# Chapter #1

# EEE8013 – EEE3001

# Linear Controller Design and State Space Analysis

# Ordinary Differential Equations

## 1. Introduction

To understand the properties (dynamics) of a system, we can model (represent) it using differential equations (DEs). The response/behaviour of the system is found by solving the DEs. In our cases, the DE is an Ordinary DE (ODE), i.e. not a partial derivative. The main purpose of this Chapter is to learn how to solve first and second order ODEs in the time domain. This will serve as a building block to model and study more complicated systems. Our ultimate goal is to control the system when it does not show a "satisfactory" behaviour. Effectively, this will be done by modifying the ODE.

## 2. First Order ODEs

The general form of a first order ODE is:

$$\frac{dx(t)}{dt} = f\left(x(t),t\right) \qquad (1)$$

where[1] $x,t \in \mathbb{R}$

**Analytical solution**: Explicit formula for $x(t)$ (a solution which can be found using various methods) which satisfies $\frac{dx}{dt} = f(x,t)$

[1] The proper notation is $x(t)$ and not $x$ but we drop the brackets in order to simplify the presentation.

**Example 1.1**: Prove that $x = e^{-3t}$ and $x = -10e^{-3t}$ are solutions of $\frac{dx}{dt} = -3x$.

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(e^{-3t}\right)}{dt} = -3\left(e^{-3t}\right) \Leftrightarrow -3e^{-3t} = -3e^{-3t}$$

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(-10e^{-3t}\right)}{dt} = -3\left(-10e^{-3t}\right) \Leftrightarrow 30e^{-3t} = 30e^{-3t}$$ ■

Obviously there are infinite solutions to an ODE and for that reason the found solution is called the **General Solution** of the ODE.

## First order Initial Value Problem : $\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$

An initial value problem is an ODE with an initial condition, hence we do not find the general solution but the **Specific Solution** that passes through $x_0$ at $t=t_0$.

**Analytical solution**: Explicit formula for $x(t)$ which satisfies $\frac{dx}{dt} = f(x,t)$ and passes through $x_0$ when $t = t_0$.

**Example 1.2:** Prove that $x = e^{-3t}$ is a solution, while $x = -10e^{-3t}$ is not a solution of $\frac{dx}{dt} = -3x, x_0 = 1$

Both expressions ($x = e^{-3t}$ and $x = -10e^{-3t}$) satisfy the $\frac{dx}{dt} = -3x$ but at $t$=0

$$x(t) = e^{-3t} \Rightarrow x(0) = 1$$

$$x(t) = -10e^{-3t} \Rightarrow x(0) = -10 \neq 1$$ ■

For that reason some books use a different symbol for the specific solution: $\phi(t, t_0, x_0)$ .

You must be clear about the difference between an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

## Linear First Order ODEs

A linear 1st order ODE is given by:

$$\begin{cases} a(t)x' + b(t)x = c(t), a(t) \neq 0 & \textit{Non autonomous} \\ ax' + bx = c, a \neq 0 & \textit{Autonomous} \end{cases} \tag{2}$$

with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

In engineering books the most common form of (2) is (since $a \neq 0$):

$$x' + k(t)x = u(t) \tag{3}$$

with $k, u \in \mathbb{R}$

Note: We say that *u* is the input to our system that is represented by (3)

The solution of (3) (using the integrating factor) is given by:

$$x(t) = e^{-kt}x(t_0) + e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1$$

The term $e^{-kt}x(t_0)$ is called transient response, while $e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1$ comes from the input signal *u*.

