Chapter #3

EEE3001 & EEE8013

State Space Analysis and Controller Design

1. General Solution of 2nd Order State Space Models

In this chapter we will study the solution of 2nd order linear state space models. This is an important step in order to describe the behaviour of our systems and then to define design targets for our control strategies. Instead of studying a generic and abstract system, in this chapter we will start with a simple 2nd order system and we will try to see how we can solve it.

1.1 Case 1: Real and unequal eigenvalues

Assume a general second order system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Not assessed material (but read it!)

By transforming the system back to a 2nd order DE:

$$\left\{ \begin{array}{l} \dot{x}_1 = -2x_1 + 2x_2 \\ \dot{x}_2 = 2x_1 - 5x_2 \end{array} \right\} \Rightarrow$$

$$x_1 = \frac{1}{2}\left(\dot{x}_2 + 5x_2\right) \Leftrightarrow$$

$$\dot{x}_1 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right)$$

But $\dot{x}_1 = -2x_1 + 2x_2$:

$$-2x_1 + 2x_2 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right)$$

But we have found that $x_1 = \frac{1}{2}\left(\dot{x}_2 + 5x_2\right)$:

$$-2\left(\frac{1}{2}\left(\dot{x}_2 + 5x_2\right)\right) + 2x_2 = \frac{1}{2}\left(\ddot{x}_2 + 5\dot{x}_2\right) \Leftrightarrow$$

$$\frac{1}{2}\ddot{x}_2 + \frac{5}{2}\dot{x}_2 = -\dot{x}_2 - 5x_2 + 2x_2 \Leftrightarrow$$

$$\ddot{x}_2 + 5\dot{x}_2 = -2\dot{x}_2 - 10x_2 + 4x_2 \Leftrightarrow$$

$$\ddot{x}_2 + 7\dot{x}_2 + 6x_2 = 0$$

Similarly for x_1:

$$\left.\begin{cases}\dot{x}_1 = -2x_1 + 2x_2 \\ \dot{x}_2 = 2x_1 - 5x_2\end{cases}\right\} \Rightarrow$$

$$x_2 = \frac{1}{2}(\dot{x}_1 + 2x_1) \Leftrightarrow$$

$$\dot{x}_2 = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - 5x_2 = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - 5\left(\frac{1}{2}(\dot{x}_1 + 2x_1)\right) = \frac{1}{2}(\ddot{x}_1 + 2\dot{x}_1) \Leftrightarrow$$

$$2x_1 - \frac{5}{2}\dot{x}_1 - 5x_1 = \frac{1}{2}\ddot{x}_1 + \dot{x}_1 \Leftrightarrow$$

$$4x_1 - 5\dot{x}_1 - 10x_1 = \ddot{x}_1 + 2\dot{x}_1 \Leftrightarrow$$

$$\ddot{x}_1 + 7\dot{x}_1 + 6x_1 = 0$$

Thus the homogeneous state space model $\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 2 \\ 2 & -5\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$ has been transformed to $\ddot{x}_2 + 7\dot{x}_2 + 6x_2 = 0$ (and $\ddot{x}_1 + 7\dot{x}_1 + 6x_1 = 0$)

Then we have a common CE which is: $r^2 + 7r + 6 = 0$

This will give two solutions: $x_{2a} = C_2 e^{-t}$ and $x_{2b} = D_2 e^{-6t}$

Similarly the ODE for x_1 will give me $x_{1a} = C_1 e^{-t}$ and $x_{1b} = D_1 e^{-6t}$

Thus two LI solutions to our state space model are:

$$x = \begin{bmatrix}C_1 \\ C_2\end{bmatrix} e^{-t} \text{ and } x = \begin{bmatrix}D_1 \\ D_2\end{bmatrix} e^{-6t} \text{ and thus } x = \begin{bmatrix}C_1 \\ C_2\end{bmatrix} e^{-t} + \begin{bmatrix}D_1 \\ D_2\end{bmatrix} e^{-6t}.$$

Obviously the choice of C_1 (or D_1) will influence the choice of C_2 (or D_2) and thus the vectors $\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$ and $\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ are not "completely arbitrary".

Thus we should say that our solution is:

$x = A \cdot e_1 \cdot e^{-t} + B \cdot e_2 \cdot e^{-6t}$, where $e_1 = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, e_2 = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$ and A, B are arbitrary constants that depend on the initial conditions.

This approach is rather cumbersome and of course we cannot (easily) find the values of the vectors e_1, e_2.

But, the important point here is the solution will be a linear combination of two vectors that are multiplied by exponentials and the 2 exponents are the 2 eigenvalues.

Now back to our system: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Let's try $x = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t}$ as a solution (similarly to what we did for 2nd order ODEs).

