Linear Controller Design and State Space Analysis

EEE3001/8013 Tutorial Exercise I

1. *By using the general form of the analytic solution try to predict the response of the following systems:*

A generic first order system can be written as $\frac{dx}{dt}+kx=u$.

- This system is stable if k is positive and unstable if k is negative.
- If the input signal u is constant (and the system is stable), u will simply shift the overall solution to u/k as at steady state $\frac{dx}{dt}=0$ (for a constant input!) and hence $0+kx_{ss}=u \Rightarrow x_{ss}=u/k$.

With these in mind I can answer the 1st question[1]:

i. $\boxed{5\frac{dx}{dt}+6x=0, \quad x(0)=0,\, x(0)=1,\, x(0)=-1}$

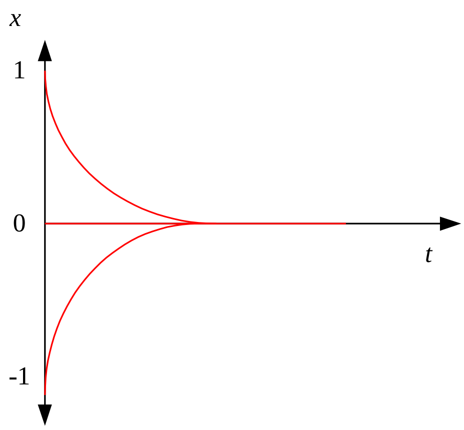
In this system the value of k is 6/5>0 so the system is stable. The input is zero (homogeneous system), so the response will converge (exponentially) to zero. When Initial Condition (IC) is +/-1, x will start from +/1 and it and will converge to zero. If it starts at 0, it will stay there $\forall\ t\in\mathbb{R}^{+}$.

From the lecture we know that $x(t)=e^{-kt}x(t_0)+e^{-kt}\int_{t_0}^{t}e^{kt_1}u(t_1)dt_1$

In this case, $x(t)=\begin{cases} e^{-\frac{6}{5}t}\cdot 1 = e^{-\frac{6}{5}t} \\ e^{-\frac{6}{5}t}\cdot 0 = 0 \\ e^{-\frac{6}{5}t}\cdot(-1) = -e^{-\frac{6}{5}t} \end{cases}$

[1] A detailed solution is given for the first question; only brief answers will be given to the rest of the questions.

Note that if you were also asked to plot the response you would need to do something like this:



ii. $\boxed{5\frac{dx}{dt} - 6x = 0, \quad x(0) = 0, x(0) = 1, x(0) = -1}$

Unstable, homogeneous. If it starts from +/-1, it diverges exponentially to +/-∞, if it starts from zero, it stays there forever.

iii. $\boxed{5\frac{dx}{dt} + 6x = 1, \quad x(0) = 0, x(0) = 1, x(0) = -1}$

Stable, converges exponentially to 1/6, regardless of ICs.

iv. $\boxed{5\frac{dx}{dt} + 6x = -1, \quad x(0) = 0, x(0) = 1, x(0) = -1}$

Similarly, for this system but now I will converge to -1/6.

v. $\boxed{\frac{dx}{dt} + 3 = 0, \quad x(0) = 0, x(0) = 1, x(0) = -1}$

This is a "trivial system" as k=0 and the output can be found by a simple integration:

$\frac{dx}{dt} = -3 \Leftrightarrow x(t) = \int 3dt = 3t + C$. The value of C can be found from the ICs as $x(0)$=C.

So the response will diverge **linearly** to +∞ and for $x(0)<0$ it will hit the axis x=0 when t=-$x(0)/3$.

2. Find the solution of $\ddot{x}+6\dot{x}+5x=0,\ x(0)=2,\ \dot{x}(0)=3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.

This is a homogeneous 2nd order DE. I will start by finding the roots of the characteristic equation:

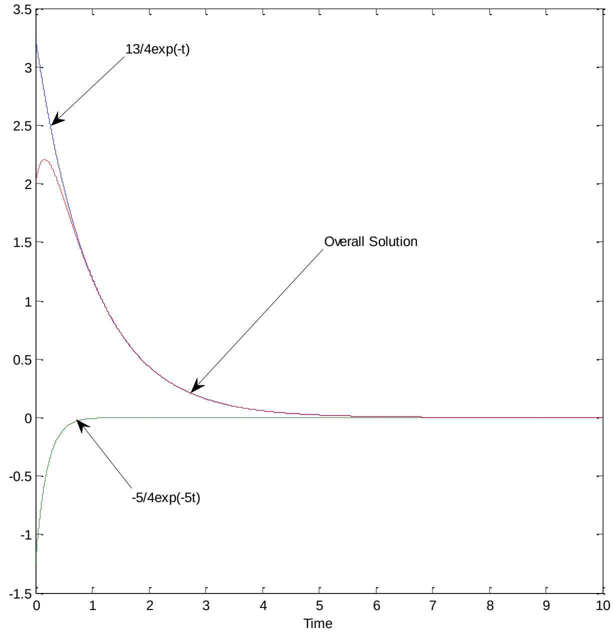
$$\Leftrightarrow r^2+6r+5=0 \Leftrightarrow (r+5)(r+1)=0 \Leftrightarrow r_2=-5 \quad r_1=-1$$

