Linear Controller Design and State Space Analysis

EEE3001/8013 Tutorial Exercise III

1. A system is given by $\dot{x}=\begin{bmatrix}-2 & 2\\ 2 & -5\end{bmatrix}x$

   (a) Find the eigenvalues and eigenvectors of this system.

   (b) Find the general solution using the previously found eigenvectors.

   (c) Find the particular solution if $x(0)=\begin{bmatrix}1\\1\end{bmatrix}$.

   (a)

```
A=[-2 2;2 -5];
[x,v]=eig(A);
e1=x(:,1)/max(x(:,1))
e2=x(:,2)/max(x(:,2))
e1 =
   -0.5000
    1.0000

e2 =
     2
     1
```

(b)

General solution: $x=C_1\begin{bmatrix}-0.5\\1\end{bmatrix}e^{-6t}+C_2\begin{bmatrix}2\\1\end{bmatrix}e^{-t}$

(c)

```
syms C1 C2
C=solve(C1*e1+C2*e2-[1;1])
C1n=double(C.C1)
C2n=double(C.C2)
C1n =
   0.4
C2n =
     0.6
```

2. A system is given by $\ddot{x}+3\dot{x}+2x=0, x(0)=1, \dot{x}(0)=-1$

(a) Find the particular solution of the differential equation.

(b) Draw a sketch of the response *x*(t).

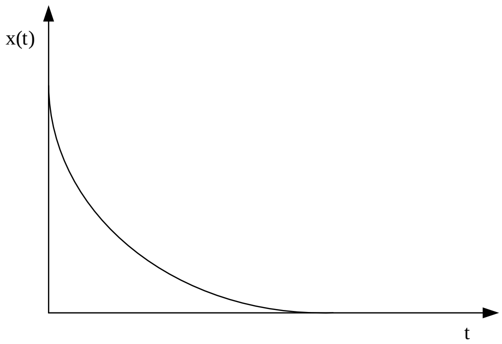
(c) Transform the system to state space form if *y* = -2*x*(t).

(d) Find the eigenvalues and eigenvectors of the system. What is the response type?

(f) Find the general solution using the eigenvectors then find the particular solution using the given initial conditions.

(a)
```
clc, clear all, syms x(t),
dx=diff(x); d2x=diff(x,2);
x_t=dsolve(d2x+3*dx+2*x, x(0)==1, dx(0)==-1)
```

(b)



(c) $\left.\begin{array}{l} x_1 = x \\ x_2 = \dot{x} \end{array}\right\} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = x_1 \Leftrightarrow y = \begin{bmatrix} -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(d) $|\lambda I - A| = 0 \Rightarrow \left\| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \right\| = 0 \Rightarrow \left\| \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix} \right\| = 0 \Rightarrow \lambda(\lambda+3)+2=0$

$$\lambda(\lambda+3)+2=0 \Rightarrow \lambda^2+3\lambda+2=0 \Rightarrow (\lambda+1)(\lambda+2)=0 \Rightarrow \lambda_{1,2}=-1,-2$$

Overdamped and stable response.

To Find the eigenvectors

$$(\lambda I - A)v = 0$$

For $\lambda_1 = -1$

$$\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow -a_1 - a_2 = 0,\ 2a_1 + 2a_2 = 0 \Rightarrow a_2 = -a_1$$

The eigenvector v_1 can be $[1\ -1]^T$

For $\lambda_2 = -2$

$$\begin{bmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow -2a_1 - a_2 = 0,\ 2a_1 + a_2 = 0 \Rightarrow a_2 = -2a_1$$

The eigenvector v_2 can be $[1\ -2]^T$

(e)

$$x(t) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 1 \\ -2 \end{bmatrix}e^{-2t}$$

$$x(0) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 + C_2 \\ -C_1 - 2C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

C_1=1, C_2=0 and the particular solution is: $x(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t}$

3. Find the state transition matrix of the homogeneous state space system that you find in 2 (c) then find the particular solution for $x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

$$M(t) = \begin{bmatrix} (2e^{-t} - e^{-2t}) & (e^{-t} - e^{-2t}) \\ -2(e^{-t} - e^{-2t}) & (-e^{-t} + 2e^{-2t}) \end{bmatrix}$$

$$x(t) = M(t)x(0) = \varphi(t)x(0)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & \left(-e^{-t} + 2e^{-2t}\right) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \left(2e^{-t} - e^{-2t}\right) & \left(e^{-t} - e^{-2t}\right) \\ -2\left(e^{-t} - e^{-2t}\right) & \left(-e^{-t} + 2e^{-2t}\right) \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$x_1(t) = -\left(2e^{-t} - e^{-2t}\right)$$
$$x_2(t) = 2\left(e^{-t} - e^{-2t}\right)$$

4. A system is given by $\dot{x} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} x$

(a) Find the state transition matrix.

(b) Find the particular solution for the homogeneous system using the state transition matrix approach for $x(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

$$M(t) = \begin{bmatrix} \left(\frac{6}{7}e^{2t} + \frac{1}{7}e^{-5t}\right) & \left(\frac{3}{7}e^{2t} - \frac{3}{7}e^{-5t}\right) \\ \left(\frac{2}{7}e^{2t} - \frac{2}{7}e^{-5t}\right) & \left(\frac{1}{7}e^{2t} + \frac{6}{7}e^{-5t}\right) \end{bmatrix}$$

$$x_1(t) = -\left(\frac{3}{7}e^{2t} - \frac{3}{7}e^{-5t}\right)$$
$$x_2(t) = -\left(\frac{1}{7}e^{2t} + \frac{6}{7}e^{-5t}\right)$$

5. A state space system is given by $\dot{x} = \begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} x$

(a) Find the state transition matrix.

(b) Find the particular solution for the homogeneous system using the state transition matrix approach for $x(0)=\begin{bmatrix}0\\1\end{bmatrix}$.

(c) Crosscheck your answer in (b) by finding the particular solution of the system using eigenvalues and eigenvectors approach.

$$M(t)=\begin{bmatrix} e^{-2t}(\cos t+2\sin t) & e^{-2t}\sin t \\ -5e^{-2t}\sin t & e^{-2t}(\cos t-2\sin t) \end{bmatrix}$$

$$x(t)=\begin{bmatrix} e^{-2t}\sin t \\ e^{-2t}(\cos t-2\sin t) \end{bmatrix}$$