So $\dot{x} = \lambda \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{\lambda t} \Rightarrow \begin{bmatrix} \lambda a_1 \\ \lambda a_2 \end{bmatrix} = \begin{bmatrix} -2a_1 + 2a_2 \\ 2a_1 - 5a_2 \end{bmatrix}$.

How can we solve that? It is a nonlinear system with 2 equations and 3 unknowns!

Assume λ is a parameter $\Rightarrow$ A homogeneous 2 by 2 linear system:

$$\begin{Bmatrix} (-2-\lambda)a_1 + 2a_2 = 0 \\ 2a_1 + (-5-\lambda)a_2 = 0 \end{Bmatrix}$$

Which always has a trivial solution $a_1=a_2=0$.

For a nontrivial solution[1] (see Cramer's rule from Linear Algebra):

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 7\lambda + 6 = 0$$

This last equation is the characteristic equation of the system and it is the same as the CE that we have from the ODEs (expected as they describe or characterise the same system given in different forms).

$\lambda^2 + 7\lambda + 6 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -6 \end{cases}$. Hence for each of these I have to find a_1, a_2:

For $\lambda_1 = -1$

$\begin{Bmatrix} -a_1 + 2a_2 = 0 \\ 2a_1 - 4a_2 = 0 \end{Bmatrix}$ the same equation twice! (why?)

I assume that $a_2=1$ which means that $a_1=2$.

I.e. for $\lambda_1 = -1$, we have that $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and hence one solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t}$.

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[1] We applied the same idea when we defined the Wronskian

For $\lambda_2 = -6$ the 2 equations are $\begin{Bmatrix} 4a_1 + 2a_2 = 0 \\ 2a_1 + a_2 = 0 \end{Bmatrix}$, I assume that a_1=1 so a_2=-2

So a second solution is $\begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$

Hence the general solution is $x(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$

Example 3.1: Find the response of the previous system when $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

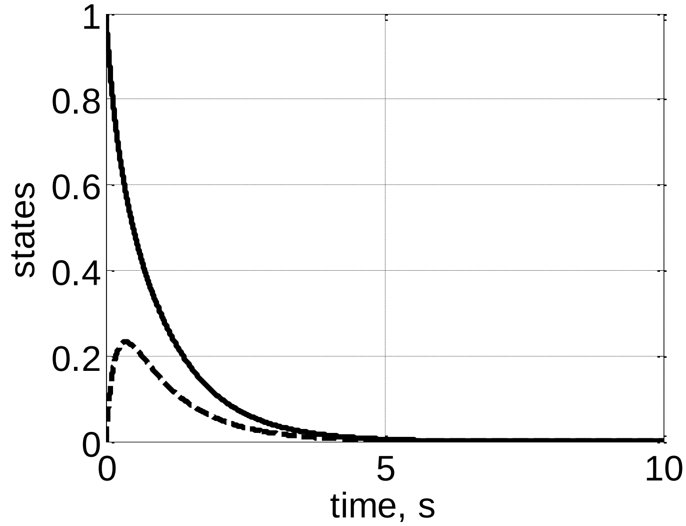
Using $x(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t}$ we have that:

$$x(0) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2C_1 + C_2 \\ C_1 - 2C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

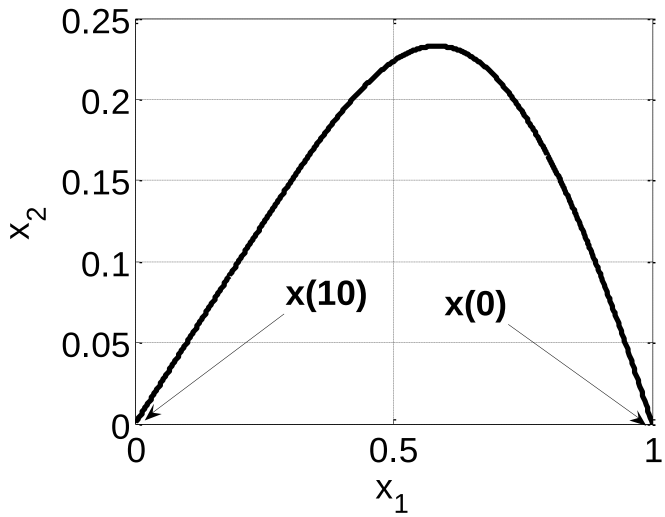
This last system of 2 algebraic equations with 2 unknowns can be solved with various methods like substitution… an easier way is:

$$\begin{bmatrix} 2C_1 + C_2 \\ C_1 - 2C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.4 \\ 0.2 \end{bmatrix}$$

$$x(t) = 0.4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + 0.2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-6t} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0.8e^{-t} + 0.2e^{-6t} \\ 0.4e^{-t} - 0.4e^{-6t} \end{bmatrix}$$



In the state space:



■

Example 3.2:

A system is given by $\ddot{x}+7\dot{x}+6x=0, x\left(0\right)=1, \dot{x}\left(0\right)=0$

i. *Find the particular solution of the DE for the given initial conditions.*

The general solution is $x=C_1e^{-t}+C_2e^{-6t}$.

