EEE 8079

$$\dot{x}(t) = x'(t) = \frac{dx(t)}{dt} = f(x(t), t)$$

A soln $x(t)$ is a function

$$x_1 = e^{-3t}$$

$$\dot{x} = -3 \cdot x$$

$$-3 \cdot x_1 = -3 \cdot e^{-3t} \rightarrow \text{RHS}$$

$$\frac{d(e^{-3t})}{dt} = -3\, e^{-3t} \rightarrow \text{LHS}$$

Try $x_2 = 10 \cdot e^{-3t}$

$$\dot{x}_2 = -30\, e^{-3t}$$

$$-3 \cdot x_2 = -3 \cdot 10\, e^{-3t} = -30\, e^{-3t}$$

$x(t) = C \cdot e^{-3t}$ is a gen. soln $\rightarrow$ ODE of $\dot{x} = -3 \cdot x$

$t = 0$

$$x(0) = x_0 = C \cdot e^{-3 \cdot 0} = C \cdot e^{0} = C \cdot 1 = C$$

$$x(t) = x_0 e^{-3t}$$

$\rightarrow$ Inf solns

ODE + I.C. = I.V.P.

Initial Value Problem

$\dot{x} = -3\cdot x$ $\qquad x_0 = 1$

$x_1 = e^{-3t}$ $\qquad x_2 = 10e^{-3t}$

$x_1(0) = e^0 = 1$

$x_2(0) = 10\cdot e^0 = 10 \neq 1 = x_0$

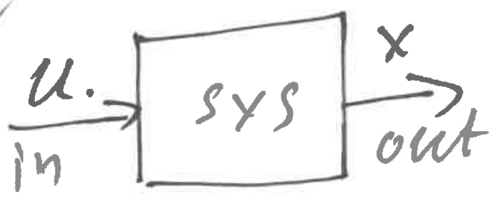
Linear F.O.

$a(t)\cdot\dot{x} + b(t)\cdot x = c(t)$

$a\cdot\dot{x} + b\cdot x = c \quad , a \neq 0$

$\downarrow$

$\dot{x} + kx = u \quad , \quad k = b/a, \; u = c/a$



Int. Factor

$$x(t) = \underbrace{e^{-kt}\cdot x_0}_{\text{I.C.}} + \underbrace{e^{-kt}\cdot \int_0^t e^{kt_1}\cdot u(t_1)\cdot dt}_{\text{Input}},$$

- $u = \text{const.}$

$\therefore \; x(t) = e^{-kt} \cdot x_0 + \frac{u}{k}\left(1 - e^{-kt}\right)$

$u = 0 \;\rightarrow\; x = e^{-kt} \cdot x_0$

$e^{-kt} = ?$

say $k = 2$. $\quad e^{-2t}$

$t = 0 \quad e^{0} = 1$

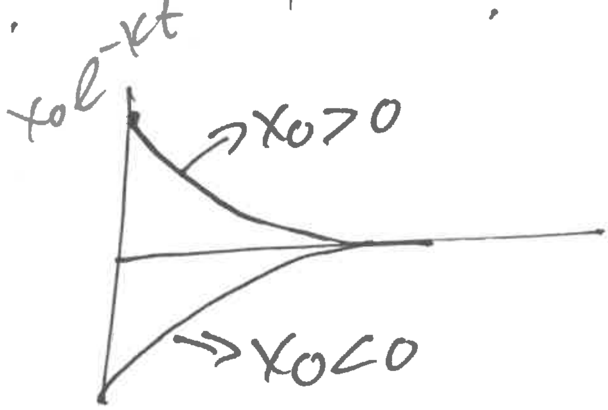
$t = 0.1 \quad e^{-0.2} = 0.8187$

$t = 0.5 \quad e^{-1} = 0.367$

$t = 1 \quad e^{-2} = 0.13533$

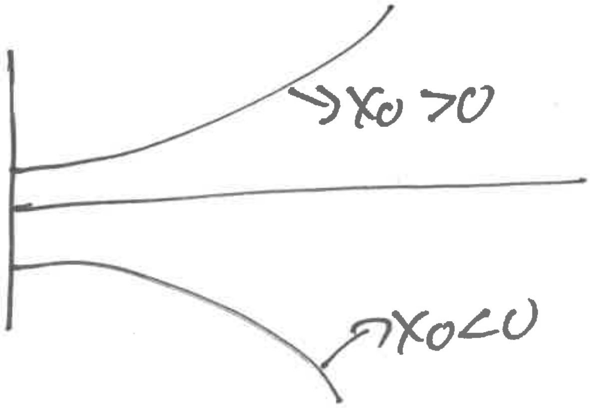
$\vdots \qquad \vdots \qquad \vdots$

$k > 0$



STABLE

$k < 0$



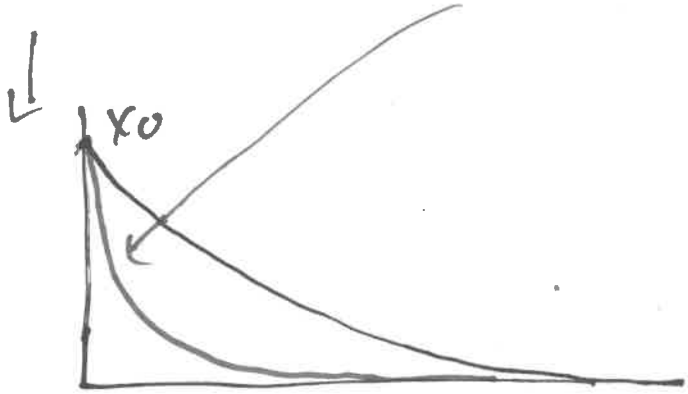
UNSTABLE

$\dot{x} = kx$

$\dot{x} = 3x$ $\qquad$ $\dot{x} = -5 \cdot x$

$k = 3 > 0$ stable $\qquad$ $k = -5 < 0$ unstable

$\dot{x} = 3x$ $\qquad$ $\dot{x} = 30 \cdot x$

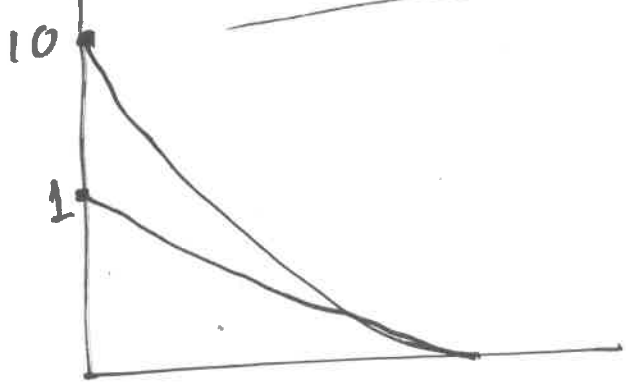


$(e^{-0.2})^{2} = 0.67$

$(e^{-1})^{2} = 0.37$

k → stability
→ speed

$\dot{x} = 3x$, $x_0 = 1$ $\qquad$ $\dot{x} = 3x$, $x_0 = 10$



$\dot{x} = kx$

$k \to$ stability

$\;\;\to$ speed

x_0

$\downarrow$

I.C.

$u \neq 0 \quad x(t) = e^{-kt} \cdot x_0 + \frac{u}{k}\left(1 - e^{-kt}\right)$

$$x(0) = e^{0} \cdot x_0 + \frac{u}{k}\left(1 - e^{0}\right)$$

$$= 1 \cdot x_0 + \frac{u}{k}\left(1 - 1\right)$$

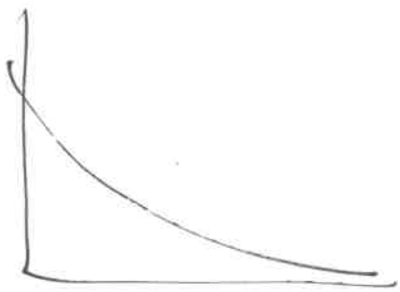
$$= x_0$$

if $k > 0 \quad x_{ss} = u/k$

$\leftarrow$ UNSTABLE | STABLE $\rightarrow k$

0

$\dot{x} = 3x$; $x_0 = 1$	S C $\emptyset$	$x_0 = -1$	S C $\emptyset$
$\dot{x} = -3x$, $x_0 = 1$	U D $+\infty$	$x_0 = -1$	U D $-\infty$
$\dot{x} = 3x + 5$, $x_0 = 1$	S C $5/3$	$x_0 = -1$	S C $5/3$
$\dot{x} = -3x + 5$, $x_0 = 1$	U	$x_0 = -1$	U
$\ddot{x} = -30 \cdot x + 5$, $x_0 = 1$	$\cancel{U}$	$x_0 = -1$	$\cancel{U}$
$\dot{x} = -30 \cdot x$, $x_0 = 1$	$\cancel{U}$	$x_0 = -1$	$\cancel{U}$



$$\ddot{x} = f(\dot{x}, x, t)$$

$$\downarrow$$

$$\ddot{x} + A\dot{x} + Bx = u$$ Linear 2nd order ODE

$$\downarrow \; u = 0$$

$$\ddot{x} + A\dot{x} + Bx = 0$$

$$\boxed{\begin{aligned}\dot{x} &= kx \\ x &= e^{kt}\end{aligned}}$$

HOPE
PRAY

$x = e^{rt} = $ soln

$\dot{x} = re^{rt}$

$\ddot{x} = r^2 e^{rt}$

$$r^2 \cancel{e^{rt}} + Ar\cancel{e^{rt}} + B\cancel{e^{rt}} = 0$$

$$r^2 + Ar + B = 0 \rightarrow \text{C.E.}$$

$r_1 = \ldots$

$r_2 = \ldots$

eigenvalues

$$r_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$\ddot{x} + 4\dot{x} + 3x = 0$

