Linear Controller Design and State Space Analysis

EEE3001/8013 Tutorial Exercise II

1. *Derive a state space representation of the mass spring system assuming that the system has 2 outputs: the displacement and the velocity.*

By choosing the states as $x_1 = x,\ x_2 = \dot{x}$ we have:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{B}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1/m\end{bmatrix}F$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}\Leftrightarrow y=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}x$$

2. *Repeat Question 1 assuming that the displacement is the only system output.*

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{B}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1/m\end{bmatrix}F$$

$$y = x_1 \Leftrightarrow y=\begin{bmatrix}1 & 0\end{bmatrix}x \Leftrightarrow y = Cx$$

3. *Find the state space model of the following system:*

$$\ddot{x}+6\dot{x}+5x=u(t)$$
$$y=4\dot{x}+x$$

By choosing the states as $x_1 = x,\ x_2 = \dot{x}$ we have:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -5 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u \Leftrightarrow \dot{x}=Ax+Bu$$

$$y=\begin{bmatrix}1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\Leftrightarrow y=Cx$$

4. *A state space model is given by*

$$\mathbf{A}=\begin{bmatrix}1 & 2 & 3 & 4 & 5\\ 6 & 7 & 8 & 9 & 10\\ 11 & 12 & 13 & 14 & 15\\ 16 & 17 & 18 & 19 & 20\\ 21 & 22 & 23 & 24 & 25\end{bmatrix},\ \mathbf{B}=\begin{bmatrix}-0.1 & -0.2 & -0.3\\ -0.4 & -0.5 & -0.6\\ -0.7 & -0.9 & -1\\ -1.1 & -1.2 & -1.3\\ -1.4 & -1.5 & -1.6\end{bmatrix},\ \mathbf{C}=\begin{bmatrix}1 & 1 & 2 & 2 & 3\\ 0 & 0 & 0 & -1 & 152\end{bmatrix},$$

$D=0$

(a) What is the order of the system? [n=5]
(b) How many inputs/ outputs do we have in this system? [in=3 and out=2]
(c) What are the dimensions of the matrix D? [2x3]

5. Find the state space model of:

(a)
$$\left.\begin{aligned}
&x^{(4)}=3x^{(3)}+4x''-3x'+x+u_1-3u_2+5u_3\\
&y_1=x^{(3)}+u_1\\
&y_2=x^{(4)}+1.2x'+u_3-u_1\\
&y_3=x
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}&=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & -3 & 4 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & -3 & 5\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}\\
\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}&=\begin{bmatrix}0 & 0 & 0 & 1\\ 1 & -1.8 & 4 & 3\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}1 & 0 & 0\\ 0 & -3 & 6\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix}
\end{aligned}\right\}$$

Order 4, 3 inputs and 3 outputs.

(b)
$$\left.\begin{aligned}
\dot{x}_1 &= 3x_1 + 3x_2 + u_1 + u_2 + u_3 + u_4 \\
\dot{x}_2 &= 3x_2 + u_1 - 2u_3 \\
y_1 &= x_1 \\
y_2 &= x_1 + 3x_2 + u_1 + u_2 \\
y_3 &= x_1 - 2x_2 + u_3 + u_4
\end{aligned}\right\}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Order 2, 4 inputs and 3 outputs.

6. *Find the transfer function of a system with:*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = 0.$$

$$G(s) = \frac{s-1}{s^2 - 5s - 2}$$

7. *Find the transfer function of a system with:*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

$$\frac{Y_1}{U_1} = G_{11} = \frac{s-1}{s^2 - 5s - 2}$$
$$\frac{Y_1}{U_2} = G_{12} = \frac{2s}{s^2 - 5s - 2}$$

8. *Find the transfer function of a system with:*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\frac{Y_1}{U_1} = G_{11} = \frac{s-1}{s^2 - 5s - 2}$$
$$\frac{Y_2}{U_1} = G_{21} = \frac{3s}{s^2 - 5s - 2}$$
$$\frac{Y_1}{U_2} = G_{12} = \frac{2s}{s^2 - 5s - 2}$$
$$\frac{Y_2}{U_2} = G_{22} = \frac{s+2}{s^2 - 5s - 2}$$

9. *What is the characteristic equation in Q.6-8? What is the system order? Is that system stable? Why?*

Q.6-8: characteristic equation is $s^2 - 5s - 2$ with roots -0.37 and 5.37 hence the system is unstable. The system is order 2.

All are OBSV, CTRB.