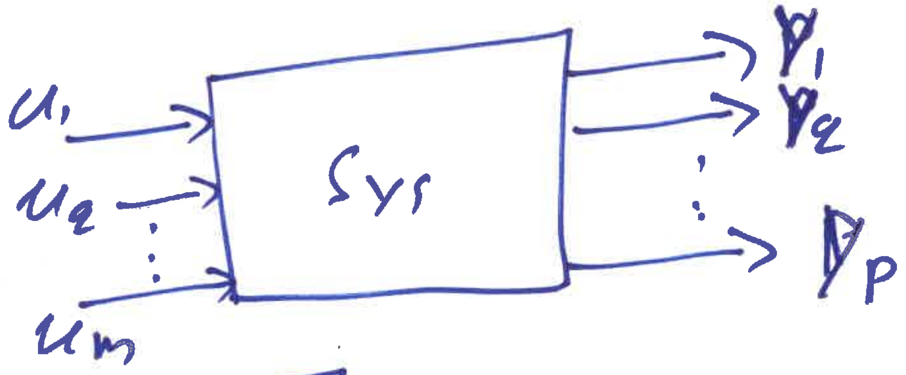
Revision

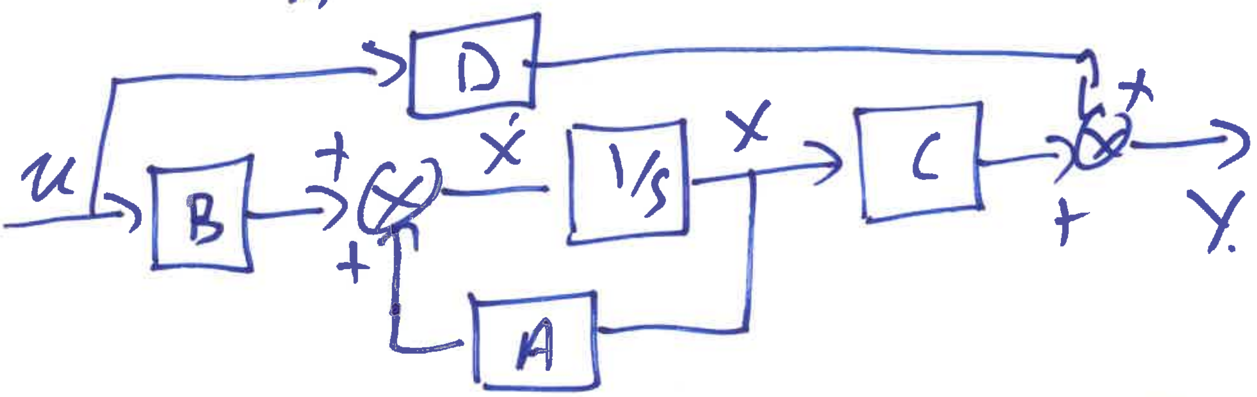
$\dot{x} = Ax + B \cdot u$ $\quad x \in \mathbb{R}^{n\times 1}$, $A \in \mathbb{R}^{n\times n}$

$y = Cx + D \cdot u$ $\quad u \in \mathbb{R}^{m\times 1}$, $y \in \mathbb{R}^{p\times 1}$

$B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$

$D \in \mathbb{R}^{p\times m}$





ODE $\xrightarrow{?}$ S.S.

$$x^{(n)} + a_{n-1} x^{(n-1)} + a_{n-2} x^{(n-2)} + \dots + a_0 x = b_1 u_1 + b_2 u_2 + \dots$$

(1) $$x^{(n)} = -a_{n-1} x^{(n-1)} - a_{n-2} x^{(n-2)} - \dots - a_0 x + b_1 u_1 + b_2 u_2 + \dots$$

2

$$\left.\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ x_3 &= \ddot{x} \\ &\vdots \\ x_n &= x^{(n-1)} \end{aligned}\right\} \overset{d/dt}{\Longrightarrow} \begin{aligned} \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} = x_3 \\ &\vdots \\ \dot{x}_n &= x^{(n)} = \end{aligned}$$

$$x^{(n)} = -a_{n-1} x^{(n-1)} - a_{n-2} x^{(n-2)} - \dots - a_0 x + b_1 u_1 + b_2 u_2 \vdots$$

$$\dot{x}_n = -a_{n-1} x_n - a_{n-2} x_{n-1} + \dots + b_1 u_1 \quad b_2 u_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \dots & 0 \\ 0 & 0 & 1 \; 0 \dots & 0 \\ - & - & - & - \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + ?$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ - & - & \dots \\ u_1 & u_2 & \dots \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