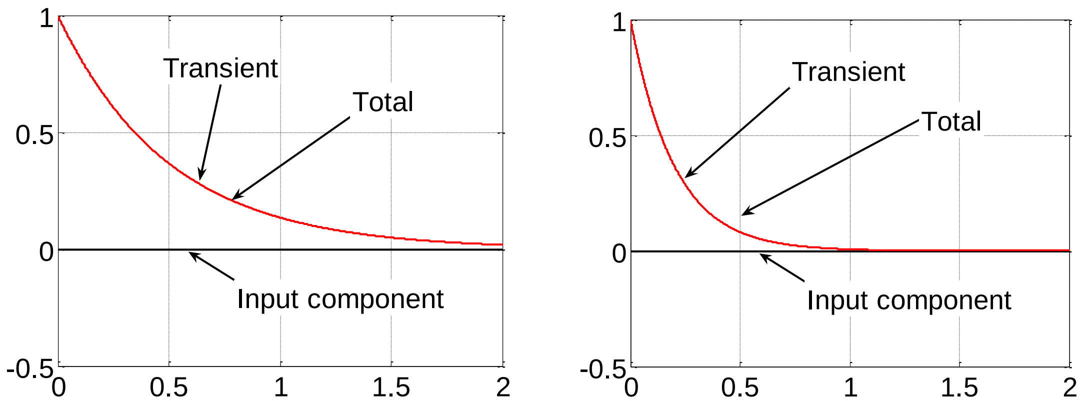
If we assume that *u* is constant:

$$x(t)=e^{-kt}x(t_0)+e^{-kt}\int_{t_0}^{t} e^{kt_1}u\,dt_1 \Leftrightarrow x(t)=e^{-kt}x(t_0)+u\frac{1}{k}\left(1-e^{-k(t-t_0)}\right)$$

Hence: $\lim_{t\to\infty} x(t)=\begin{cases} 0+u\frac{1}{k}(1-0)=u/k, & k>0 \\ \pm\infty, & k<0 \end{cases}$

Thus we say that if *k*>0 the system is stable (and the solution converges exponentially at *u*/*k*) while if *k*<0 the system is unstable (and the solution diverges exponentially to ±∞, ).

**Example 1.3:** *u*=0 and *k*=2 & 5, $x_0$=1

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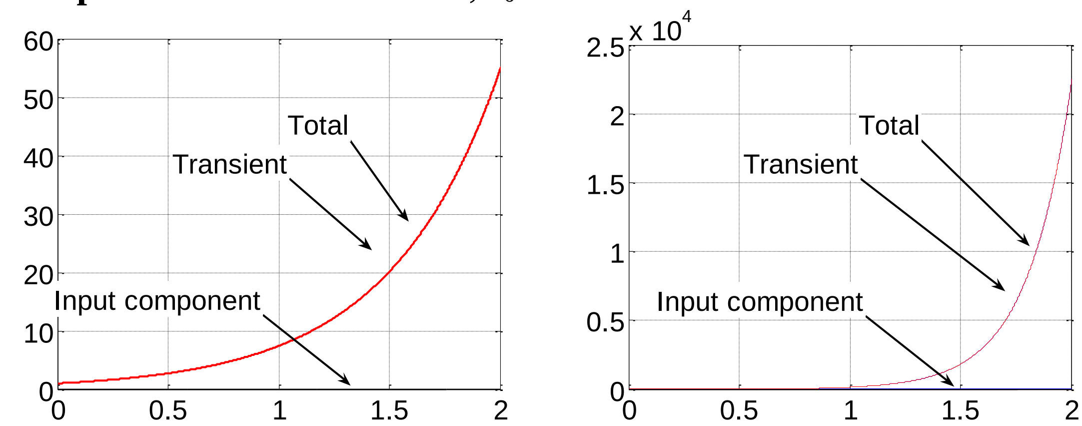
**Example 1.4:** $u$=0 and $k$=-2 & 5, $x_0$=1