Since the 2 roots are negative the solution is stable and will decay exponentially to zero (since it's a homogeneous system)

This is an overdamped system and the solution can be written as: $x=C_1e^{-5t}+C_2e^{-t}$

Using the given initial conditions I can calculate C_1 and C_2:

$C_1=-5/4$, $C_2=13/4$ => $x=-5/4e^{-5t}+{}^{13}\!/_{4}e^{-t}$



For MSc students:

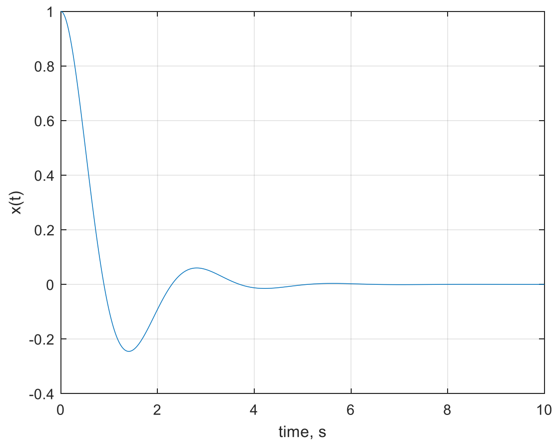
```
clc, clear all, close all,
syms x(t); Dx=diff(x,1); D2x=diff(x,2); ODE=D2x+6*Dx+5*x;
x_sol=dsolve(ODE, x(0)==2, Dx(0)==3)
x_num=double(subs(x_sol,t,0:0.01:10));
plot(0:0.01:10,x_num);
xlabel('time, s'), ylabel('x(t)'), grid
```

3. Find the solution of $\ddot{x}+2\dot{x}+6x=0,\ x(0)=1,\ \dot{x}(0)=0$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.

Homogeneous system, with eigenvalues: -1±2.2361i, so stable, converging to zero with oscillations.

Particular solution: exp(-t)*cos(5^(1/2)*t) + (5^(1/2)*exp(-t)*sin(5^(1/2)*t))/5

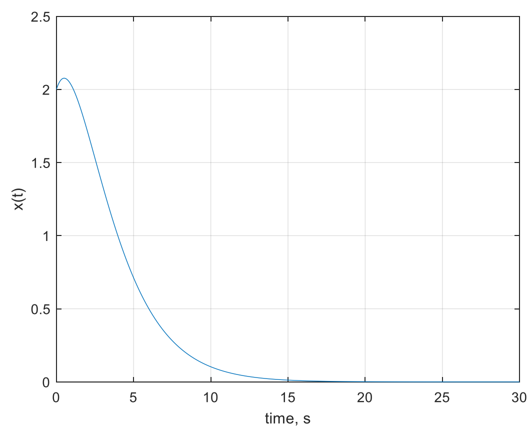
Response:



4. Find the solution of $\ddot{x}+\dot{x}+0.25x=0,\ x(0)=2,\ \dot{x}(0)=1/3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.

Eigenvalues -0.5 (twice), so stable, exponentially converging to zero.

Particular solution: 2*exp(-t/2) + (4*t*exp(-t/2))/3



5. Find the Wronskian matrices of the solutions of Q2-5.

For Q2:

$$x_1 = e^{-5t},\ x_2 = e^{-t} \Rightarrow W = \begin{bmatrix} e^{-5t} & e^{-t} \\ -5e^{-5t} & -e^{-t} \end{bmatrix} \Rightarrow |W| = e^{-5t}\left(-e^{-t}\right) - e^{-t}\left(-5e^{-5t}\right) = -e^{-6t} + 5e^{-6t} = 4e^{-6t} \neq 0$$

For MSc students:

```
clc, clear all, close all, syms t,
x1=exp(-5*t); x2=exp(-t); Dx1=diff(x1); Dx2=diff(x2);
W=[x1, x2; Dx1, Dx2], det(W)
```

Q3[2]: $|W| = -\sqrt{5}\cdot 2i\cdot e^{-2t}$

Q4: $|W| = e^{-t}$

[2] `clc, clear all, close all, syms t, x=roots([1 2 6]); x1=exp(x(1)*t); x2=exp(x(2)*t); Dx1=diff(x1); Dx2=diff(x2); W=[x1, x2; Dx1, Dx2]; det(W); simplify(det(W)), pretty(ans)`