Laplace Transform

$f(t)$

$$F(s) = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt$$

$$s = a + bi$$

$$f(t) \longrightarrow F(s)$$

$$\dot{f}(t) \longrightarrow s \cdot F(s) + \cancel{I.C.} \to 0$$

$$\ddot{f}(t) \longrightarrow s^2 F(s)$$

$$\dddot{f}(t) \longrightarrow s^3 F(s)$$

$$\vdots$$

$$f^{(n)}(t) \longrightarrow s^n \cdot F(s)$$

$$\dot{x} + 5x = u \xrightarrow{C.E.} \boxed{r + 5 = 0}, \quad r = -5 < 0$$

$$x(t) \longrightarrow X(s)$$

$$\dot{x}(t) \longrightarrow sX(s)$$

$$u(t) \longrightarrow U(s)$$

$$sX(s) + 5X(s) = U(s)$$

$$\frac{X(s)}{U(s)} = \frac{1}{\boxed{s+5}}$$

$$\frac{Out(s)}{In(s)} = G(s)$$

$$Out(s) = G(s) \cdot In(s)$$

T.F.

$$\dddot{x} + 3\ddot{x} + 5\dot{x} + x = u + \dot{u}$$

$x(t) \rightarrow X(s)$

$\dot{x}(t) \rightarrow s \cdot X(s)$

$\ddot{x} \rightarrow s^2 X(s)$

$\dddot{x} \rightarrow s^3 X(s)$

$u \rightarrow U(s)$

$\dot{u} \rightarrow s\, U(s)$

$$s^3 X(s) + 3 s^2 X(s) + 5 \cdot s \cdot X(s) + X(s) = U(s) + s U(s)$$

$$X(s)\,(s^3 + 3s^2 + 5s + 1) = U(s)\,(s+1)$$

$$\frac{X(s)}{U(s)} = \frac{s+1}{s^3 + 3s^2 + 5 \cdot s + 1} \quad \text{C.E.}$$

$u = \dot{u} = 0$, $x = e^{rt}$

C.E. $r^3 + 3r^2 + 5 \cdot r + 1 = 0$

ODE $\rightarrow$ S.S.

$\downarrow$

T.F. $\leftarrow$?

$$\dot{x} = Ax + Bu$$
$$y = C \cdot x$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \xrightarrow{L} \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_n(s) \end{bmatrix}$$

$$x(t) \rightarrow X(s)$$
$$u(t) \rightarrow U(s)$$
$$\dot{x}(t) \rightarrow s \cdot X(s)$$

$$s\,X(s) = A \cdot X(s) + B \cdot U(s)$$
$$(sI - A)\;X(s) = B \cdot U(s)$$
$$X(s) = (sI - A)^{-1} \cdot B \cdot U(s)$$

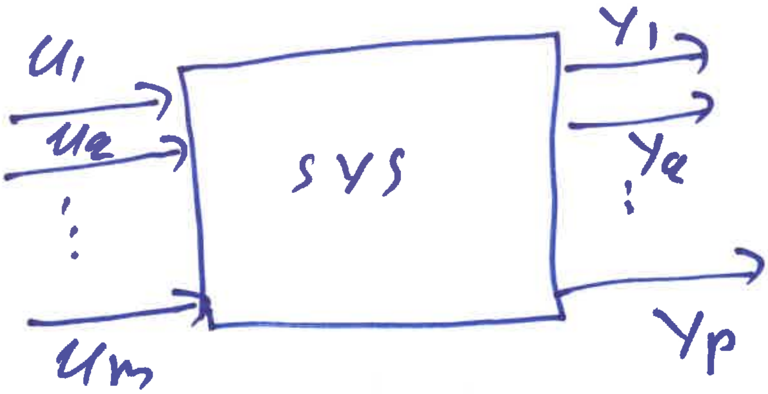
$$Y(s) = C \cdot X(s)$$
$$Y(s) = \underbrace{C \cdot (sI - A)^{-1} \cdot B}_{} \cdot U(s)$$

$$\text{OUT} = \underbrace{\qquad\qquad\qquad}_{} \cdot \text{In}$$

$$G(s) = C \cdot (sI - A)^{-1} \cdot B$$

$$Y(s) = C \cdot (sI - A)^{-1} \cdot B \cdot U(s)$$

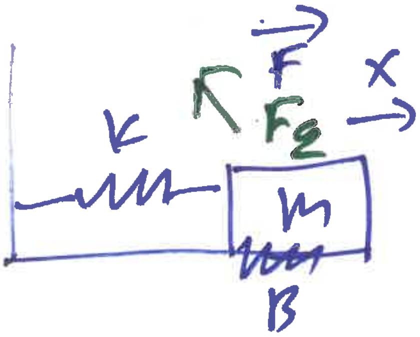
$$G(s) = C \cdot (sI - A)^{-1} \cdot B$$



$$\begin{bmatrix} \frac{y_1}{u_1} = \ldots & \frac{y_1}{u_2} = \ldots & \frac{y_1}{u_3} \ldots\ldots \\ \frac{y_2}{u_1} = \ldots\ldots & & \\ \vdots & & \\ \frac{y_p}{u_1} \ldots\ldots & & \frac{y_p}{u_m} = \ldots \end{bmatrix}$$

$$G(s) = \underbrace{C}_{p \times n} \cdot \underbrace{(sI - A)^{-1}}_{n \times n} \cdot \underbrace{B}_{n \times m}$$

$$p \times m$$



$Y_1 = \text{disp.} = x$

$Y_2 = \text{speed.} = \dot{x}$

$u_1 = F$

$\frac{Y_1}{u_1} =$ How F inf. x

$\frac{Y_2}{u_1} =$ How F inf. $\dot{x}$

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n = 2 $\quad$ p = 2 $\quad$ m = 2

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$B \rightarrow$ 1 input $\quad C \rightarrow$ 1 output

T.F. = ?