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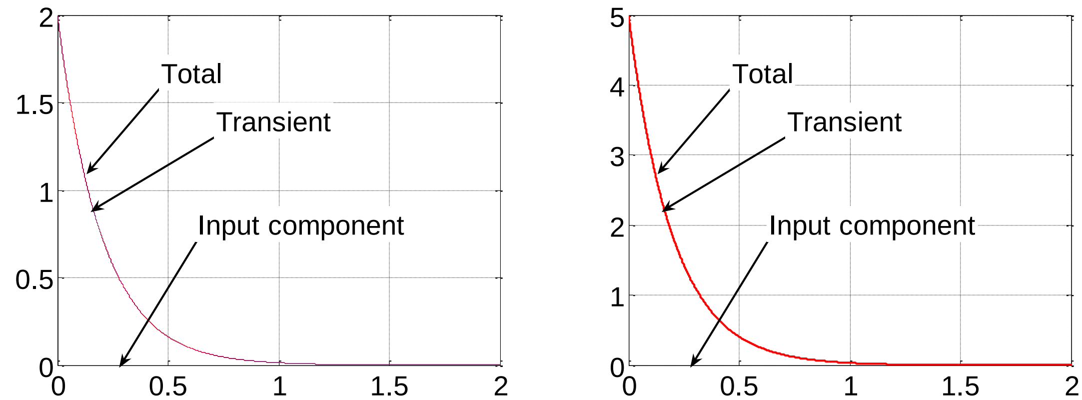
**Example 1.5:** $u$=0 and $k$=5, $x_0$=1 & 5

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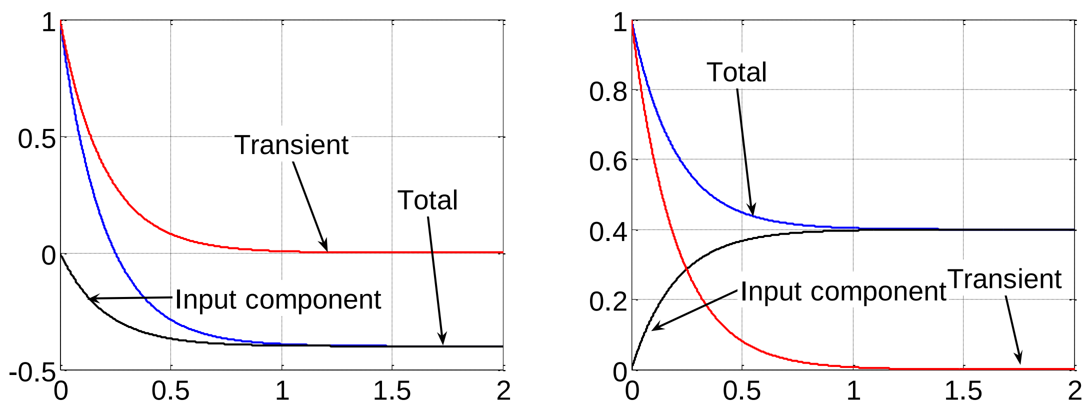
**Example 1.6:** $u$=-2 & 2 and $k$=5, $x_0$=1

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Comments:

- In real systems we cannot have a state (say the speed of a mass-spring system) that becomes infinite, obviously the system will be destroyed when *x* gets to a high value.
- For the dynamics (settling time, stability…) of the system we should only focus on the homogenous ODE: $x' + k(t)x = 0$

## 3. Second Order ODEs

### 3.1 General Material

A second order ODE has as a general form:

$$\frac{d^2x(t)}{dt^2} = f\left(x'(t), x(t), t\right) \tag{4}$$

A linear 2nd order ODE is given by:

$$\begin{cases} x''(t) + A(t)x'(t) + B(t)x(t) = u(t), & \textit{Non autonomous} \\ x''(t) + Ax'(t) + Bx(t) = u(t), & \textit{Autonomous} \end{cases} \tag{5}$$

And again we focus on autonomous homogeneous systems:

$$x''(t) + A(t)x'(t) + B(t)x(t) = 0 \tag{6}$$

Again we define as an analytical solution of (6) an expression that satisfies it.

**Example 1.7:** Given $x''-2x'-3x=0$ prove that $x=e^{3t}$ and $x=e^{-t}$ are two solutions:

$$\left(e^{3t}\right)''-2\left(e^{3t}\right)'-3\left(e^{3t}\right)=0 \Leftrightarrow$$
$$9e^{3t}-6e^{3t}-3e^{3t}=0 \Leftrightarrow$$
$$0=0$$
$$\left(e^{-t}\right)''-2\left(e^{-t}\right)'-3\left(e^{-t}\right)=0 \Leftrightarrow$$
$$e^{-t}+e^{-t}-3e^{-t}=0 \Leftrightarrow$$
$$0=0$$

