Final Revision

EXAMS !!!!

- **Past exam papers**

- 2017/18 3q comp. 120 mins

  40 min/q.

  Questions like prove $G(s) = C\cdot(sI - A)^{-1}B$

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$|A - \lambda I| = 0 \quad \begin{vmatrix} 5-\lambda & 4 \\ 3 & 1-\lambda \end{vmatrix} = 0 \Rightarrow$$

$$\lambda^2 - 6\lambda - 7 = 0 \Rightarrow \begin{matrix} \lambda_1 = 7 \\ \lambda_2 = -1 \end{matrix}$$

<u>unstable</u>

poles $= -1$, -2.

CTRB

$$M_c = [\,B \quad AB\,]$$

$$= \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 5+4 \\ 1 & 3+1 \end{bmatrix} = \begin{bmatrix} 1 & 9 \\ 1 & 4 \end{bmatrix}$$

$$|M_c| = 1 \cdot 4 - 1 \cdot 9 = -5 \neq 0$$

if $\quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$M_c = \left[\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right]$$

$$= \begin{bmatrix} 1 & 1 & 9 & 5 \\ 1 & 0 & 4 & 3 \end{bmatrix}$$

$$A_{CL} = A - B \cdot K.$$

$$= \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

$$= \begin{bmatrix} 5-k_1 & 4-k_2 \\ 3-k_1 & 1-k_2 \end{bmatrix}.$$

$$|A_{CL} - \lambda I| = 0$$

$$\begin{vmatrix} 5-k_1-\lambda & 4-k_2 \\ 3-k_1 & 1-k_2-\lambda \end{vmatrix} = 0$$

- $\lambda_1 = -1 \Rightarrow \begin{vmatrix} 5-k_1+1 & 4-k_2 \\ 3-k_1 & 1-k_2+1 \end{vmatrix} = 0 \Rightarrow \lambda ..$

- $\lambda_2 = -2 \Rightarrow \ldots$

$$k_1 = 5.4$$

$$k_2 = 3.6$$

~~Des e-cef 1 & estimator~~

Estimator

$$M_o = \begin{bmatrix} C \\ CA \end{bmatrix}$$

$$CA = \begin{bmatrix} 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 11 & 6 \end{bmatrix}$$

$$M_o = \begin{bmatrix} 1 & 2 \\ 11 & 6 \end{bmatrix} \Rightarrow \text{rank} = 2 \Rightarrow \underline{\text{obsv}}$$

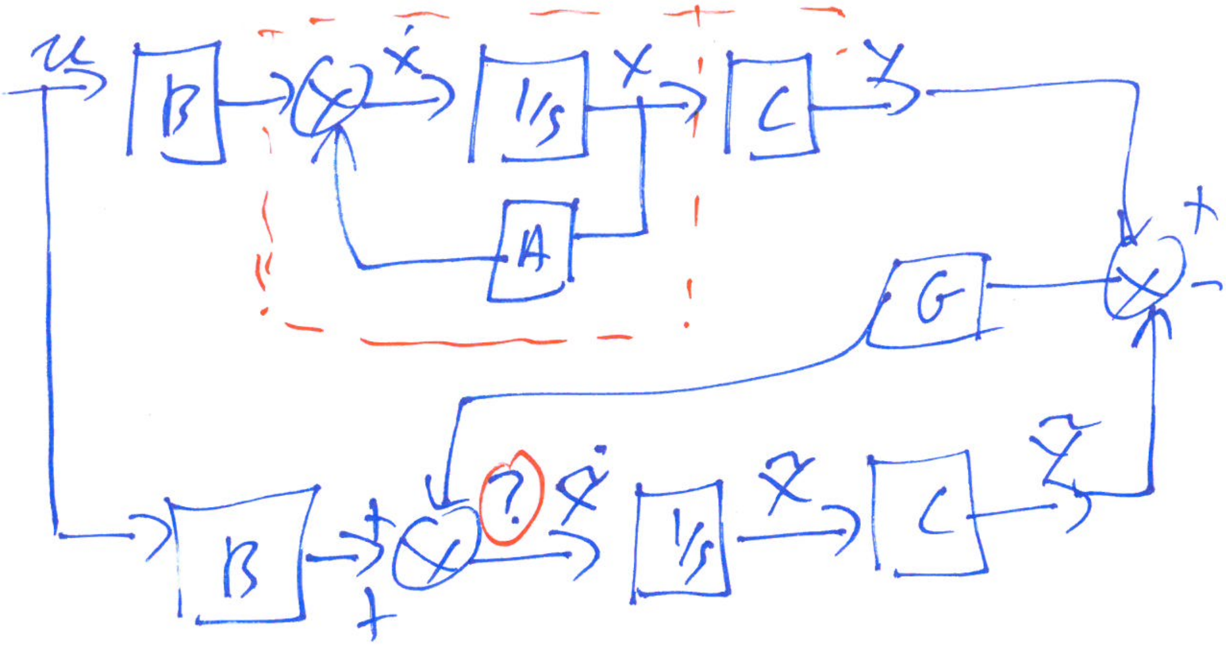
$$A - GC = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \end{bmatrix} = \dots$$