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ +1 & s+0.5 \end{bmatrix}$$

$$(sI - A)^{-1} \longrightarrow |sI - A| = s(s+0.5)+1$$

$$= s^2 + 0.5s + 1$$

$$\hookrightarrow = \frac{1}{s^2 + 0.5s + 1} \begin{bmatrix} s+0.5 & 1 \\ -1 & s \end{bmatrix}$$

$$C \cdot (sI - A)^{-1} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{1\times 2} \cdot \frac{1}{(\quad)} \underbrace{\begin{bmatrix} s+0.5 & 1 \\ -1 & s \end{bmatrix}}_{2\times 2}$$

$$\cancel{= \frac{1}{(\quad)} \begin{bmatrix} s+0.5 \end{bmatrix}}$$

$$= \frac{1}{(\quad)} \begin{bmatrix} s+0.5 & 1 \end{bmatrix}.$$

$$C \cdot (sI - A)^{-1} \cdot B$$

$$\frac{1}{(\quad)} \underset{1\times 2}{\begin{bmatrix} s+0.5 & 1 \end{bmatrix}} \cdot \underset{2\times 1}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

$$= \frac{1}{s^2+0.5s+1} \cdot 1$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$Y_1 \quad Y_2$

$\downarrow u_1$

$$\frac{Y_1}{u_1} \qquad p = 2 \quad m = 1$$

$$\frac{Y_2}{u_1} \qquad G \in \mathbb{R}^{2\times 1}$$

$$C\cdot \underbrace{(sI-A)^{-1}}\cdot B$$

$$\begin{bmatrix}1&0\\0&2\end{bmatrix}\frac{1}{(\ \ )}\cdot\begin{bmatrix}s+0.5&1\\-1&s\end{bmatrix}\cdot\begin{bmatrix}0\\1\end{bmatrix}$$

2×2 $\quad$ 2×1

$$\begin{bmatrix}1&0\\0&2\end{bmatrix}\cdot\frac{1}{(\ \ )}\begin{bmatrix}0+1\\0+s\end{bmatrix}$$

2×2 $\quad$ 2×1

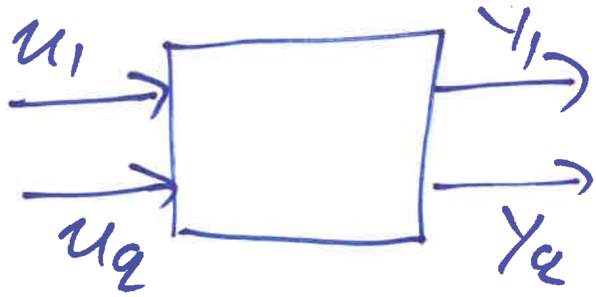
$$=\frac{1}{(\ \ )}\cdot\begin{bmatrix}1\\2s\end{bmatrix}=\frac{1}{s^2+0.5s+1}\cdot\begin{bmatrix}1\\2s\end{bmatrix}$$

$$G_{11}=\frac{1}{s^2+0.5s+1}$$

$$G_{21}=\frac{2s}{s^2+0.5s+1}$$

$$A=\begin{bmatrix}0 & 1\\ -1 & -0.5\end{bmatrix} \quad B=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \quad C=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$$ 45

2in 2out



$$\frac{y_1}{u_1}=G_{11}$$

$$\frac{y_1}{u_2}=G_{12}$$

$$\frac{y_2}{u_1}=G_{21}$$

$$\frac{y_2}{u_2}=G_{22}$$

$$G(s)=C\cdot(sI-A)^{-1}\cdot B$$

$$=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\cdot\frac{1}{(\ \ )}\begin{bmatrix}s+0.5 & 1\\ -1 & s\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{(\ \ )}\begin{bmatrix}s+0.5 & 1\\ -2 & 2s\end{bmatrix}\cdot\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{(\ \ )}\begin{bmatrix}s+0.5 & s+1.5\\ -2 & -2+2s\end{bmatrix}$$

$$G_{11} = \frac{s+0.5}{s^2+0.5s+1}$$

$$G_{12} = \frac{s+1.5}{s^2+0.5s+1}$$

$$G_{21} = \frac{-2}{s^2+0.5s+1}$$

$$G_{2,2} = \frac{-2+2s}{s^2+0.5s+1}$$

$$G_{i,j}(s) = \frac{\begin{vmatrix} sI-A & -B_i \\ C_j & D \end{vmatrix}}{|sI-A|}$$

$B_i \rightarrow i^{th}$ colum of B

$C_j \rightarrow j^{th}$ row of C.