■

Assume that you have 2 solutions for a 2nd order ODE $x_1$ and $x_2$ (we will see later how to get these two solutions), then:

$$\left.\begin{array}{l} x_1''(t)+A(t)x_1'(t)+B(t)x_1(t)=0 \\ x_2''(t)+A(t)x_2'(t)+B(t)x_2(t)=0 \end{array}\right\}$$

obviously I can multiply these two equations with arbitrary constants:

$$\left.\begin{array}{l} C_1x_1''(t)+C_1A(t)x_1'(t)+C_1B(t)x_1(t)=0 \\ C_2x_2''(t)+C_2A(t)x_2'(t)+C_2B(t)x_2(t)=0 \end{array}\right\}$$

and now I can add them and collect similar terms:

$$\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}''+A(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}'+B(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}=0$$

which means that $C_1x_1(t)+C_2x_2(t)$ (i.e. the linear combination of $x_1$ and $x_2$) is also a solution of the ODE.

**Example 1.8:** Given $x''-2x'-3x=0$ prove that $x=e^{3t}+2e^{-t}$ is a solution:

$$\left(e^{3t}+2e^{-t}\right)''-2\left(e^{3t}+2e^{-t}\right)'-3\left(e^{3t}+2e^{-t}\right)=0 \Leftrightarrow$$
$$9e^{3t}+2e^{-t}-2\left(3e^{3t}-2e^{-t}\right)-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$
$$9e^{3t}+2e^{-t}-6e^{3t}+4e^{-t}-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$
$$9e^{3t}-6e^{3t}-3e^{3t}+2e^{-t}+4e^{-t}-6e^{-t}=0 \Leftrightarrow$$
$$0=0$$

■

Now, the question is, if we have $x_1$ and $x_2$, can ALL other solutions of the ODE, be expressed as a linear combination of $x_1$ and $x_2$? So assume a third solution $\varphi(t)$:

$$\varphi''(t)+A(t)\varphi'(t)+B(t)\varphi(t)=0$$

Now, the question can be written as, can we find constants $C_1$ and $C_2$ such as:

$$\left\{\begin{array}{l}\varphi(t)=C_1x_1(t)+C_2x_2(t)\\ \varphi'(t)=C_1x_1'(t)+C_2x_2'(t)\end{array}\right\}$$

This equation can be seen as a 2by2 system with unknowns $C_1$ and $C_2$ as:

$$\begin{bmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{bmatrix}\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}\varphi(t)\\ \varphi'(t)\end{bmatrix}$$

From linear algebra this system of equations has a unique solution if:

$$\begin{vmatrix}x_1(t) & x_2(t)\\ x_1'(t) & x_2'(t)\end{vmatrix}=x_1(t)x_2'(t)-x_2(t)x_1'(t)\neq 0$$

Note: The matrix $W\left(x_1(t),x_2(t)\right)=\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$ is called the Wronskian[2] of the ODE.

We also know from linear algebra that the determinant is not zero if:

$$\begin{bmatrix} x_1(t) \\ x_1'(t) \end{bmatrix} \neq C \begin{bmatrix} x_2(t) \\ x_2'(t) \end{bmatrix}$$

So if the two solutions $x_1$ and $x_2$ are linear independent (LI) then ANY other solution can be described by the linear combination of $x_1$ and $x_2$. So now we have to look for two LI solutions for the 2nd order ODE.

**Example 1.9:** Prove that two solutions of $x''-2x'-3x=0, x_1=e^{3t}$ and $x_2=e^{-t}$ are linear independent.

$$W\left(x_1(t),x_2(t)\right)=\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}=\begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix}\Rightarrow |W|=\begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix}\Rightarrow$$
$$|W|=e^{3t}\left(-e^{-t}\right)-3e^{3t}e^{-t}=-e^{2t}-3e^{2t}=-4e^{2t}$$ ■

**Example 1.10:** Prove that two solutions of $x''-2x'-3x=0, x_1=e^{3t}$ and $x_2=2e^{3t}$ are NOT linear independent.

[2] From the Polish mathematician Józef Maria Hoëne-Wroński

$$W\left(x_1(t), x_2(t)\right) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{bmatrix} \Rightarrow$$

$$|W| = \begin{vmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t} \end{vmatrix} = 6e^{6t} - 6e^{6t} = 0$$ ■

**Example 1.11:** For the ODE $x'' - 2x' - 3x = 0$ prove that the solution $x = -e^{3t} + 2e^{t}$ cannot be written as any combination of $x_1 = e^{3t}$ and $x_2 = 2e^{3t}$.