$x = e^{rt} \Rightarrow$ C.E.: $r^2 + 4\cdot r + 3 = 0$

$\Delta = 4^2 - 4\cdot 3 = 16 - 12 = 2^2$

$r_{1,2} = \dfrac{-4 \pm 2}{2}$ $\begin{cases} r_1 = -3 \\ r_2 = -1 \end{cases} \Rightarrow$

$x_1 = e^{-3t}$

$\dot{x}_1 = -3e^{-3t}$

$\ddot{x}_1 = 9e^{-3t}$

$x_2 = e^{-t}$

$9\cancel{e^{-3t}} + 4(-3\cancel{e^{-3t}}) + 3\cancel{e^{-3t}} = 0$

$0 = 0$

so $x_1 = e^{-3t}$ is a soln.

is $x_3 = 10e^{-3t}$ $\quad \dot{x}_3 = -30e^{-3t}$, $\ddot{x}_3 = 90e^{-3t}$

$9\cdot 90\cancel{e^{-3t}} + 4\cdot(-30e^{-3t}) + 3\cdot 10\cancel{e^{-3t}} = 0$

$0 = 0$

---

if x_1, x_2 are solns, so does $c_1\cdot x_1$ and $c_2 x_2$, also $x = c_1\cdot x_1 + c_2 x_2$ is a soln

General soln

2nd ORDER ODE

$$\Big\downarrow x = e^{rt}$$

2nd order polynomial eqn.

$$\Downarrow$$

r_1, r_2

$$\Downarrow$$

$x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$

$$\Downarrow$$

Gen. soln $x = c_1 x_1 + c_2 x_2$

$= c_1 e^{r_1 t} + c_2 e^{r_2 t}$

$\ddot{x} + 4\dot{x} + 3x = 0 \Rightarrow r^2 + 4r + 3 = 0 \Rightarrow$ $r_1 = -1$, $r_2 = -3$

$\Rightarrow x(t) = c_1 e^{-t} + c_2 e^{-3t}$

$\dot{x} = -c_1 e^{-t} - 3 c_2 e^{-3t}$ $\Big\} \Rightarrow$

$+$ $x_0 = 1$, $\dot{x}_0 = 0$

$\left.\begin{aligned} c_1 + c_2 &= 1 \\ -c_1 - 3c_2 &= 0 \end{aligned}\right\} \Rightarrow$ $c_2 = -0.5$, $c_1 = 1.5$

$\Rightarrow$ specific soln: $x = 1.5 e^{-t} - 0.5 e^{-3t}$

$\ddot{x} + 2\dot{x} + x = 0 \quad x_0 = 1, \dot{x}_0 = 0$ ⑩

$\Downarrow$

$r^2 + 2r + 1 = 0 \Rightarrow \Delta = 4 - 4 = 0$

$r_{1,2} = \frac{-2 \pm 0}{2}$

$r_1 = r_2 = r = -1 \rightarrow x_1 = e^{-1t}$

$x_2 = t e^{-t}$

Gen soln $x = c_1 x_1 + c_2 x_2$

$= c_1 e^{-t} + c_2 t e^{-t}$ (1)

$\dot{x} = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$ ②

①, ② $\xRightarrow{t=0}$ $x = c_1 \cdot 1 + c_2 \cdot 0 \cdot 1 = c_1 = 1$

$-c_1 \cdot 1 + c_2 \cdot 1 - c_2 \cdot 0 \cdot 1 = 0$

$c_2 = 1.$

spec. soln $\boxed{x = e^{-t} + t e^{-t}}$

$\ddot{x} + 7\dot{x} + 6x = 0$, $x_0 = 1$
$\dot{x}_0 = 2$

1) stability
2) Gen soln
3) spec. soln

1) C.E. $r^2 + 7 \cdot r + 6 = 0$ $\Delta = 49 - 4 \cdot 6 = 25$

$r_{1,2} = \dfrac{-7 \pm 5}{2}$ $\to r_1 = -1$, $\to r_2 = -6$ stable

2) $x_1 = e^{-t}$. $x_2 = e^{-6t}$

$$\boxed{x = c_1 e^{-t} + c_2 e^{-6t}}$$

3) $\dot{x} = -c_1 e^{-t} - 6 c_2 e^{-6t}$ $\Rightarrow$ I am c_2

$$\left.\begin{aligned} x(0) &= c_1 + c_2 = 1 \\ \dot{x}(0) &= -c_1 - 6c_2 = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_2 &= -3/5 \\ c_1 &= 8/5 \end{aligned}$$

$x = -\frac{3}{5} e^{-6t} + \frac{8}{5} e^{-t}$

~~Assume $cx =$~~

$$c_1 + c_2 = 1$$
$$-c_1 - 6c_2 = 2$$

$$0c_1 + (-5c_2) = 3$$

$$c_2 = -3/5$$

Assume ODE, $x_0 \Rightarrow \ldots$

$$x = c_1 e^t + c_2 e^{-t}$$

Assume $x_0 = 1 \quad \dot{x}_0 = -1$

$$\dot{x} = c_1 e^t - c_2 e^{-t}$$

$$\left.\begin{aligned} x(0) &= c_1 + c_2 = 1 \\ \dot{x}(0) &= c_1 - c_2 = -1 \end{aligned}\right\} \Rightarrow 2c_1 + 0c_2 = 0$$

$$c_1 = 0$$
$$\Rightarrow c_2 = 1$$

$$x(t) = 0 \cdot e^t + 1 \cdot e^{-t}$$
$$= e^{-t}$$

$x_0 = 1, \ \dot{x}_0 = -1 \Rightarrow c_1 = 0, \ c_2 = 1.$

$x = e^{-t}$

$x_0 = 1.0000001 \Rightarrow \ldots c_1 = 0.000001$

$c_2 = 1$

$$x = \underbrace{0.00001 \cdot \underbrace{e^t}_{\substack{+\infty \\ 10^{15}}}}_{10^{10}} + \underbrace{e^{-t}}_{\to 0}$$

$\ddot{x} + \dot{x} + x = 0$ $\qquad \Delta = 1 - 4 = -3$

Assume $= -4$

$$r_{1,2} = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm \sqrt{4 \cdot i^2}}{2}$$

$$= \frac{-1 \pm \sqrt{(2 \cdot i)^2}}{2}$$

$$= \frac{-1 \pm 2i}{2} \quad \begin{cases} r_1 = -\frac{1}{2} + i \\ r_2 = -\frac{1}{2} - i \end{cases}$$

Method. A: $\left.\begin{array}{l} r_1 = -1/2 + i \\ r_2 = -1/2 - i \end{array}\right\} \Rightarrow$

$$x_1 = e^{(-1/2 + i)\cdot t}$$

$$x_2 = e^{(-1/2 - i)\cdot t}$$

$$X = C_1 \cdot x_1 + C_2 \cdot x_2$$

Method B

$$e^{j\theta} = e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{(a+bi)t} = e^{at} \cdot e^{bit}$$

$$= e^{at} \cdot (\cos bt + i \sin bt)$$

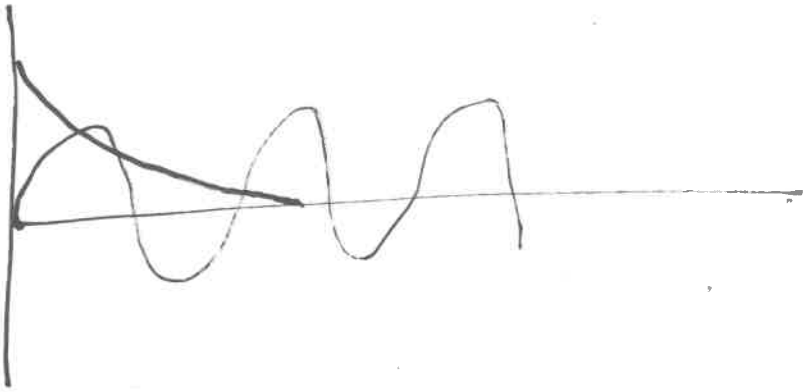
$$= \underbrace{e^{at} \cdot \cos bt}_{\Downarrow} + i \cdot e^{at} \sin bt$$