Using the initial conditions we get C_1=6/5, C_2=-1/5 and the particular solution is: $x = \frac{6}{5}e^{-t} - \frac{1}{5}e^{-6t}$

Note: since r_1 *and* r_2 are negative the response is stable and it converges exponentially to zero (homogeneous system) without oscillations.

ii. *Transform the system to state space form if y=x(t)*

You can solve this part as in chapter 2. By defining $x_1 = x$, $x_2 = \dot{x}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ y = x_1 \Leftrightarrow y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

iii. *Find the eigenvalues. Is the system stable? What will be the response type?*

$$|\lambda I - A| = 0 \Rightarrow \left\| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -7 \end{bmatrix} \right\| = 0 \Rightarrow \left\| \begin{bmatrix} \lambda & -1 \\ 6 & \lambda + 7 \end{bmatrix} \right\| = 0 \Rightarrow \lambda(\lambda + 7) + 6 = 0$$

Hence the characteristic equation is:

$$\lambda(\lambda + 7) + 6 = 0 \Rightarrow \lambda^2 + 7\lambda + 6 = 0 \Rightarrow (\lambda + 1)(\lambda + 6) = 0$$

The eigenvalues of the system are: -1, -6 hence the system is stable with overdamped response.

iv. *Find the eigenvectors*

$$(\lambda I - A)v = 0$$

For $\lambda_1 = -1$

$$\begin{bmatrix} \lambda & -1 \\ 6 & \lambda+7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -1 & -1 \\ 6 & 6 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow -a_1 - a_2 = 0, 6a_1 + 6a_2 = 0 \Rightarrow a_2 = -a_1$$

The eigenvector v_1 can be $[1\ -1]^T$

For $\lambda_2 = -6$

$$\begin{bmatrix} \lambda & -1 \\ 6 & \lambda+7 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow \begin{bmatrix} -6 & -1 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = 0 \Rightarrow -6a_1 - a_2 = 0, 6a_1 + a_2 = 0 \Rightarrow a_2 = -6a_1$$

The eigenvector v_2 can be $[1\ -6]^T$

v. *Find the general solution using the eigenvalues and eigenvectors*

$$x(t) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} + C_2\begin{bmatrix} 1 \\ -6 \end{bmatrix}e^{-6t}$$

vi. *Find the particular solution. (Compare your answer with i.)*

$$x(0) = C_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2\begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} C_1 + C_2 \\ -C_1 - 6C_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

C_1=6/5, C_2=-1/5 and the particular solution is:

$$x(t) = 6/5\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-t} - 1/5\begin{bmatrix} 1 \\ -6 \end{bmatrix}e^{-6t} = C_1 \times v_1 \times e^{-t} + C_2 \times v_2 \times e^{-6t}$$

$$\Rightarrow x(t) = \begin{bmatrix} 6/5 \\ -6/5 \end{bmatrix}e^{-t} + \begin{bmatrix} -1/5 \\ 6/5 \end{bmatrix}e^{-6t} = \begin{bmatrix} 6/5e^{-t} - 1/5e^{-6t} \\ -6/5e^{-t} + 6/5e^{-6t} \end{bmatrix}$$

$$\Rightarrow x_1(t) = 6/5e^{-t} - 1/5e^{-6t}, x_2(t) = -6/5e^{-t} + 6/5e^{-6t}$$ ■

1.2 Case 2: Repeated Eigenvalues

Now it is possible to have 2 sub-cases:

- I can find 2 LI vectors (a rather artificial case)

$$A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\Rightarrow \lambda_{1,2}=2\Rightarrow\begin{cases}e_1=\begin{bmatrix}1 & 0\end{bmatrix}^T\\ e_2=\begin{bmatrix}0 & 1\end{bmatrix}^T\end{cases}$$

  Hence we have 2 uncoupled 1st order ODEs which can be solved separately.

- I **cannot** find 2 LI vectors

$$A=\begin{bmatrix}1 & -1\\ 1 & 3\end{bmatrix}\Rightarrow \lambda_{1,2}=2\Rightarrow e_{1,2}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

In that case I have that $(A-\lambda I)^2 b=0$ (b is called the generalised eigenvector of A), which can be written as $(A-\lambda I)(A-\lambda I)b=0$. Now I substitute $v=(A-\lambda I)b$ and I have $(A-\lambda I)v=0$, i.e. v is one eigenvector of A for the eigenvalue λ. Now it can be proved that the solution is $x(t)=C_1(vt+b)e^{\lambda t}+C_2 ve^{\lambda t}$.