$x = C_1 x_1 + C_2 x_2 \Leftrightarrow -e^{3t} + 2e^{t} = C_1 e^{3t} + C_2 e^{3t} = (C_1 + C_2)e^{3t}$

From this expression we have that $C_1 + C_2 = -1$ (and hence we have the term $-e^{3t}$) but there is no term $e^{t}$ for $2e^{t}$. ■

But how can we find two LI solutions? For homogeneous 1st order ODEs with *u*=0 the solution was: $x(t) = e^{-kt}C$ so we will try a similar approach for 2nd order ODEs:

$x'' + Ax' + Bx = 0$, assume[3] $x = e^{rt} \Rightarrow x' = re^{rt}$ & $x'' = r^2 e^{rt} \Rightarrow$

$$x'' + Ax' + Bx = 0 \Leftrightarrow r^2 e^{rt} + Are^{rt} + Be^{rt} = 0 \Leftrightarrow$$

$$r^2 + Ar + B = 0 \tag{7}$$

This is called the Characteristic Equation (CE) and we have to check its roots:

$r = \dfrac{-A \pm \sqrt{A^2 - 4B}}{2}$, these are the Characteristic values or Eigenvalues.

[3] Notice that we do NOT know what is the value of *r*.

### 3.2 Roots are real and unequal

If $A^2 > 4B$ the system is called **Overdamped** and the two roots are $r_1$ and $r_2$ with $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$. Then $x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are two linear independent solutions as:

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} \neq 0$$

hence the general solution is

$$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t} \tag{8}$$

If $r_1$ and $r_2 < 0$ then $x \to 0$ and the system is stable.

If $r_1$ or $r_2 > 0$ then $x \to \pm\infty$ and the system is unstable.

**Example 1.12:** The CE of $x'' + 11x' + 30x = 0$ is $r^2 + 11r + 30 = 0$ which means that the two roots are: $r_{1,2} = \dfrac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2} = \dfrac{-11 \pm 1}{2} \Rightarrow \begin{cases} r_1 = -5 \\ r_2 = -6 \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{-5t} \\ x_2 = e^{r_2 t} = e^{-6t} \end{cases}$

This means that the general solution is $x = C_1 e^{-5t} + C_2 e^{-6t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & e^{-6t} \\ -5e^{-5t} & -6e^{-6t} \end{vmatrix} = -6e^{-5t}e^{-6t} + 5e^{-6t}e^{-5t} = -\mathrm{e}^{-11t} \neq 0$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ -5C_1 - 6C_2 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} C_1 &= 6 \\ C_2 &= -5 \end{aligned}\right\} \Rightarrow x = 6e^{-5t} - 5e^{-6t}$$

■

### 3.4 Roots are Complex (and hence not equal)

If $A^2 < 4B$ then the system is called **Underdamped** and the two roots are $r_1 = a + bj$ and $r_2 = \overline{r}_1 = a - bj$ with $r_1 \neq r_2$, $r_1$, $r_2 \in \mathbb{C}$. Then $x_1 = e^{r_1 t} = e^{(a+bj)t}$ and $x_2 = e^{r_2 t} = e^{(a-bj)t}$ are two linear independent solutions as

$$\begin{vmatrix} e^{(a+bj)t} & e^{(a-bj)t} \\ (a+bj)e^{(a+bj)t} & (a-bj)e^{(a-bj)t} \end{vmatrix} = e^{(a+bj)t}(a-bj)e^{(a-bj)t} - e^{(a-bj)t}(a+bj)e^{(a+bj)t} =$$
$$(a-bj)e^{2at} - (a+bj)e^{2at} = e^{2at}(a-bj-a-bj) = -2e^{2at}bj \neq 0$$

Hence the general solution is

$$x = C_1 x_1 + C_2 x_1 = C_1 e^{rt} + C_2 e^{\overline{r}t} \tag{9}$$

but remember that $C_1$ and $C_2$ are complex now variables such as $x \in \mathbb{R}$.