$x_1 = e^{at} \cdot \cos bt$ Re

$x_2 = e^{at} \cdot \sin bt$ Im

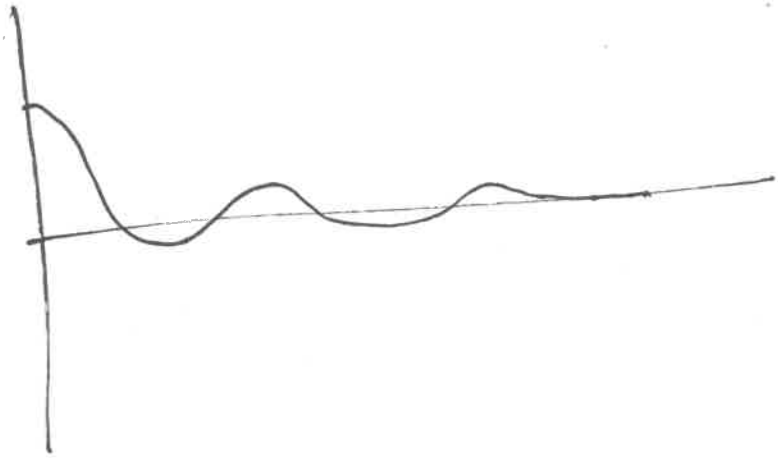
$$X = \cancel{C_1 \cdot x_1 + C_2 x_2}$$

$$= e^{at} \cdot (C_1 \cos bt + C_2 \sin bt).$$



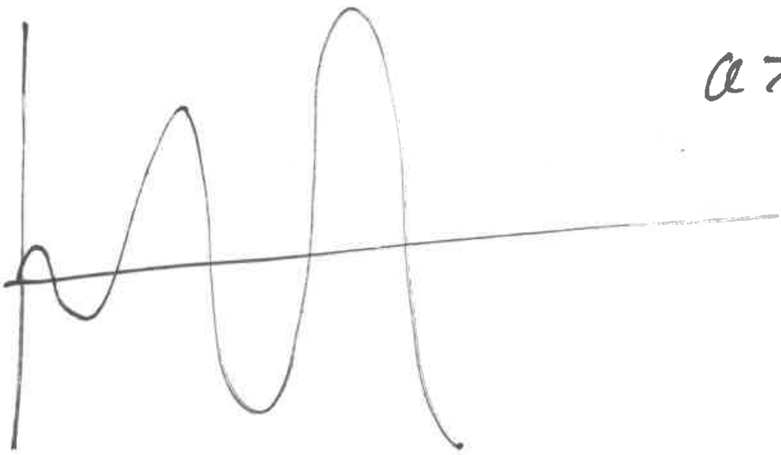
$a < 0$

stable



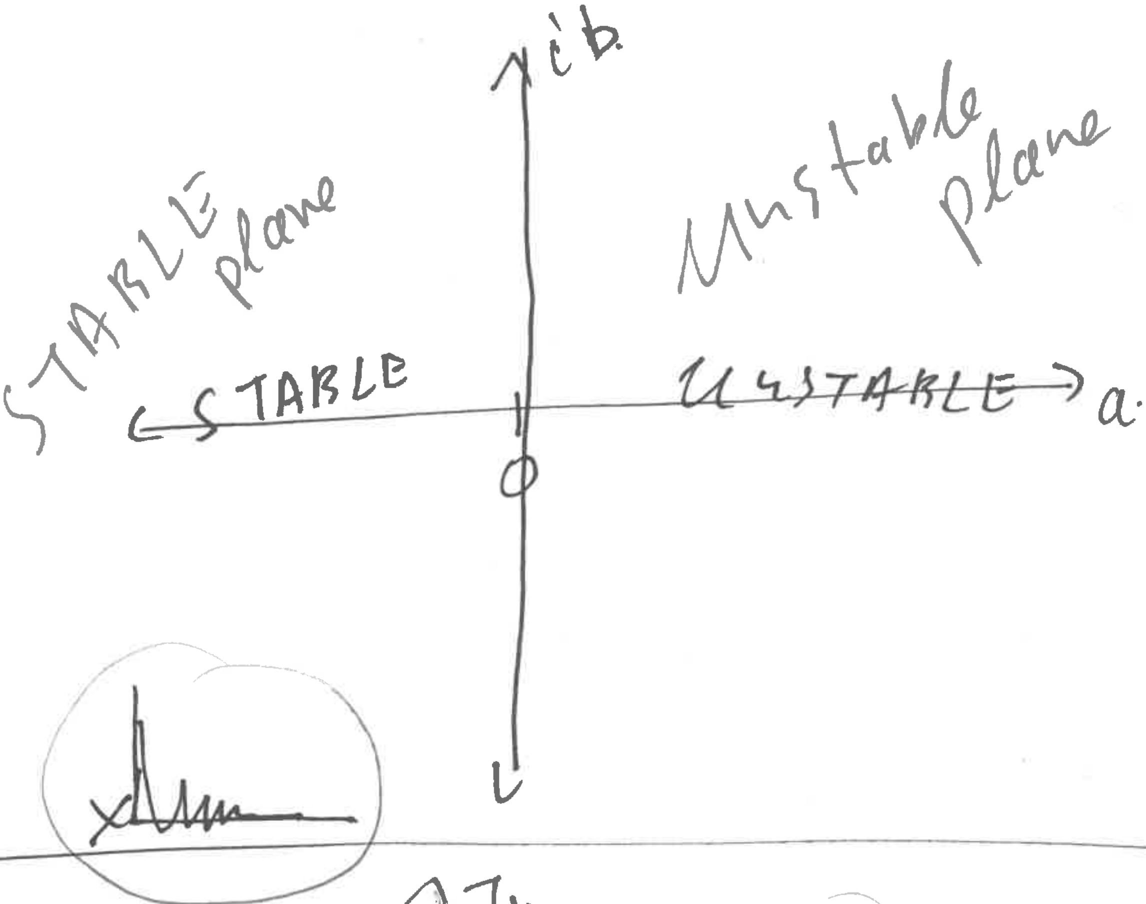
unstable

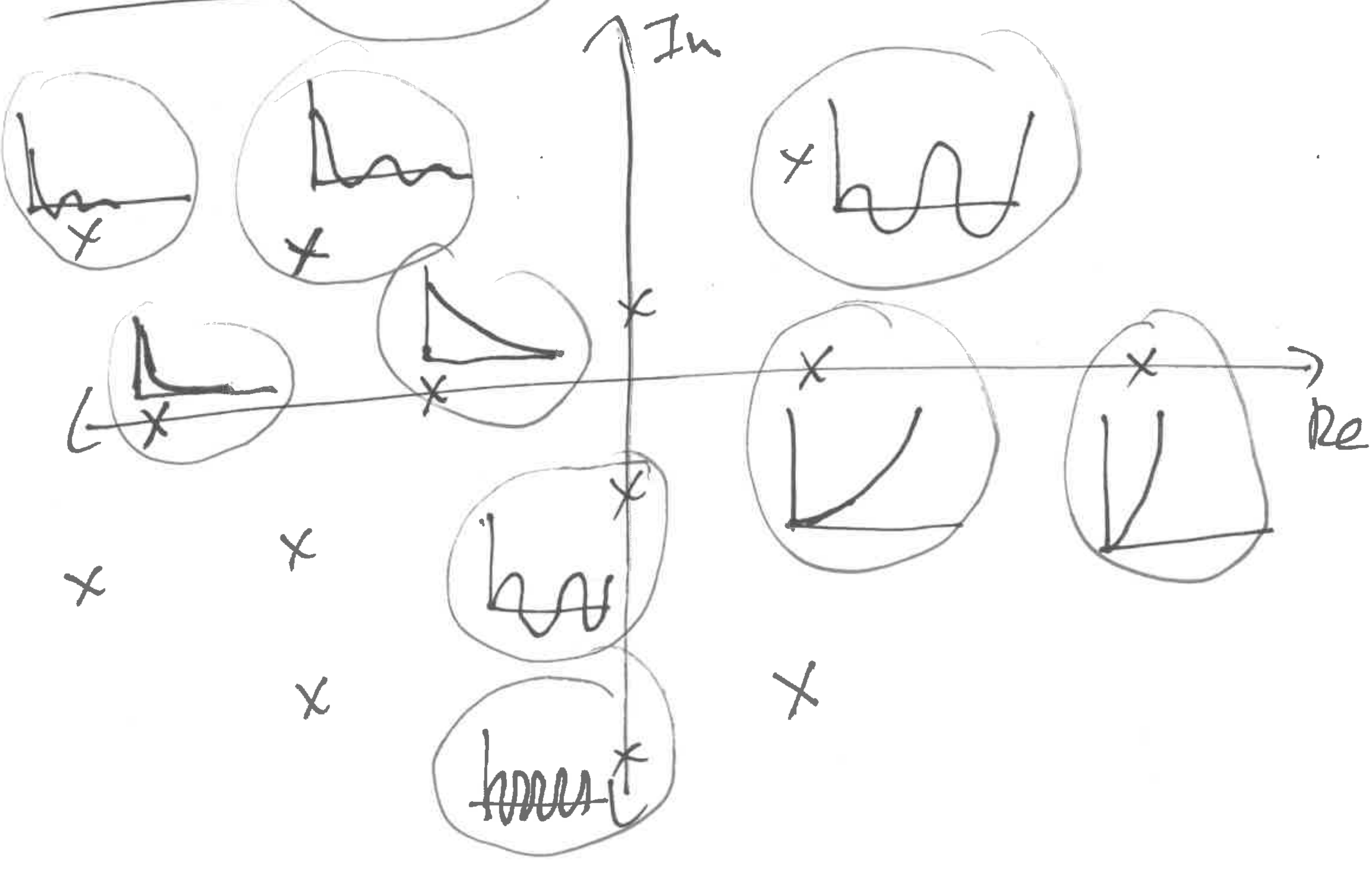
$a > 0$



ODE $\rightarrow$ $r = a \pm bi$ (stable; how fast I osc.)