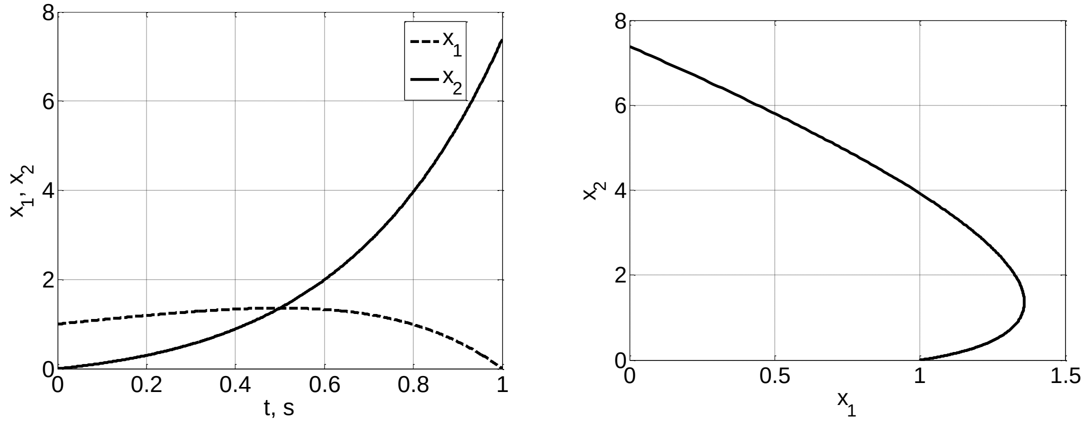
So in that case:

$$\begin{bmatrix}1\\ -1\end{bmatrix}=(A-2I)b\Leftrightarrow\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}b_1\\ b_2\end{bmatrix}\Leftrightarrow\begin{bmatrix}1\\ -1\end{bmatrix}=\begin{bmatrix}-b_1-b_2\\ b_1+b_2\end{bmatrix}\Leftrightarrow\begin{bmatrix}b_1\\ b_2\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}$$

Hence the solution is: $x(t)=C_1\left(\begin{bmatrix}1\\ -1\end{bmatrix}t+\begin{bmatrix}0\\ -1\end{bmatrix}\right)e^{\lambda t}+C_2\begin{bmatrix}1\\ -1\end{bmatrix}e^{\lambda t}$

If we are given the same initial conditions:

$$x(0)=\begin{bmatrix}0\\-C_1\end{bmatrix}+\begin{bmatrix}C_2\\-C_2\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow\begin{cases}C_2=1\\C_1=-1\end{cases}$$



Example 3.3 Find the general solution for the homogeneous system:

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}3&-18\\2&-9\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Solution:

$$\left|\lambda I-\begin{bmatrix}3&-18\\2&-9\end{bmatrix}\right|=\begin{vmatrix}\lambda-3&18\\-2&\lambda+9\end{vmatrix}=(\lambda-3)(\lambda+9)+36=0\Rightarrow$$
$$\lambda^2+6\lambda+9=0\Rightarrow\lambda_1=\lambda_2=-3$$

$$(\lambda I-A)v=\left(-3I-\begin{bmatrix}3&-18\\2&-9\end{bmatrix}\right)v=\left(\begin{bmatrix}-3&0\\0&-3\end{bmatrix}-\begin{bmatrix}3&-18\\2&-9\end{bmatrix}\right)v\Rightarrow$$
$$\begin{bmatrix}-6&18\\-2&6\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}=0\Leftrightarrow v_1-3v_2=0\overset{v_2=1}{\Rightarrow}v_1=3\Rightarrow v=\begin{bmatrix}3\\1\end{bmatrix}$$

$$\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}6 & -18\\2 & -6\end{bmatrix}\begin{bmatrix}b_1\\b_2\end{bmatrix}\Leftrightarrow 1=2b_1-6b_2\overset{b_2=0}{\Leftrightarrow} b_1=1/2\Rightarrow$$

$$x(t)=C_1\left(\begin{bmatrix}3\\1\end{bmatrix}t+\begin{bmatrix}\frac{1}{2}\\0\end{bmatrix}\right)e^{-3t}+C_2\begin{bmatrix}3\\1\end{bmatrix}e^{-3t}$$ ■[2]