**Example 1.13:** The CE of $x'' + 2x' + 5x = 0$ is $r^2 + 2r + 5 = 0$ which means that the two roots are: $r_{1,2} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{(-1+2j)t} \\ x_2 = e^{r_2 t} = e^{(-1-2j)t} \end{cases}$

This means that the general solution is $x = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{(-1+2j)t} & e^{(-1-2j)t} \\ (-1+2j)e^{(-1+2j)t} & (-1-2j)e^{(-1-2j)t} \end{vmatrix} =$$
$$(-1-2j)e^{(-1+2j)t}e^{(-1-2j)t} - (-1+2j)e^{(-1+2j)t}e^{(-1-2j)t} =$$
$$(-1-2j)e^{-2t} - (-1+2j)e^{-2t} = (-1-2j+1-2j)e^{-2t} =$$
$$-4je^{-2t} \neq 0$$

If the initial condition is $x(0)=1, x'(0)=0$ then:

$$\left.\begin{array}{l} C_1+C_2=1 \\ (-1+2j)C_1+(-1-2j)C_2=0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1=\frac{1}{2}+\frac{1}{4}j \\ C_2=\frac{1}{2}-\frac{1}{4}j \end{array}\right\} \Rightarrow$$

$$x=\left(\frac{1}{2}+\frac{1}{4}j\right)e^{(-1+2j)t}+\left(\frac{1}{2}-\frac{1}{4}j\right)e^{(-1-2j)t}$$ ■

An alternative approach is not to use $x_1$ & $x_2$ but a linear combination of them:

$$y_1=e^{rt}+e^{\bar{r}t},\ y_2=e^{rt}-e^{\bar{r}t}$$

Note that $\begin{vmatrix} e^{rt}+e^{\bar{r}t} & e^{rt}-e^{\bar{r}t} \\ re^{rt}+\bar{r}e^{\bar{r}t} & re^{rt}-\bar{r}e^{\bar{r}t} \end{vmatrix} \neq 0$

Using Euler's formula: $e^{(a+bj)t}=e^{at}\left(\cos bt+j\sin bt\right)$ and hence:

$$y_1=e^{(a+bj)t}+e^{(a-bj)t}=e^{at}\left(\cos bt+j\sin bt+\cos bt-j\sin bt\right)=2e^{at}\cos bt$$
$$y_2=e^{(a+bj)t}-e^{(a-bj)t}=e^{at}\left(\cos bt+j\sin bt-\cos bt+j\sin bt\right)=j2e^{at}\sin bt$$

As $y_1$ and $y_2$ are solutions so do $y_1\times\frac{1}{2}, y_2\times\frac{1}{2j}$. So the general solution when we have complex roots is:

$$x(t)=e^{at}\left(C_1\cos bt+C_2\sin bt\right), C_1, C_2\in\mathbb{R} \tag{10}$$

**Example 1.14:** The CE of $x''+2x'+5x=0$ is $r^2+2r+5=0$ which means that the two roots are: $r_{1,2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm4j}{2}=-1\pm2j\Rightarrow\begin{cases} r_1=-1+2j \\ r_2=-1-2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t}\cos(2t) \\ x_2 = e^{-t}\sin(2t) \end{cases}$

This means that the general solution is $x = e^{-t}\left(C_1\cos 2t + C_2\sin 2t\right)$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) - 2e^{-t}\sin(2t) & -e^{-t}\sin(2t) + 2e^{-t}\cos(2t) \end{vmatrix} \neq 2e^{-2t}$$

If the initial condition is $x(0)=1, x'(0)=0$ then:

$$\left.\begin{matrix} C_1 = 1 \\ -C_1 + 2C_2 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} C_1 = 1 \\ C_2 = 0.5 \end{matrix}\right\} \Rightarrow$$
$$x = e^{-t}\left(\cos 2t + 0.5\sin 2t\right)$$

