{x_6 = c_1 x_4 + c_2 x_5}$$

$$x_6 = c_1 x_2 + c_2 x_5$$

$$= c_1 e^{-t} + c_2 (e^{-t} + 5e^{-t})$$

$$3e^{-t} + \pi e^{3t} = (c_1 + 6 \cdot c_2) e^{-t} + 0\, e^{3t}$$

$$\cancel{\pi} \neq 0$$

$$x_5 = 6 \cdot e^{-t}$$

$$x_2 = 1 \cdot e^{-t}$$

$$x_2 = m \cdot x_5$$

x_1, x_2 sol's $\ddot{x} + A\dot{x} + Bx = 0$

$y \nearrow$

?

$$y = c_1 x_1 + c_2 x_2$$

$$\dot{y} = c_1 \dot{x}_1 + c_2 \dot{x}_2$$

$$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow$$

$$\underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{\text{unknown}} = \underbrace{\begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{-1}}_{\text{only if } \begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} \neq 0} \cdot \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ \dot{x}_1 \end{bmatrix} \neq m \begin{bmatrix} x_2 \\ \dot{x}_2 \end{bmatrix}$$

x_1, x_2 are L.I.

2nd ODE

↓

Inf. solns.

↙ ↘

x_1 x_2

$x_1 \neq m \cdot x_2$ → const.

↓

$$y = c_1 x_1 + c_2 x_2$$

$\ddot{x} + A\dot{x} + Bx = 0$

I want to find 2 L.I sols x_1, x_2

All sols $y = c_1 x_1 + c_2 x_2$

$\dot{x} + kx = 0 \rightarrow x = e^{-kt} \cdot x_0$

$e^{rt} \cdot c.$

$\ddot{x} + A\dot{x} + Bx = 0$ Assume $x = e^{rt}$

r = unknown

$r^2 + Ar + B = 0 \rightarrow$ C.E.

$r = \dots$ eigenvalues

$\downarrow$

$r^2 + Ar + B = 0$

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2} \qquad \Delta = A^2 - 4 \cdot B$$

- $\Delta > 0$ $\quad r_1 = \dfrac{-A + \sqrt{\Delta}}{2} \qquad r_2 = \dfrac{-A - \sqrt{\Delta}}{2}$

$r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$

$\Downarrow$

$x_1 = e^{r_1 t} \qquad x_2 = e^{r_2 t}$

$$\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \dots = (r_1 - r_2) e^{(r_1 + r_2)t} \neq 0$$

so x_1, x_2 are L.I.

so ALL sol^ns

$x = c_1 x_1 + c_2 x_2$

e.g. $\ddot{x} + 11\dot{x} + 30x = 0 \quad x_0 = 1, \dot{x}_0 = 0$

$x = e^{rt}$

$r^2 + 11r + 30 = 0 \longrightarrow r_1 = -5 \rightarrow x_1 = e^{-5t}$

$r_2 = -6 \rightarrow x_2 = e^{-6t}$

$x = c_1 e^{-5t} + c_2 e^{-6t}$

$\hookrightarrow$ ALL.

stable

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$\ddot{x} + A\dot{x} + Bx = 0 \rightarrow r_1 = 5$

$r_2\ 6$

unstable.

$\rightarrow r_1 = 5$

$\rightarrow r_2 = -6$

unstable

$x = c_1 e^{-5t} + c_2 e^{-6t}$

$\left.\begin{array}{l} x_0 = 1 \\ x(0) = c_1 \cdot 1 + c_2 \cdot 1 \end{array}\right\} \Rightarrow c_1 + c_2 = 1 \quad (1)$

$\left.\begin{array}{l} \dot{x} = -5 c_1 e^{-5t} - 6 c_2 e^{-6t} \\ \dot{x}_0 = 0 \end{array}\right\} \Rightarrow$

$-5 c_1 - 6 c_2 = 0 \quad (2)$

$\Rightarrow c_1 = 6$
$c_2 = -5$

$x = 6 e^{-5t} - 5 e^{-6t}$

$$x = c_1 e^{t} + c_2 e^{-t}$$

$$x_0 = 1 \qquad \dot{x}_0 = -1$$

$$\left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 - c_2 = -1 \end{array}\right\} \begin{array}{l} c_2 = 1 \\ c_1 = 0 \end{array}$$

$$x = e^{-t}$$

real life $\quad x_0 = 1$ exactly

$x_0 = 1.00001$

$c_1 \neq 0$

$c_1 = 10^{-1000}$

- $\Delta < 0$

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2}$$

$$= \frac{-A \pm \sqrt{(-\Delta)\cdot i^2}}{2}$$

$$= \frac{-A \pm i\sqrt{-\Delta}}{2} = a \pm bi$$

$$a = -A/2$$

$$b = \frac{\sqrt{-\Delta}}{2}.$$

$$r_1 = a + bi \qquad r_2 = a - bi$$

$$x_1 = e^{(a+bi)t} \qquad x_2 = e^{(a-bi)t}$$

$$\begin{vmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{vmatrix} = \ldots = -2e^{at}\cdot bi \neq 0$$