. . . . . .

Des poles at $[-10 \quad -20] \Rightarrow \dots$

$\dots \quad G_1 = 39.37$

$G_2 = -1.6875$



Tracking

- Prove $K_1 = N_u + K \cdot N_x$

- Using ↗ $K_1 = ?$

$$\begin{bmatrix} N_u \\ N_x \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 1 \\ 3 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

LQR

-) $A \quad B \quad C \quad Q \quad R \Rightarrow P$

$\uparrow$ Given

$K = \cancel{B}^{-1} \cdot R^{-1} \cdot B^T \cdot P \Rightarrow \ldots$

- A, B, C …

$Q_A = I_2 \quad R_A = 10 \cdot \Rightarrow K_A = \ldots$

$Q_B = 10 \cdot I_2 \quad R_B = 1 \Rightarrow K_B = \ldots$

- A, B, C ….
- PP $[-1 \quad -2] \Rightarrow \ldots K_A$
- PP $[-10 \quad -20] \Rightarrow \ldots K_B$

- $A\,B\,C$ ….

$Q, R, P \Rightarrow K_A = \ldots$

$P.P. = [-1 \quad -2] \Rightarrow K_B = \ldots$

$$A = \begin{bmatrix} -12 & 5 \\ -25 & 10 \end{bmatrix} \Rightarrow \lambda = -1 \pm 2j$$

$$(A - \lambda I)e = 0$$

$$\begin{bmatrix} -12-\lambda & 5 \\ -25 & 10-\lambda \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12-(-1+2j) & 5 \\ -25 & 10-(-1+2j) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} (-11-2j)e_1 + 5e_2 = 0 \\ -25e_1 + (11-2i)e_2 = 0 \end{cases}$$

$$\downarrow$$

$$-25e_1 + (11-2i)e_2 = 0$$

$$\frac{-25}{11-2i} e_1 + e_2 = 0$$

$$\frac{(11+2i)}{(11+2i)} \cdot \frac{(-25)}{11-2i} e_1 + e_2 = 0$$

$$\frac{-25(11+2i)}{11^2+2^2} e_1 + e_2 = 0$$

$$-\frac{25(11+2i)}{121+4} e_1 + e_2 = 0$$

$$\frac{-25(11+2i)}{125} e_1 + e_2 = 0$$

$$\frac{-(11+2i)}{5} e_1 + e_2 = 0$$

$$(-11-2i) e_1 + 5e_2 = 0$$

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix}, \quad \dot{x} = Ax$$

$$\bullet\ \lambda_1 = -1 \Rightarrow e_1 = \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} \qquad \lambda = 7 \quad e_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x = c_1 e^{\lambda_1 t} e_1 + c_2 e^{\lambda_2 t} e_2$$

$$= c_1 \underbrace{e^{-t} \begin{bmatrix} -1 \\ 1.5 \end{bmatrix}}_{x_1} + c_2 \underbrace{e^{7t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{x_2}$$

$$x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = -1/4$$

$$c_2 = 1.5/4.$$

$$x(t) = -\frac{1}{4} \begin{bmatrix} -1 \\ 1.5 \end{bmatrix} e^{-t} + \frac{1.5}{4} \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t}$$

$$= -\frac{1}{4} \left(\begin{bmatrix} -e^{-t} - 3e^{7t} \\ 1.5e^{-t} - 1.5e^{7t} \end{bmatrix} \right).$$

$$\dot{x} = Ax \quad , \quad A = \begin{bmatrix} 5 & 4 \\ 3 & 1 \end{bmatrix} \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- STM. = ?
- x = ?