1.3 Case 3: Complex Eigenvalues

If I have complex eigenvalues then $\lambda_1=\overline{\lambda_2}$ and the corresponding eigenvectors are $e_1=\overline{e_2}$. In that case the general solution is given by:

$$x(t)=C_1e_1e^{\lambda_1 t}+C_2e_2e^{\lambda_2 t}$$

Example 3.4

$$A=\begin{bmatrix}-1/2 & 1\\-1 & -1/2\end{bmatrix}\Rightarrow \lambda=-0.5+j\Rightarrow e=\begin{bmatrix}1\\j\end{bmatrix}\Rightarrow$$

$$x(t)=C_1e^{(-0.5+j)t}\begin{bmatrix}1\\j\end{bmatrix}+C_2e^{(-0.5-j)t}\begin{bmatrix}1\\-j\end{bmatrix}$$

Assuming $x(0)=\begin{bmatrix}1\\0\end{bmatrix}$ we have that:

$$\begin{bmatrix}1\\0\end{bmatrix}=C_1\begin{bmatrix}1\\j\end{bmatrix}+C_2\begin{bmatrix}1\\-j\end{bmatrix}\Leftrightarrow\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}C_1\\jC_1\end{bmatrix}+\begin{bmatrix}C_2\\-jC_2\end{bmatrix}\Rightarrow$$

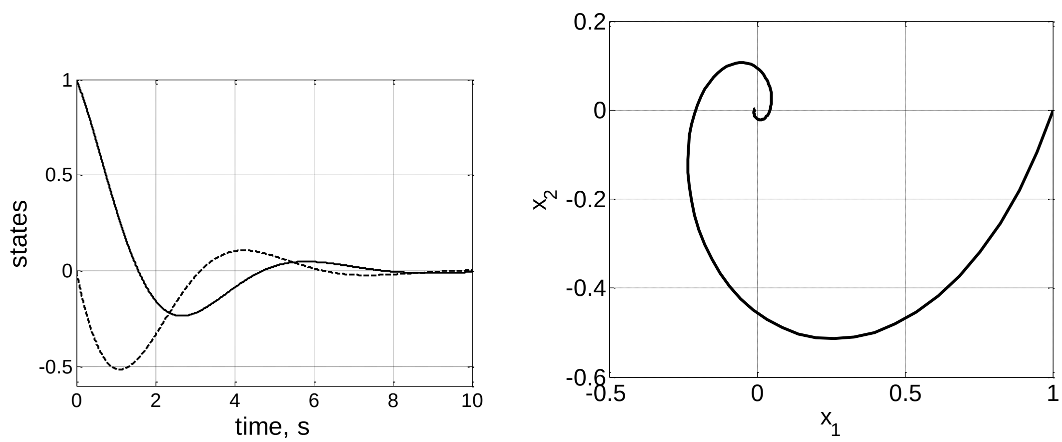
[2] Why was I allowed to choose $b_2=0$?

$$\begin{bmatrix} C_1 + C_2 = 1 \\ C_1 - C_2 = 0 \end{bmatrix} \Rightarrow C_1 = C_2 = 0.5$$

$$x(t) = 0.5e^{(-0.5+j)t}\begin{bmatrix} 1 \\ j \end{bmatrix} + 0.5e^{(-0.5-j)t}\begin{bmatrix} 1 \\ -j \end{bmatrix} =$$

$$0.5e^{-0.5t}\left(e^{jt}\begin{bmatrix} 1 \\ j \end{bmatrix} + e^{-jt}\begin{bmatrix} 1 \\ -j \end{bmatrix}\right) = 0.5e^{-0.5t}\left(\begin{bmatrix} e^{jt} \\ je^{jt} \end{bmatrix} + \begin{bmatrix} e^{-jt} \\ -je^{-jt} \end{bmatrix}\right) =$$

$$0.5e^{-0.5t}\left(\begin{bmatrix} e^{jt} + e^{-jt} \\ je^{jt} - je^{-jt} \end{bmatrix}\right) = 0.5e^{-0.5t}\left(\begin{bmatrix} 2\cos t \\ -2\sin t \end{bmatrix}\right) = e^{-0.5t}\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$



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2. Solution Matrices and Solution of Linear Systems

2.1 2nd order systems

For second order systems $\dot{x} = Ax$ we have seen that we have 2 solutions (x_1, x_2) depending on the eigenvalues of A (real and distinct, repeated and complex):

$$x_1 = e_1 e^{\lambda_1 t},\ x_2 = e_2 e^{\lambda_2 t} \text{ if } \lambda_1 \neq \lambda_2, \lambda_1, \lambda_2 \in \mathbb{R}.$$

$$x_1 = e e^{\lambda t},\ x_2 = \left(et + b\right) e^{\lambda t} \text{ if } \lambda_1 = \lambda_2 = \lambda, \lambda \in \mathbb{R}.$$

$$x_1 = e e^{\lambda t},\ x_2 = \bar{e} e^{\bar{\lambda} t} \text{ if } \lambda_1 = \overline{\lambda_2} = \lambda, \lambda \in \mathbb{C}.$$

Or $x_1 = \text{Re}\left(e e^{\lambda t}\right), x_2 = \text{Im}\left(e e^{\lambda t}\right)$ if $\lambda_1 = \overline{\lambda_2} = \lambda, \lambda \in \mathbb{C}$.