$$x = c_1 \cdot x_1 + c_2 x_2$$

$$c_1, c_2 \in \mathbb{C}$$

$$\ddot{x} + \dot{x} + x = 0$$

$$r^2 + r + 1 = 0 \quad \begin{cases} r_1 = -0.5 + 0.866 \cdot i \\ r_2 = -0.5 - 0.866 \cdot i \end{cases}$$

$$x_1 = e^{(-0.5+0.866 i)t}$$

$$x_2 = e^{(-0.5-0.866 i)t}$$

$$x = c_1 x_1 + c_2 x_2 \qquad x_0 = 1, \quad \dot{x}_0 = 0$$

real

$$c_1 = 0.5 - \frac{\sqrt{3} \cdot i}{6}$$

$$c_2 = -\frac{\sqrt{3}\,(-i) + \sqrt{3}\, i) \cdot i}{6}$$

$x_1 = e^{r_1 t}$ $\quad r_1 = a + bi$

$x_2 = e^{r_2 t}$ $\quad r_2 = \bar{r}_1$

$y_1 = \frac{1}{2}(x_1 + x_2)$

$$x_1 + x_2 = e^{(a+bi)t} + e^{(a-bi)t}$$

$$= e^{at} e^{bit} + e^{at} \cdot e^{-bit}$$

$$e^{at}\left(\underbrace{e^{bit} + e^{-bit}}\right)$$

$\cos bt + i\sin bt + \cos(-bt) + i\sin(-bt)$

$\cos bt + \cancel{i\sin bt} + \cos(bt) - \cancel{i\sin bt}$

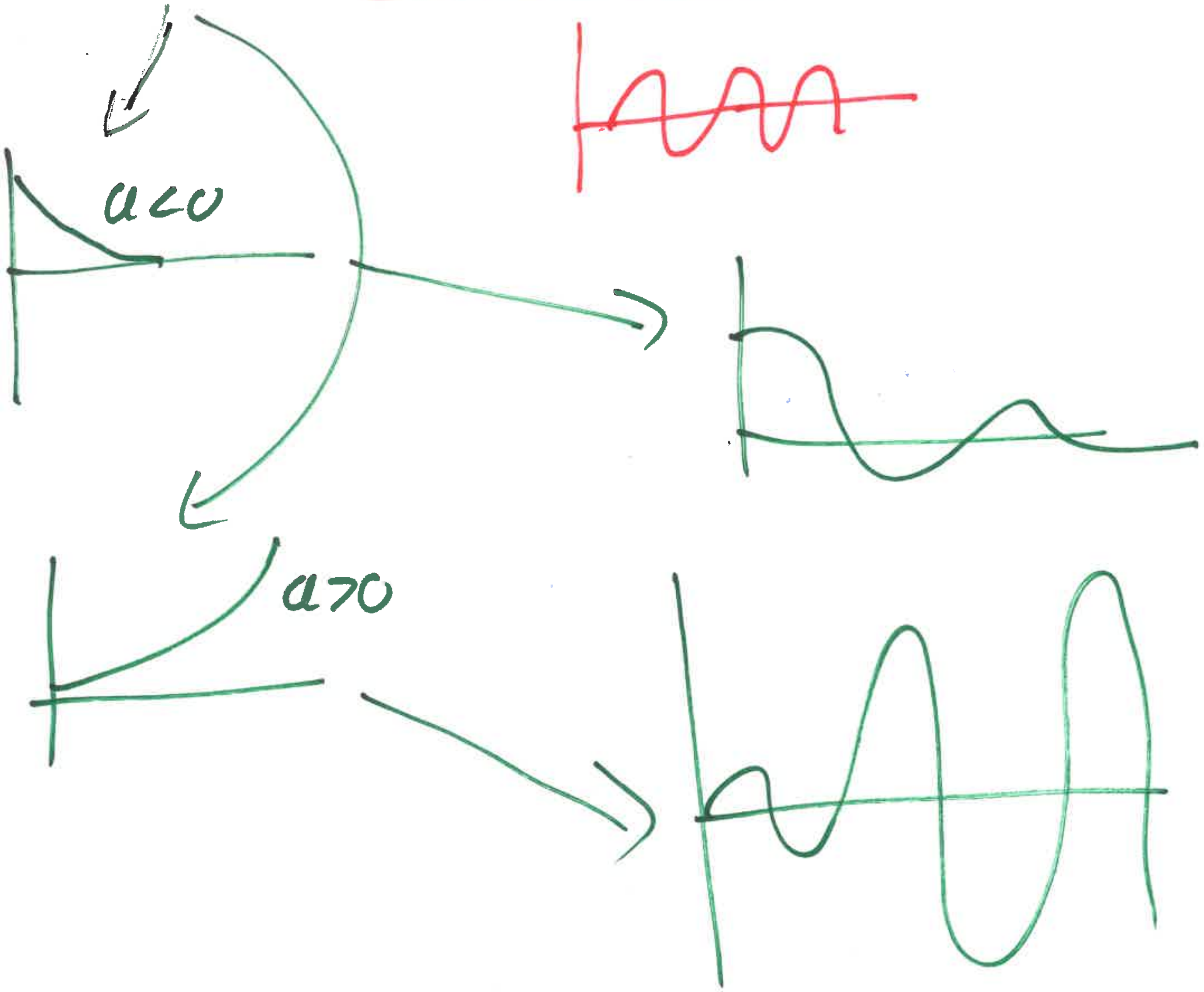
$2 \cdot \cos bt$

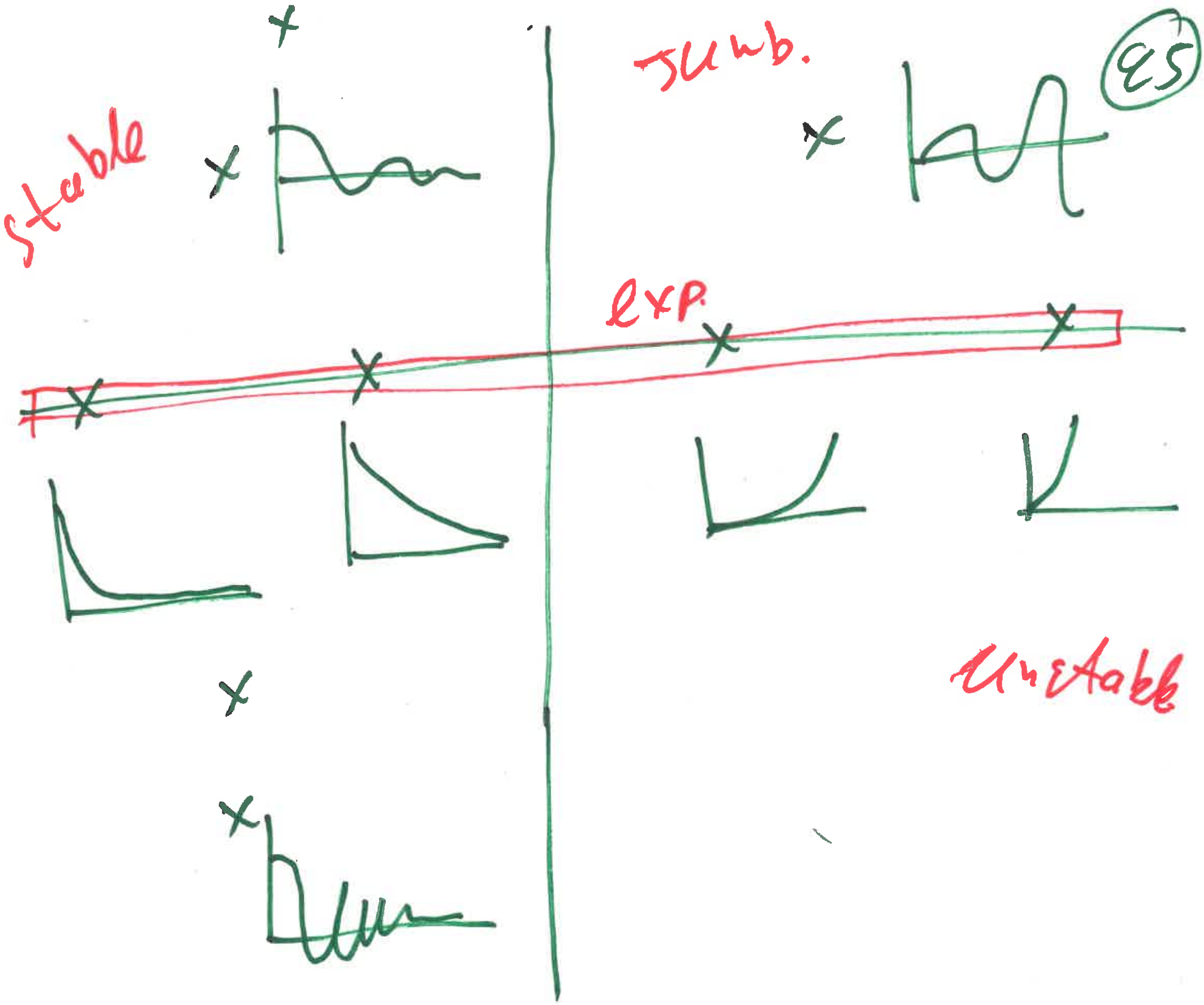
$y_1 = e^{at} \cdot \cos bt$

$y_2 = \frac{1}{2i}(x_1 - x_2) = \dots = e^{at}\sin bt$

$x = c_1 y_1 + c_2 y_2$

$$x = e^{at}\left(c_1 \cos bt + c_2 \sin bt\right)$$

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exp.
unstable