■

### 3.3 Roots are real and equal

If $A^2 = 4B$ then the system is called **Critically damped** and the two roots are $r = r_1 = r_2$ with $r \in \mathbb{R}$. One solution is $x_1 = e^{rt}$ but how about $x_2$? We can use $x_2 = te^{rt}$ and the general solution:

$$x = C_1x_1 + C_2x_2 = C_1e^{r_1t} + C_2te^{r_1t} \qquad (11)$$

The Wronskian is:

$$\begin{vmatrix} e^{r_1t} & te^{r_1t} \\ r_1e^{r_1t} & r_1te^{r_1t} + e^{r_1t} \end{vmatrix} = e^{r_1t}\left(r_1te^{r_1t} + e^{r_1t}\right) - r_1e^{r_1t}te^{r_1t} = r_1te^{2r_1t} + e^{2r_1t} - r_1te^{2r_1t} = e^{2r_1t} \neq 0$$

**Example 1.15:** The CE of $x'' + 2x' + x = 0$ is $r^2 + 2r + 1 = 0$ which means that the two roots are: $r_{1,2} = \dfrac{-2 \pm \sqrt{0}}{2} \Rightarrow \begin{cases} r_1 = -1 \\ r_2 = -1 \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t} \\ x_2 = te^{-t} \end{cases}$

This means that the general solution is $x = C_1 e^{-t} + C_2 te^{-t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{2t} \neq 0$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

$$\left.\begin{matrix} C_1 = 1 \\ -C_1 + C_2 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} C_1 = 1 \\ C_2 = 1 \end{matrix}\right\}$$

$$x = e^{-t} + te^{-t}$$ ■

**Not assessed material**

To see why $x_2 = te^{rt}$ is the 2nd solution go to the ODE and place $x = e^{rt}$:

$$\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt}\left(r^2 + Ar + B\right)$$

Since $r_1$ is a double root of the CE: $r^2 + Ar + B = a\left(r - r_1\right)^2$ for some constant $a$. So: $\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt} a\left(r - r_1\right)^2$

Taking the time derivative wrt $r$:

$$\frac{d\left(\left(e^{rt}\right)''\right)}{dr} + A\frac{d\left(\left(e^{rt}\right)'\right)}{dr} + B\frac{d\left(e^{rt}\right)}{dr} = \frac{d\left(e^{rt} a\left(r - r_1\right)^2\right)}{dr}$$

And as we can change the sequence of the differentiation:

$$\left(\frac{d\left(e^{rt}\right)}{dr}\right)'' + A\left(\frac{d\left(e^{rt}\right)}{dr}\right)' + B\frac{d\left(e^{rt}\right)}{dr} = \frac{d\left(e^{rt} a\left(r - r_1\right)^2\right)}{dr}$$

By using simple calculus:

$$\left(e^{rt}t\right)'' + A\left(e^{rt}t\right)' + Be^{rt}t = \frac{d\left(e^{rt}\right)}{dr}a\left(r-r_1\right)^2 + e^{rt}\frac{d\left(a\left(r-r_1\right)^2\right)}{dr} \Leftrightarrow$$

$$\left(e^{rt}t\right)'' + A\left(e^{rt}t\right)' + Be^{rt}t = e^{rt}ta\left(r-r_1\right)^2 + e^{rt}2a\left(r-r_1\right)$$

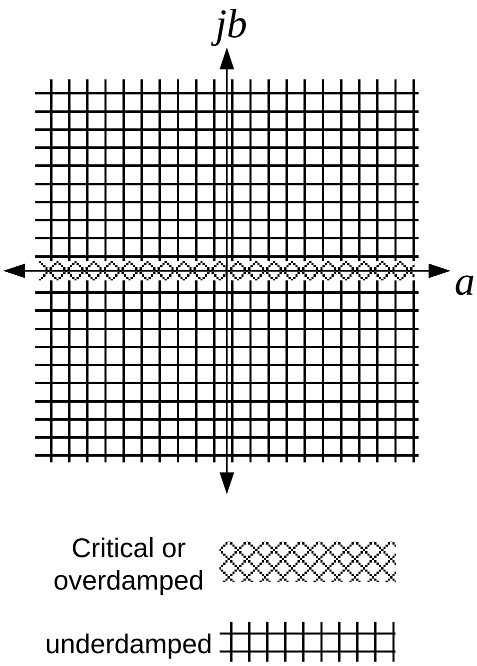
By placing now where $r=r_1$: $\left(e^{rt}t\right)'' + A\left(e^{rt}t\right)' + Be^{rt}t = 0$