$$X = \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} -e^{-t} & 2e^{7t} \\ 1.5e^{-t} & e^{7t} \end{bmatrix}$$

$$\Phi = X(t) \cdot X^{-1}(0) = STM$$

$$X^{-1}(0) = \begin{bmatrix} -1 & 2 \\ 1.5 & 1 \end{bmatrix}$$

$$|X(0)| = -1 - 3 = -4.$$

$$X^{-1}(0) = \frac{1}{-4} \cdot \begin{bmatrix} 1 & -2 \\ -1.5 & -1 \end{bmatrix}$$

$$STM = \frac{1}{-4} \begin{bmatrix} -e^{-t} & 2e^{7t} \\ 1.5e^{-t} & e^{7t} \end{bmatrix} \cdot \begin{bmatrix} 1 & -2 \\ -1.5 & -1 \end{bmatrix}$$

$$x(t) = STM \cdot x_0 = S.T.M. \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -1.5 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

$$x = -\frac{1}{4} \begin{bmatrix} -e^{-t} & 2e^{7t} \\ 1.5e^{-t} & e^{7t} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1.5 \end{bmatrix}$$

$$= -\frac{1}{4} \cdot \begin{bmatrix} -e^{-t} - 3e^{7t} \\ 1.5e^{-t} - 1.5e^{7t} \end{bmatrix}$$

- $e^{At} \cdot x_0 = x(t)$

$$e^{At} = I + At + \frac{(At)^2}{2} + \frac{(At)^3}{3!} + \dots$$

Since $\lambda_1 \neq \lambda_2$, and $\lambda_1, \lambda_2 \in \mathbb{R}$.

$$e^{At} = T \cdot e^{\Lambda t} T^{-1}, \quad T = [e_1 \quad e_2]$$

$$e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}$$

$$T = \begin{bmatrix} -1 & 2 \\ 1.5 & 1 \end{bmatrix}, \quad T^{-1} = -\frac{1}{4} \begin{bmatrix} 1 & -2 \\ -1.5 & -1 \end{bmatrix}$$

$$e^{At}\cdot x_0 = \underbrace{\begin{bmatrix} -1 & 2 \\ 1.5 & 1 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{7t} \end{bmatrix}}_{e^{nt}} * \underbrace{\left(-\frac{1}{4}\right) \begin{bmatrix} 1 & -2 \\ -1.5 & -1 \end{bmatrix}}_{T^{-1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \cdots = -\frac{1}{4} \begin{bmatrix} -e^{-t} - 3e^{7t} \\ 1.5e^{-t} - 1.5e^{7t} \end{bmatrix}$$

Extra material on S.T.M. with complex eigs.

$$A=\begin{bmatrix} 0 & 1 \\ -5 & -4 \end{bmatrix} \Rightarrow \lambda = -2 \pm i \Rightarrow e = \begin{bmatrix} -0.4 \mp 0.2i \\ 1 \end{bmatrix}$$

$$x_1 = e^{(-2+i)t} \cdot \begin{bmatrix} -0.4-0.2i \\ 1 \end{bmatrix}$$

$$x_2 = e^{(-2-i)t} \begin{bmatrix} -0.4+0.2i \\ 1 \end{bmatrix}$$

$$X = [x_1 \quad x_2] \quad , \quad X(0) = \begin{bmatrix} -0.4-0.2i & -0.4+0.2i \\ 1 & 1 \end{bmatrix}$$

$$|X(0)| = -0.4i$$

$$X^{-1}(0) = \frac{1}{-0.4i} \cdot \begin{bmatrix} 1 & 0.4-0.2i \\ -1 & -0.4-0.2i \end{bmatrix}$$

$$= \begin{bmatrix} 2.5i & 0.5+i \\ -2.5i & 0.5-i \end{bmatrix}$$

$$X^{-1}(0)\cdot X_0 = \begin{bmatrix} 2.5i & 0.5+i \\ -2.5i & 0.5-i \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5+i \\ 0.5-i \end{bmatrix}.$$

$$X = \begin{bmatrix} e^{\lambda_1 t}(-0.4-0.2i) & e^{\lambda_2 t}\cdot(-0.4+0.2i) \\ e^{\lambda_1 t} & e^{\lambda_2 t} \end{bmatrix} \cdot \begin{bmatrix} 0.5+i \\ 0.5-i \end{bmatrix}$$

$$X = \begin{bmatrix} e^{\lambda_1 t}\cdot(-0.4-0.2i)\cdot(0.5+i) + e^{\lambda_2 t}\cdot(-0.4+0.2i)\cdot(0.5-i) \\ e^{\lambda_1 t}\cdot(0.5+i) + e^{\lambda_2 t}\cdot(0.5-i) \end{bmatrix}.$$