$\rightarrow e^{at} \cdot (C_1 \cdot \cos bt + C_2 \cdot \sin bt)$





$\ddot{x} + A\dot{x} + Bx = u.$

$\downarrow u = 0$

$\ddot{x} + A\dot{x} + Bx = 0$

$\downarrow x = e^{rt}$

$r^2 + Ar + B = 0 \longrightarrow \Delta = A^2 - 4 \cdot B$

C.E. → eigenvalues

- $\Delta > 0$ $r_1 \neq r_2 \in \mathbb{R}$.
  $x_1 = e^{r_1 t}, \ x_2 = e^{r_2 t}$
  $x = C_1 \cdot x_1 + C_2 \cdot x_2$
  r_1 and $r_2 < 0 \to$ stable
  r_1 or $r_2 > 0 \longrightarrow$ unstable

- $\Delta = 0$ $r_1 = r_2 = r \in \mathbb{R}$
  $x_1 = e^{rt}, \ x_2 = t e^{rt}$
  $r < 0 \to$ stable
  $r > 0 \longrightarrow$ unst.

- $\Delta < 0$ $r = \alpha \pm bi$
  $x_1 = e^{rt}, \ x_2 = e^{\bar{r}t}$
  or $x_1 = e^{\alpha t} \cdot \cos bt$
  $x_2 = e^{\alpha t} \cdot \sin bt$
  $x = C_1 x_1 + C_2 x_2$
  $\alpha < 0 \to$ stable
  $\alpha > 0 \to$ unst.

1st $\dot{x} + kx = u.$

$k > 0$, $k < 0$

$\Downarrow$ stable $\Downarrow$ unst.

$x = e^{rt}$

$r + k = 0 \Rightarrow r = -k$

$\ddot{x} + A\dot{x} + Bx = u.$

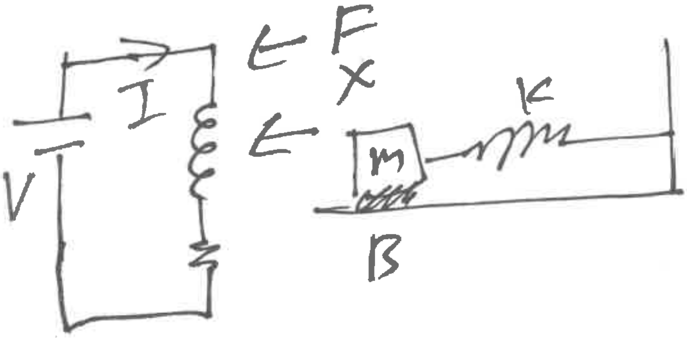
Assume $u = \text{const}$ $r < 0 \Rightarrow x \to x_{ss}$

$\dot{x}_{ss} = 0$

$\ddot{x}_{ss} = 0$

$0 + A \cdot 0 + B \cdot x_{ss} = u.$

$x_{ss} = u/B$



$V = ?$

$x = ?$

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$V = -iR + L\, di/dt \qquad F = K_A \cdot i$

$F - B\dot{x} - Kx = m\ddot{x}$

$\frac{di}{dt} = \cancel{+} R/L \cdot i + \frac{V}{L}$

$K \qquad Vu$

$i_{ss} = u/K = \frac{V/L}{R/L} = \frac{V}{R}$

$i(t) = e^{-Kt} \cdot x_0 + \frac{u}{K}\left(1 - e^{-Kt}\right)$

$x_0 = 0$

$i(0) = 0$

$i(t) = \frac{V/\cancel{L}}{R/\cancel{L}}\left(1 - e^{-R/L \cdot t}\right)$

$$F - B\dot{x} - kx = m\cdot\ddot{x}$$

$$\ddot{x} + \underbrace{\frac{B}{m}}_{A}\dot{x} + \underbrace{\frac{k}{m}}_{B}x = F = k_A\cdot i$$

$$= k_A\cdot\left(1 - e^{-R_L t}\right)\cdot\frac{V}{R}$$

LAPLACE TRANSFORM

$$f(t) \rightarrow F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$s = 6 + i\omega$$

$$f(t) \rightarrow F(s)$$
$$1 \rightarrow 1/s$$
$$t \rightarrow 1/s^2$$
$$f'(t) \rightarrow s\cdot F(s)$$
$$f''(t) \rightarrow s^2\cdot F(s)$$
$$\vdots$$
$$f^{(k)}(t) \rightarrow s^k\cdot F(s)$$

$\dot{x} + 5x = u. \longrightarrow r+5=0 \ , r=-5$

C.E.

$u(t) \longrightarrow U(s)$

$x(t) \longrightarrow X(s)$

$\dot{x}(t) \longrightarrow s \cdot X(s)$

$sX(s) + 5X(s) = U(s)$

$X(s)\,(s+5) = U(s)$

$$\frac{X(s)}{U(s)} = \frac{1}{s+5} \longrightarrow \text{T.F.}$$

Den of T.F. $\boxed{s+5=0}$

$$x'' + 5x' + 6x = u(t) + 3 \cdot u'(t)$$

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$$x \longrightarrow X(s)$$
$$\dot{x} \longrightarrow s\,X(s)$$
$$\ddot{x} \longrightarrow s^2 \cdot X(s)$$
$$u \longrightarrow U(s)$$
$$\dot{u} \longrightarrow s\,U(s)$$

$$s^2 \cdot X(s) + 5 \cdot s \cdot X(s) + 6 \cdot X(s) = U(s) + 3 \cdot s \cdot U(s)$$
$$X(s)\left(s^2 + 5s + 6\right) = U(s)(3s+1)$$

$$\frac{X(s)}{U(s)} = \frac{3s+1}{\boxed{s^2 + 5 \cdot s + 6}} \longrightarrow \text{C.E.}$$

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$v = iR + L\, di/dt$

$m\ddot{x} + B\dot{x} + kx = k_A \cdot i$

$i(t) \rightarrow I(s)$

$v(t) \rightarrow V(s)$

$x(t) \rightarrow X(s)$

$di/dt \rightarrow s\,I(s)$

$\dot{x} \rightarrow s\,X(s)$

$\ddot{x} \rightarrow s^2 X(s)$

$V(s) = I(s)\,R + Ls\,I(s)$

$V(s) = I(s) \cdot (Ls + R)$ $\quad I(s) = \dfrac{V(s)}{Ls+R}$

$\dfrac{I(s)}{V(s)} = \dfrac{1}{Ls+R}$ T.F. beet. V-I

$m \cdot s^2 X(s) + B \cdot s X(s) + k X(s) = k_A I(s)$

$X(s) \cdot (ms^2 + Bs + k) = k_A \cdot I(s)$

$\dfrac{X(s)}{I(s)} = \dfrac{k_A}{ms^2 + Bs + k}$ T. F. beet X-I.

$\dfrac{X(s)}{V(s)} = \dfrac{k_A}{(Ls+R)(ms^2+Bs+k)} \rightarrow$ C.E $\rightarrow$

eigenvalues

$$(Ls+R)(ms^2+Bs+k)=0$$

- $s=-R/L$
- $ms^2+Bs+k=0 \Rightarrow \ldots$

$$f(t) \longrightarrow F(s)$$

final value of $f(t)$?

$$\lim_{t\to\infty} f(t)$$

F.V.T.

$$F(s) \rightarrow F_{ss} = \lim_{s\to 0} s\,F(s)$$

$$X(s) = V(s)\,\frac{KA}{(Ls+R)\cdot(ms^2+Bs+k)}$$

$$V(t)=1 \longrightarrow V(s)=1/s$$

$$X(s)=\frac{1}{s}\,\frac{KA}{(Ls+R)(ms^2+Bs+k)}$$

$$X_{ss}=\lim_{s\to 0} s\,X(s)=\lim_{s\to 0} \cancel{s}\,\frac{1}{\cancel{s}}\,\frac{KA}{(Ls+R)(ms^2+Bs+k)}$$

$$=\frac{KA}{R\cdot k}$$

$$x''' + 5 \cdot x'' + 3 \cdot x' + 2x = u''' + 3u'' + 3u$$ 25

Find X_{ss} $\quad u = 1 \Rightarrow U(s) = 1/s$

$$\left.\begin{aligned}
x &\longrightarrow X(s)\\
\dot{x} &\longrightarrow s \cdot X(s)\\
\ddot{x} &\longrightarrow s^2 \cdot X(s)\\
\dddot{x} &\longrightarrow s^3 X(s)\\
u &\longrightarrow U(s)\\
\ddot{u} &\longrightarrow s^2 U(s)\\
\dddot{u} &\longrightarrow s^3 U(s)
\end{aligned}\right\} \Rightarrow$$

$$s^3 \cdot X(s) + 5 \cdot s^2 \cdot X(s) + 3 \cdot s X(s) + 2X(s) = s^3 U(s) + 3s^2 U(s) + 3U(s)$$

$$(s^3 + 5s^2 + 3s + 2) X(s) = U(s)(s^3 + 3s^2 + 3)$$

$$\frac{X(s)}{U(s)} = \frac{s^3 + 3s^2 + 3}{s^3 + 5s^2 + 3s + 2}$$

$$X_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{s^3 + 3s^2 + 3}{s^3 + 5s^2 + 3s + 2} = 1.5$$

T.F. $\frac{OUT(S)}{In(S)} = \frac{P(S)}{Q(S)}$

C.E. $Q(S) = 0 \rightarrow$ poles of T.F.