Now any combination $x = c_1 x_1 + c_2 x_2$ is also a solution (principle of superposition) and also any other solution can be expressed by the above linear combination. This effectively means that to describe the behaviour of a 2nd order system we just need x_1 and x_2. When we are given an initial condition x_0 effectively we are asked to find a specific solution that passes (starts) through x_0, and this can be done by finding the appropriate values of c_1, c_2 (this is what we have done before).

Now, x_1 and x_2 are 2 2by1 column vectors. If we put them together in one matrix (this matrix is called "**<u>Fundamental Solution Matrix (FSM)</u>**") we have $X = \left[x_1 \quad x_2\right]$ which is 2by2. It will be better if we write as: $X(t) = \left[x_1(t) \quad x_2(t)\right]$.

Thus $x = c_1 x_1 + c_2 x_2$ can be written as: $x(t) = X(t) \times c$, where $c = \begin{bmatrix} c_1 & c_2 \end{bmatrix}^T$

We are given the value at t=0 as x_0: $x_0 = X(0) \times c$ or $c = X^{-1}(0) x_0$. Hence going back to $x(t) = X(t) \times c$ we have: $\boxed{x(t) = X(t) \times X^{-1}(0) x_0}$

The product $X(t) \times X^{-1}(0)$ is called the State Transition Matrix (STM)[3].

This means that if we know $X(t)$ we can easily find $X(0)$ and $X^{-1}(0)$. Then using $x(t) = X(t) \times X^{-1}(0) x_0$ we can find any other solution. This is effectively what we previously did but now it is in a more compact form, it can easily be extended to high order systems and above all it can be used in time varying systems (i.e. where the state matrix *A* is not constant). Before we see how it can be used for time varying systems let's see how it can be used for the systems that we previously studied:

Example 3.5:

$$x_1 = e_1 e^{\lambda_1 t}, x_2 = e_2 e^{\lambda_2 t} \Rightarrow X(t) = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

Hence $x = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} \times \begin{bmatrix} e_1 & e_2 \end{bmatrix}^{-1} x_0$ ■

Example 3.6:

$$x_1 = e e^{\lambda t}, x_2 = (et + b) e^{\lambda t} \Rightarrow X(t) = \begin{bmatrix} e e^{\lambda t} & (et + b) e^{\lambda t} \end{bmatrix} \Rightarrow X(0) = \begin{bmatrix} e & b \end{bmatrix}$$

[3] You must be clear about the difference between the FSM and the STM

Hence $x = \begin{bmatrix} ee^{\lambda t} & (et+b)e^{\lambda t} \end{bmatrix} \times \begin{bmatrix} e & b \end{bmatrix}^{-1} x_0$ ■

Example 3.7:

$$x_2 = \mathrm{Re}\left(ee^{\lambda t}\right), x_2 = \mathrm{Im}\left(ee^{\lambda t}\right) \Rightarrow X(t) = \begin{bmatrix} \mathrm{Re}\left(ee^{\lambda t}\right) & \mathrm{Im}\left(ee^{\lambda t}\right) \end{bmatrix} \Rightarrow$$
$$X(0) = \begin{bmatrix} \mathrm{Re}(e) & \mathrm{Im}(e) \end{bmatrix}$$

Hence $x = \begin{bmatrix} \mathrm{Re}\left(ee^{\lambda t}\right) & \mathrm{Im}\left(ee^{\lambda t}\right) \end{bmatrix} \times \begin{bmatrix} \mathrm{Re}(e) & \mathrm{Im}(e) \end{bmatrix}^{-1} x_0$ ■

Example 3.8:

We know that for $A = \begin{bmatrix} -8 & -4 \\ 1.5 & -3 \end{bmatrix}$ we have $\begin{cases} \lambda_1 = -6, e_1 = \begin{bmatrix} -2 & 1 \end{bmatrix}^T \\ \lambda_2 = -5, e_2 = \begin{bmatrix} 1 & -3/4 \end{bmatrix}^T \end{cases}$. Hence the FSM is: $X(t) = \begin{bmatrix} e_1 e^{\lambda_1 t} & e_2 e^{\lambda_2 t} \end{bmatrix} = \begin{bmatrix} -2e^{-6t} & e^{-5t} \\ e^{-6t} & -\frac{3}{4}e^{-5t} \end{bmatrix}$

$$\Rightarrow X(0) = \begin{bmatrix} -2 & 1 \\ 1 & -\frac{3}{4} \end{bmatrix} \Rightarrow X^{-1}(0) = \begin{bmatrix} -1.5 & -2 \\ -2 & -4 \end{bmatrix}$$