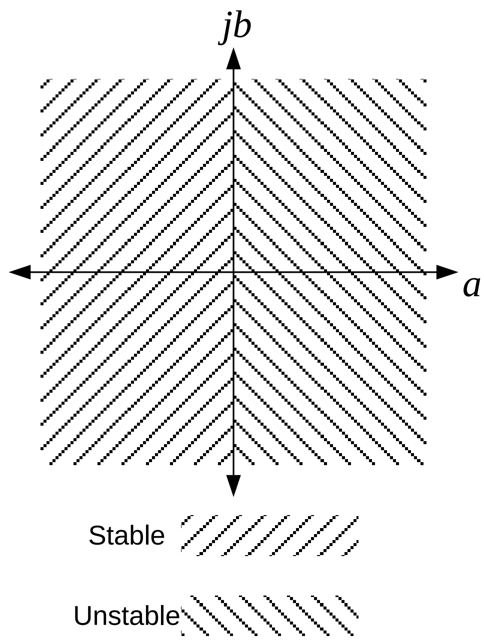
Which means that $e^{rt}t$ must be a solution of my ODE and:

$$\begin{vmatrix} e^{r_1t} & te^{r_1t} \\ r_1e^{r_1t} & tr_1e^{r_1t} + e^{r_1t} \end{vmatrix} = e^{r_1t}\cdot\left(tr_1e^{r_1t} + e^{r_1t}\right) - te^{r_1t}\cdot r_1e^{r_1t} = tr_1e^{2r_1t} + e^{2r_1t} - tr_1e^{2r_1t} = e^{2r_1t} \neq 0$$

And hence $x_2\left(t\right) = e^{rt}t$ is my second solution.

Root Space




| Name | Oscillations? | Components of solution |
|---|---|---|
| Overdamped | No | Two exponentials: $e^{k_1 t}, e^{k_2 t},\ k_1, k_2 < 0$ |
| Critically damped | No | Two exponentials: $e^{kt}, te^{kt},\ k < 0$ |
| Underdamped | Yes | One exponential and one cosine $e^{kt}, \cos(\omega t),\ k < 0$ |
| Undamped | Yes | one cosine $\cos(\omega t)$ |

## 4. Tutorial Exercise I

1. By using the general form of the analytic solution try to predict the response of the following systems. Your answer must describe the system as stable/unstable, convergent to zero/nonzero value. Crosscheck your answer by solving the DE:
    - $5\frac{dx}{dt}+6x=0, \quad x(0)=0,\, x(0)=1,\, x(0)=-1$
    - $5\frac{dx}{dt}-6x=0, \quad x(0)=0,\, x(0)=1,\, x(0)=-1$
    - $5\frac{dx}{dt}+6x=1, \quad x(0)=0,\, x(0)=1,\, x(0)=-1$
    - $5\frac{dx}{dt}+6x=-1, \quad x(0)=0,\, x(0)=1,\, x(0)=-1$
    - $\frac{dx}{dt}-3=0, \quad x(0)=0,\, x(0)=1,\, x(0)=-1$
2. Find the solution of $\ddot{x}+6\dot{x}+5x=0,\, x(0)=2,\, \dot{x}(0)=3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
3. Find the solution of $\ddot{x}+2\dot{x}+6x=0,\, x(0)=1,\, \dot{x}(0)=0$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
4. Find the solution of $\ddot{x}-\dot{x}+0.25x=0,\, x(0)=2,\, \dot{x}(0)=1/3$. Briefly describe how the solution behaves for these initial conditions. Draw a sketch of the response.
5. Find the Wronskian matrices of the solutions of Q2-5.