$P(S) = 0 \rightarrow$ zeros of T.F.

Revision

$$\dot{x}+kx=u \Longrightarrow x=e^{-kt}\cdot x_0+e^{-kt}\int_0^t e^{kt_1}u(t_1)\,dt_1$$

$$\xRightarrow{u=\text{const}} x=e^{-kt}\cdot x_0+\frac{u}{k}\left(1-e^{-kt}\right)$$

- $k>0 \to e^{-kt}\to 0 \Longrightarrow x\to u/k$
- $k<0 \to e^{-kt}\to \infty \Longrightarrow x\to \pm\infty$

$x_0 \to$ I.C.

$u \to$ S.S.

$u=0$

C.E.

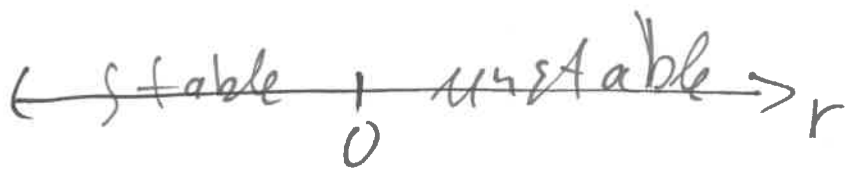
eigenvalue

$$\dot{x}+kx=0 \xRightarrow{x=e^{rt}} r+k=0 \Leftrightarrow r=-k$$

$r<0 \to$ stable

$r>0 \to$ unstable

eigenplane



$\ddot{x} + A\dot{x} + Bx = u$

$\downarrow u = 0$

$\ddot{x} + A\dot{x} + Bx = 0 \xRightarrow{x = e^{rt}} r^2 + Ar + B = 0$ → C.E

eigs

$\Delta = A^2 - 4 \cdot B$

$x_0 = \dots \quad \dot{x}_0 = \dots$

$c_1 = \dots$

$c_2 = \dots$

☒ $\Delta > 0 \quad r_1, r_2 \in \mathbb{R}, \; r_1 \neq r_2$

$x_1 = e^{r_1 t} \quad x_2 = e^{r_2 t} \Rightarrow x = c_1 x_1 + c_2 x_2$

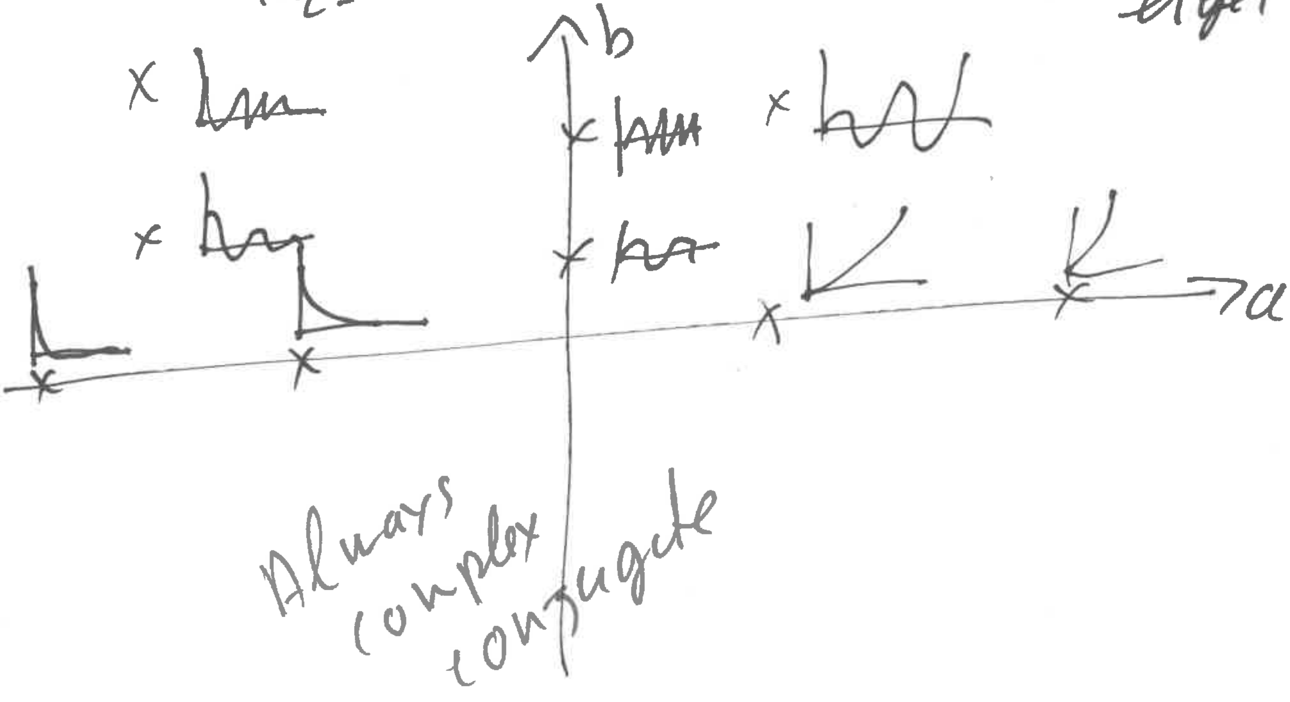
☒ $\Delta = 0 \quad r_1 = r_2 = r \in \mathbb{R}$

$x_1 = e^{rt}, \; x_2 = t e^{rt}$

☒ $\Delta < 0 \quad r_1 = a + bi, \quad r_2 = a - bi$

- $x_1 = e^{r_1 t}, \; x_2 = e^{r_2 t}$
- $x_1 = e^{at} \cos bt$
  $x_2 = e^{at} \sin bt$

eigen plane



$$\dddot{x} + A\ddot{x} + B\dot{x} + Cx = 0$$

$$\Big\downarrow e^{rt}$$

$$r^3 + Ar^2 + Br + C = 0 \Rightarrow \begin{array}{l} r_1 = \dots \\ r_2 = \dots \\ r_3 = \dots \end{array}$$

e.g. $x = C_1 e^{-t} + C_2 e^{-2t} + C_3 e^{+5t}$

$x = C_1 e^{-t} + C_2 t e^{-t} + C_3 e^{6t}$

L.T.

$f(t) \to F(s)$

$f'(t) \to s \cdot F(s)$

$f''(t) \to s^2 F(s)$

$\vdots$

$f(t) \to f_{ss} = \lim_{t \to \infty} f(t)$

$F(s) \to F_{ss} = \lim_{s \to 0} s\, F(s)$

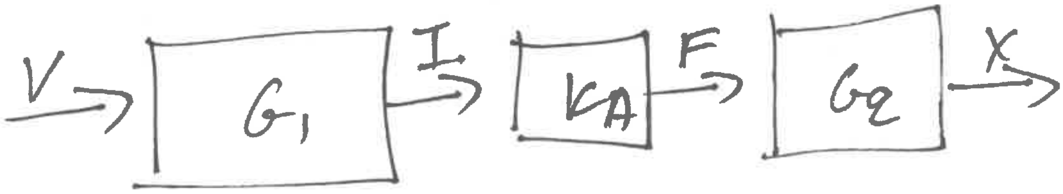
F.V.T.

$\downarrow$

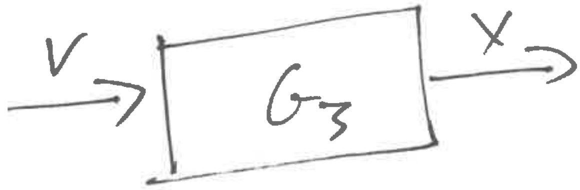
T.F.

$$\frac{Out(s)}{In(s)} = \frac{P(s) \to \text{zeros}}{Q(s) \to \text{eigs or poles}}$$

$Q(s) = 0 \leftarrow$ C.E

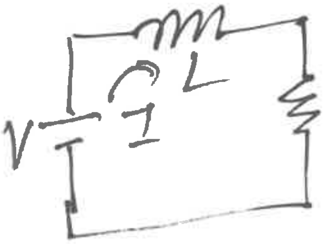


$$G_1(s) = \frac{1}{Ls+R} \qquad G_2 = \frac{KA}{ms^2+Bs+k}$$



$$G_3 = \frac{KA}{(Ls+R)(ms^2+Bs+k)}$$

$$G_3 = G_1 \cdot KA \cdot G_2$$



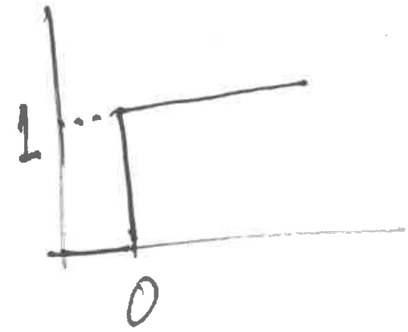
$\rightarrow \; V = iR + L\, di/dt$

$i(0) = 0 \implies$

$i(t) = \frac{V}{R}\left(1 - e^{-R/L\, t}\right)$

$V \rightarrow$ step Function

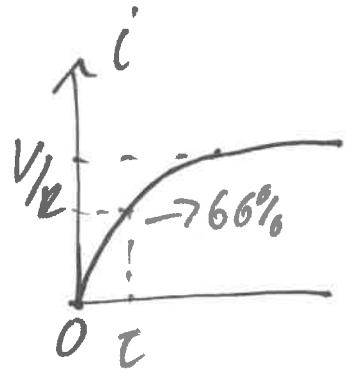
$v(t) = V \rightarrow V(s) = V/s$



$t = 0 \quad e^{-R/L\, t} = 1 \implies i(0) = 0$

$\vdots$

$t = \ldots \quad e^{-R/L\, t} \implies 0 \qquad i_{ss} = V/R$



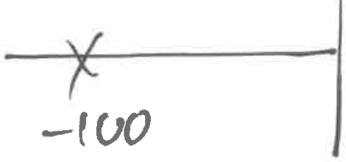
$I(s) = V(s) \cdot \frac{1}{Ls + R}$

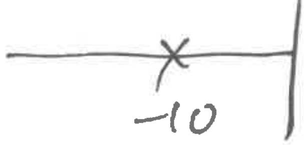
$I_{ss} = \lim_{s \to 0} \cancel{s} \frac{V}{\cancel{s}} \frac{1}{\cancel{Ls}_{\to 0} + R} = V/R$