Thus if $x_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$:

$$x(t) = X(t) X^{-1}(0) x_0 = \begin{bmatrix} -2e^{-6t} & e^{-5t} \\ e^{-6t} & -\frac{3}{4}e^{-5t} \end{bmatrix} \begin{bmatrix} -1.5 & -2 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -5.5 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-6t} - 10 \begin{bmatrix} 1 \\ -3/4 \end{bmatrix} e^{-5t}$$

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Not assessed material

Unfortunately we cannot follow a similar strategy when A is time varying, for example $A(t)=\begin{bmatrix} -t & 1 \\ -e^{-t} & -e^{-2t} \end{bmatrix}$. In these cases we have to rely on numerical solutions.

Even though we cannot find x_1 and x_2 we know that they exist. Hence we know that the FSM exists as well: $X(t)=\begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}$

and of course at t=0 we have a constant matrix $X(0)=\begin{bmatrix} x_{1_A}(0) & x_{2_A}(0) \\ x_{1_B}(0) & x_{2_B}(0) \end{bmatrix}$ with the inverse $X^{-1}(0)=\begin{bmatrix} x_{1_A}(0) & x_{2_A}(0) \\ x_{1_B}(0) & x_{2_B}(0) \end{bmatrix}^{-1}$ also being constant. Let's assume that for our case $X^{-1}(0)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for some constants a, b, c, d. Then:

$$X(t)\times X^{-1}(0)=\begin{bmatrix} x_{1_A}(t) & x_{2_A}(t) \\ x_{1_B}(t) & x_{2_B}(t) \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} ax_{1_A}(t)+cx_{2_A}(t) & bx_{1_A}(t)+dx_{2_A}(t) \\ ax_{1_B}(t)+cx_{2_B}(t) & bx_{1_B}(t)+dx_{2_B}(t) \end{bmatrix}$$

Or: $\begin{bmatrix} ax_1(t)+cx_2(t) & bx_1(t)+dx_2(t) \end{bmatrix}$

Now, since x_1 and x_2 are solutions of $\dot{x}=Ax$ then so must be $x_3=ax_1(t)+cx_2(t)$ and $x_4=bx_1(t)+dx_2(t)$. This means that $\dot{x}_3=Ax_3$ and $\dot{x}_4=Ax_4$.

Also $X(0)\times X^{-1}(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and hence $\begin{bmatrix}x_3(0) & x_4(0)\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ or $x_3(0)=\begin{bmatrix}1\\ 0\end{bmatrix}$ and $x_4(0)=\begin{bmatrix}0\\ 1\end{bmatrix}$. **<u>Be careful</u>** we do not yet know the functions $x_3(t)$ and $x_4(t)$ since we do not know $x_1(t)$ and $x_2(t)$.

In order for us to find $\mathbf{x}_3(t)$ and $\mathbf{x}_4(t)$ we simply have to numerically solve $\dot{x}_3=Ax_3$ and $\dot{x}_4=Ax_4$ for $x_3(0)=\begin{bmatrix}1\\ 0\end{bmatrix}$ and $x_4(0)=\begin{bmatrix}0\\ 1\end{bmatrix}$.

Now, in general our solutions $x_1(t)$ and $x_2(t)$ also depend on t_0 which may not be zero as in the previous case, hence we should have written $x_1(t,t_0)$ and $x_2(t,t_0)$. To avoid confusion and to comply with various other authors we will use $x_1(t)$ and $x_2(t)$ for most cases and $\varphi_1(t,t_0)$ and $\varphi_2(t,t_0)$ when we want to say that our solutions also depend on the initial time. Hence the FSM is $\Phi(t,t_0)$ and not $X(t)$. Also since $x(t)=\Phi(t,t_0)\times\Phi^{-1}(t_0,t_0)x_0$, i.e. our solution to the IVP also depend on the initial condition: $\varphi(t,t_0,x_0)=\Phi(t,t_0)\times\Phi^{-1}(t_0,t_0)x_0$.

2.3 State transition matrix for LTI systems

For a scalar ODE: $\dot{x} = ax$ the solution was $x(t) = e^{at}x(0)$ (no special cases) so can we do the same with $\dot{x} = Ax$, i.e. $x(t) = e^{At}x(0)$?