$\frac{I(s)}{V(s)} = \frac{1}{Ls + R} = \frac{1/R}{\frac{L}{R}s + 1} = \frac{K}{\tau s + 1} \rightarrow \text{pole} = -\frac{1}{\tau}$

$\tau \rightarrow$ Time constant

$\tau = 0.01$ s-plane $\tau = 0.1$





Smaller $\tau \rightarrow$ Faster sys

2nd. order sys.

Generic C.E.

$$s^2 + 2 \cdot Z \cdot w_n \cdot s + w_n^2 = 0$$

2nd order pol wrt s

$$\Delta = 4 \cdot Z^2 \cdot w_n^2 - 4 w_n^2$$

$$= 4 \cdot w_n^2 \cdot (Z^2 - 1)$$

$$\Delta > 0 \Rightarrow Z^2 - 1 > 0 \Rightarrow Z > 1$$

$$s_{1,2} = \frac{-2 \cdot Z \cdot w_n \pm \sqrt{\Delta}}{2}$$

$$= \frac{-2 \cdot Z \cdot w_n \pm \sqrt{4 \cdot w_n^2 \cdot (Z^2 - 1)}}{2}$$

$$= \frac{-\cancel{2} \cdot Z \cdot w_n \pm \cancel{2} \cdot w_n \sqrt{Z^2 - 1}}{\cancel{2}}$$

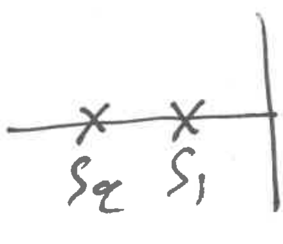
$$= -Z \cdot w_n \pm w_n \sqrt{Z^2 - 1} \Rightarrow$$

$$S_1 = -Z \cdot w_n + w_n \sqrt{Z^2 - 1}$$

$$S_2 = -Z \cdot w_n - w_n \sqrt{Z^2 - 1}$$

overdamped

stable



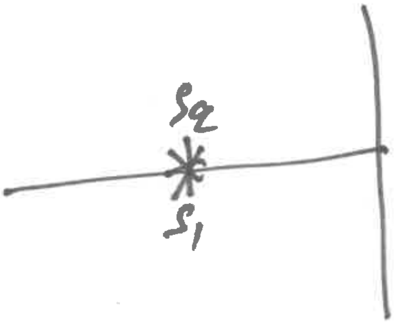
$\Delta = 0 \qquad \Delta = 4 \cdot \omega_n^2 (z^2 - 1)$

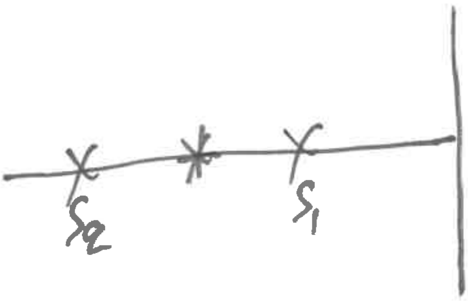
$\Rightarrow z = 1.$

$$s_{1,2} = \frac{-z \cdot \omega_n 2 \pm \cancel{\sqrt{\Delta}} \to 0}{2}$$

$$s_{1,2} = -z \cdot \omega_n = -\omega_n$$

stable
critically
damped





$D = 4\cdot w_n^2\cdot(Z^2-1)$

$D<0 \quad 0<Z<1$

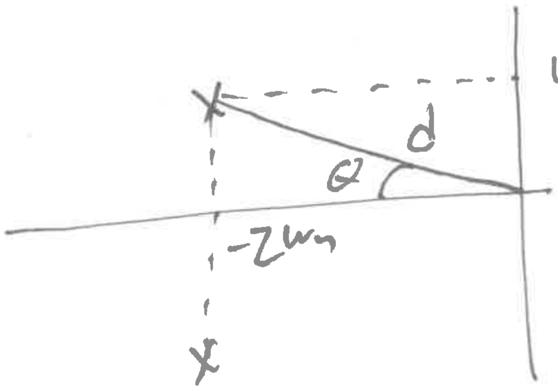
$s_{1,2} = -Z\cdot w_n \pm w_n\sqrt{Z^2-1}$

$= -Z\cdot w_n \pm \sqrt{(1-Z^2)\cdot i^2}\cdot w_n.$

$= -Z\cdot w_n \pm i\cdot \underbrace{w_n\sqrt{1-Z^2}}_{w_d}$ Under damped.

$= \underbrace{-Z\cdot w_n}_{a} \pm \underbrace{i\cdot w_d}_{b}$

$-Z\cdot w_n < 0 \rightarrow$ stable.



$\cos\theta = \frac{+Z\cdot w}{d} = \frac{+Z\cdot w_n}{w_n}$

$= Z.$

$\theta = \cos^{-1} Z$

$d^2 = (Zw_n)^2 + w_d^2$

$= Z^2\cdot w_n^2 + w_n^2(1-Z^2)$

$= \cancel{Z^2\cdot w_n^2} + w_n^2 - \cancel{Z^2\cdot w_n^2}$

$d^2 = w_n^2 \Rightarrow d = w_n$

$z > 1 \rightarrow$ over

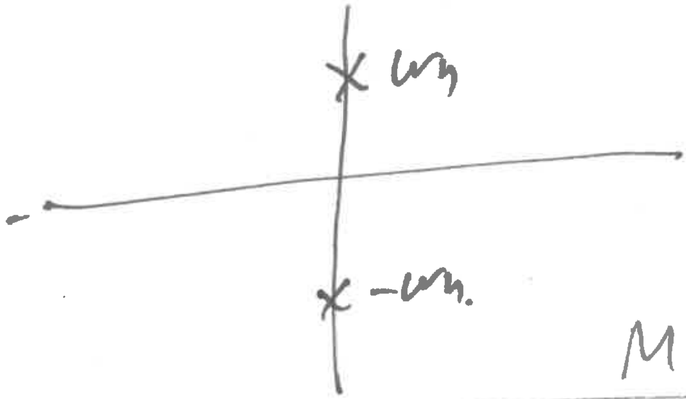
$z = 1 \rightarrow$ critically

$0 < z < 1 \rightarrow$ under

$z = 0$

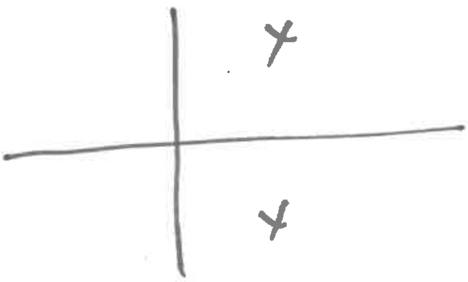
$$S_{1,2} = -z \cdot \omega_n \pm \omega_n \cdot i \sqrt{1-z^2}$$

$$= \underbrace{0}_{\text{real part}} \pm \underbrace{\omega_n \cdot i}_{\text{Imag.}}$$



Marginally stable

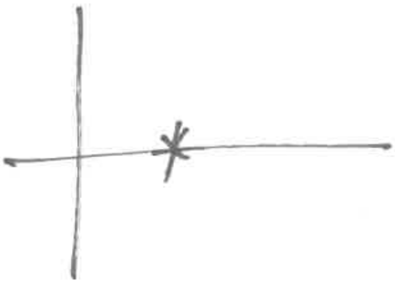
- $z \in [-1, 0]$



Unstable

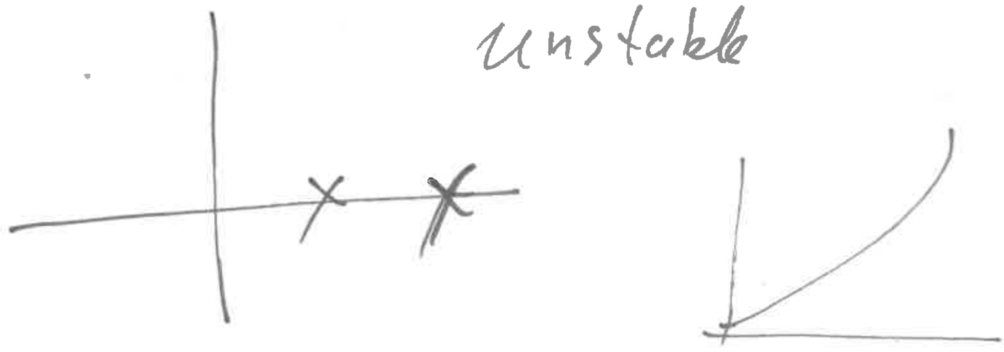
$z = -1$ $\quad S_{1,2} = +\omega_n$

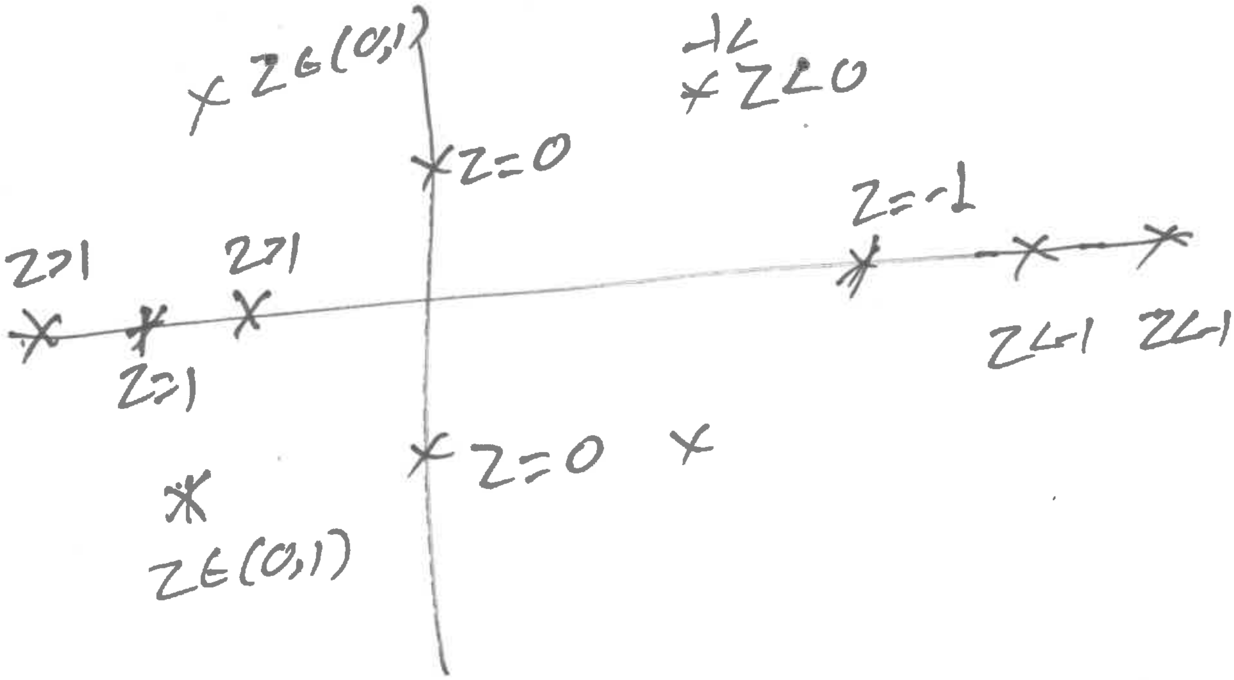
Unstable



$z < -1$

unstable





$G(s) = \frac{3s+1}{s^2+s+5}$ $\quad In = 1/s$

$t_p = ?$ $\quad t_r = ?$ $\quad M_p = ?$ $\quad t_{s_{2\%}} = ?$

S.S. output

Drawing.

I need to find $\zeta, \omega_n = ?$

C.E. $\quad s^2 + 2\cdot\zeta\cdot\omega_n\cdot s + \omega_n^2 = 0$

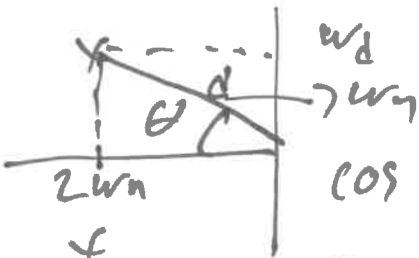
C.E $\quad s^2 + 1\cdot s + 5 \qquad = 0$ $\quad \Big\} \Rightarrow$

$2\cdot\zeta\cdot\omega_n = 1$

$\omega_n^2 = 5 \Rightarrow \omega_n = 2.23$ rad/s

$\zeta = 0.223$

$\omega_d = \omega_n \cdot \sqrt{1-\zeta^2} = 2.179$ rad/s



$\cos\theta = \zeta \Rightarrow \theta = \cos^{-1}\zeta = \ldots.$

$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.179} = 1.44s$

$t_r = \frac{\pi - \theta}{\omega_d} = \ldots = 0.82s$

$t_s = \frac{4}{\zeta\omega_n} = 8.04s$

$M_p = \exp\left(-\zeta\pi/\sqrt{1-\zeta^2}\right)$

$= 0.486$ or

48.6%

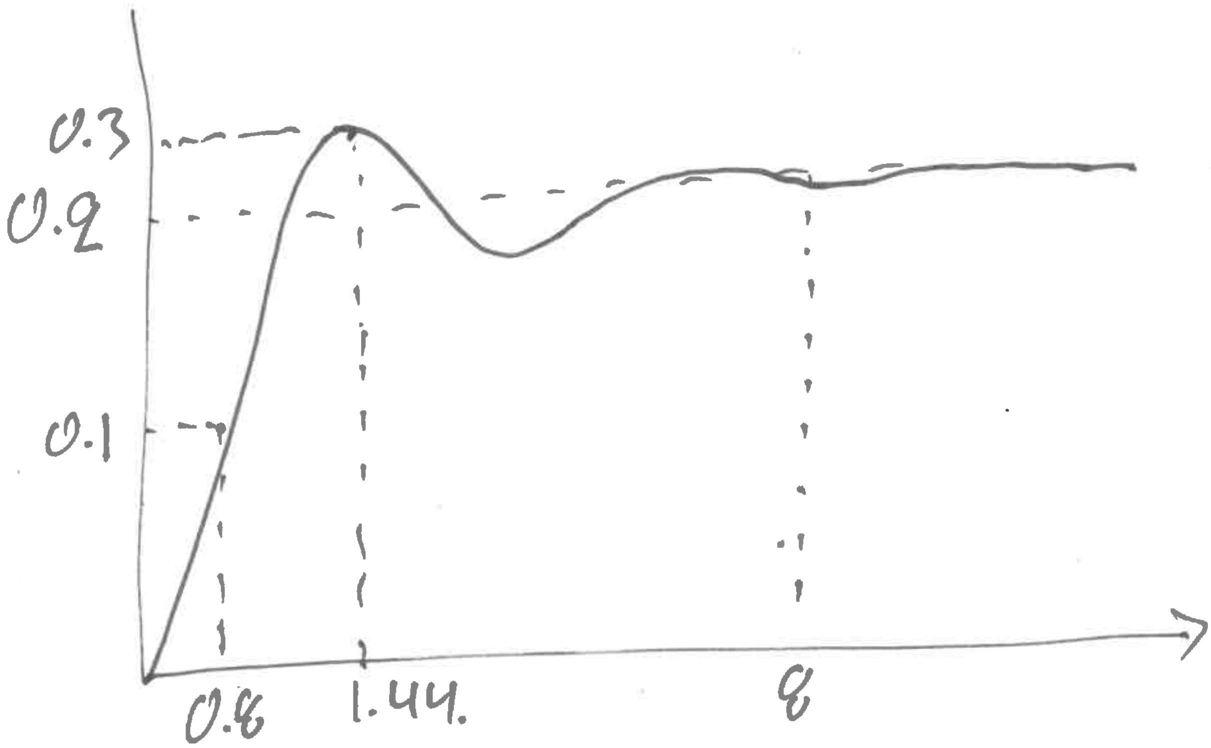
$U(s) = 1/s$

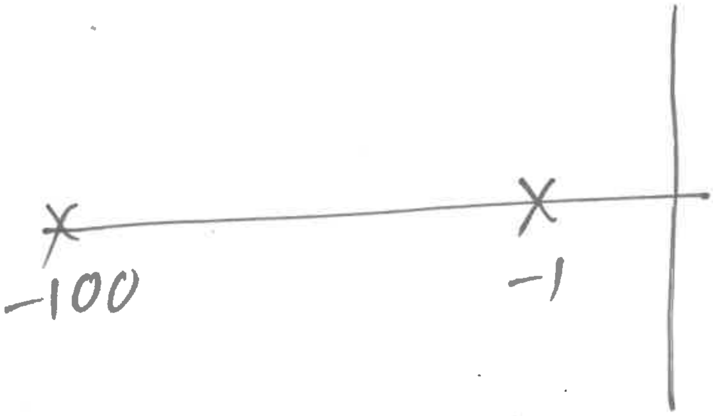
$$\frac{C(s)}{U(s)} = \frac{3s+1}{s^2+s+5}$$