If only we knew how to calculate e^{At} => No special cases are needed then.

It can be proved [4]that $e^{At} = \mathrm{I} + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$

How[5] $x(t) = e^{At}x(0)$ is associated with $x(t) = C_1 e_1 e^{\lambda_1 t} + C_2 e_2 e^{\lambda_2 t}$?

Remember: $Ae_i = e_i \lambda_i$ or in a matrix notation:

$$\underset{2\times2}{A}\underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{2\times2} = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{2\times2}\underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{2\times2}$$

$$\underset{2\times2}{A}\;\underset{2\times2}{T} = \underset{2\times2}{T}\;\underset{2\times2}{\Lambda} \quad \text{or} \quad A = T\Lambda \mathrm{T}^{-1}$$

So:

$$e^{At} = \mathrm{I} + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots = \mathrm{I} + \left(T\Lambda\mathrm{T}^{-1}\right)t + \frac{1}{2!}\left(\left(T\Lambda\mathrm{T}^{-1}\right)t\right)^2 + \frac{1}{3!}\left(\left(T\Lambda\mathrm{T}^{-1}\right)t\right)^3 + \ldots$$

But $\left(T\Lambda\mathrm{T}^{-1}\right)^2 = \left(T\Lambda\mathrm{T}^{-1}\right)\left(T\Lambda\mathrm{T}^{-1}\right) = T\Lambda^2\mathrm{T}^{-1}$

so:

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[4]Taylor Series

[5] Similar analysis can be done also for the other 2 cases, but it is rather harder and outside the scope of this module.

$$e^{At} = TIT^{-1} + \left(T\Lambda T^{-1}\right)t + \frac{1}{2!}\left(T\Lambda^2 T^{-1}\right)t^2 + \frac{1}{3!}\left(T\Lambda^3 T^{-1}\right)t^3 + \ldots =$$
$$= T\left(I + \Lambda t + \frac{1}{2!}\left(\Lambda t\right)^2 + \frac{1}{3!}\left(\Lambda t\right)^3 + \ldots \right)T^{-1} = Te^{\Lambda t}T^{-1}$$

And $e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$

Hence: $\underset{2\times1}{x(t)} = \underset{2\times2}{e^{At}}\,\underset{2\times1}{x(0)} = \underset{2\times2}{T}\;\underset{2\times2}{e^{\Lambda t}}\;\underset{2\times2}{T^{-1}}\;\underset{2\times1}{x(0)} =$

$$\underset{2\times1}{x(t)} = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}^{-1}}_{2\times2} \underset{2\times1}{x(0)}$$

$$\underset{2\times1}{x(t)} = \underbrace{\begin{bmatrix} e_1 & e_2 \end{bmatrix}}_{2\times2} \underbrace{\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}}_{2\times2} \underset{2\times2}{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}} \underset{2\times1}{x(0)}$$

Note: the vectors *e* are 2x1 vectors, and the vectors *w* are 1x2 vectors.

$$x(t) = e_1 e^{\lambda_1 t} w_1 x(0) + e_2 e^{\lambda_2 t} w_2 x(0)$$

This is similar to $x(t) = C_1 e_1 e^{\lambda_1 t} + C_2 e_2 e^{\lambda_2 t}$.

Tutorial Exercise III

1. A system is given by $\dot{x} = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} x$

   (a) Find the eigenvalues and eigenvectors of this system.

   (b) Find the general solution using the previously found eigenvectors.

   (c) Find the particular solution if $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. A system is given by $\ddot{x} + 3\dot{x} + 2x = 0, x(0) = 1, \dot{x}(0) = -1$

   (a) Find the particular solution of the differential equation.

   (b) Draw a sketch of the response x(t).

   (c) Transform the system to state space form if y = -2x(t).

   (d) Find the eigenvalues and eigenvectors of the system. What is the response type?

   (f) Find the general solution using the eigenvectors then find the particular solution using the given initial conditions.

3. Find the state transition matrix of the homogeneous state space system that you find in 2 (c) then find the particular solution for $x(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

4. A system is given by $\dot{x} = \begin{bmatrix} 1 & 3 \\ 2 & -4 \end{bmatrix} x$

(a) Find the state transition matrix.

(b) Find the particular solution for the homogeneous system using the state transition matrix approach for $x(0)=\begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

(c) Find the system state response for a unit step input if $x(0)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $B=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Comment on the system stability.

5. A state space system is given by $\dot{x}=\begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix}x$

(a) Find the state transition matrix.

(b) Find the particular solution for the homogeneous system using the state transition matrix approach for $x(0)=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(c) Crosscheck your answer in (b) by finding the particular solution of the system using eigenvalues and eigenvectors approach.