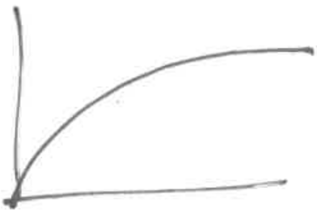
$$C(s) = U(s)\,\frac{3s+1}{s^2+s+5}$$

$$C_{ss} = \lim_{s\to 0} s\,U(s)\,\frac{3s+1}{s^2+s+5}$$

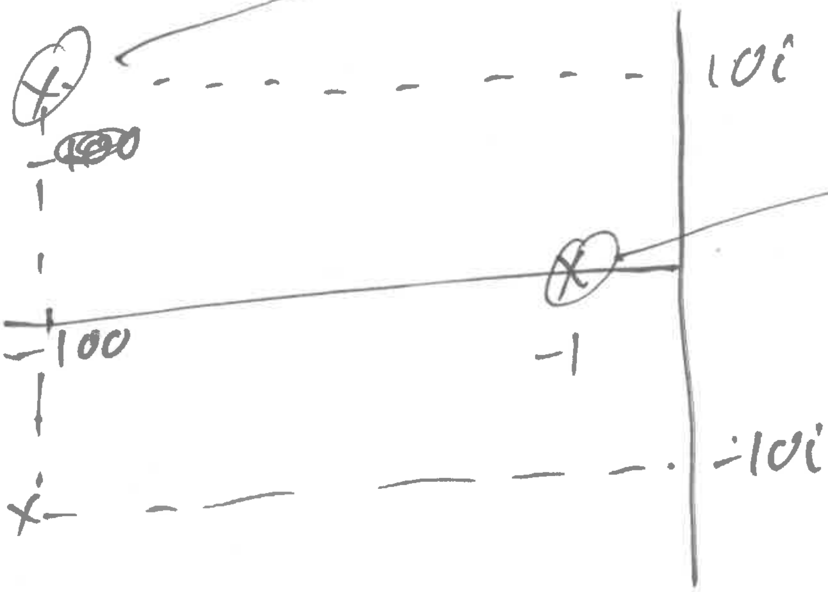
$$= \lim_{s\to 0} \cancel{s}\,\frac{1}{\cancel{s}}\,\frac{3s+1}{s^2+s+5} = 1/5 = 0.2$$



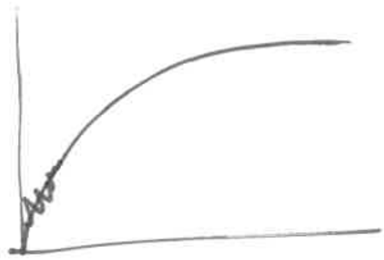


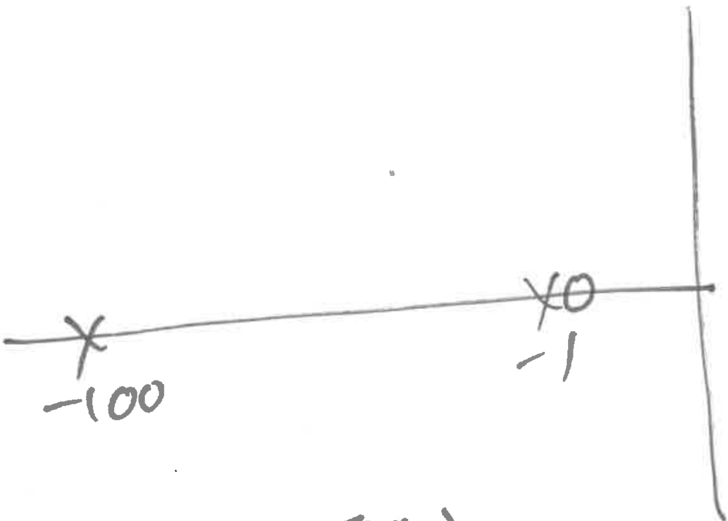


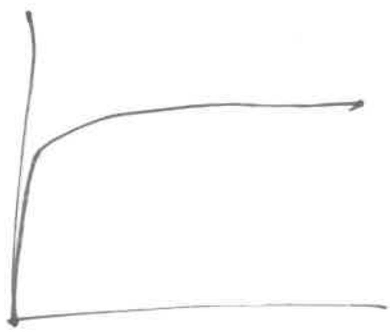
$\rightarrow e^{-100t} \cdot \cos 10t$



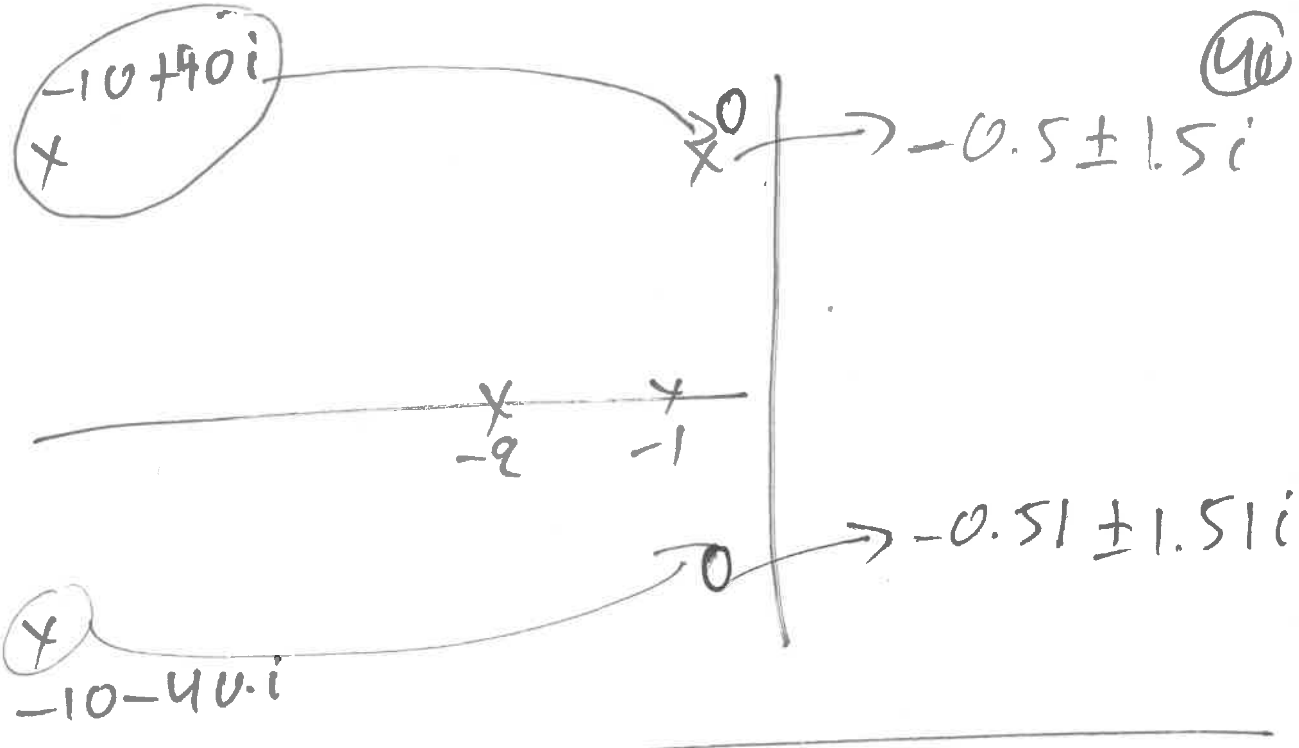
$\rightarrow e^{-t}$

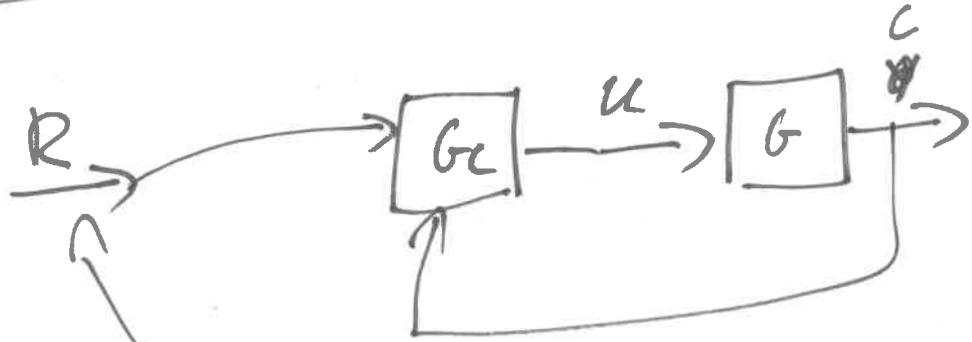






$$\frac{(\cancel{s+0.999})}{(s+100)(\cancel{s+1})} \simeq \frac{1}{s+100}$$

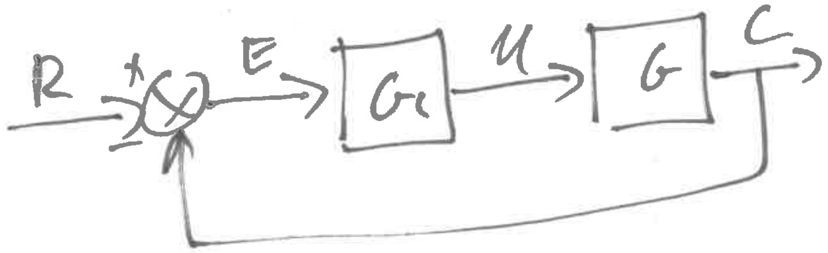




$u = ?$: $C \rightarrow$ Ref.

$\frac{C}{u} = G \rightarrow$ O.L.T.F

$\frac{C}{R} \rightarrow$ C.L.T.F.



$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c}$$

$u = ?$: $E = 0$

$$\left.\begin{array}{l} C = G \cdot u \\ u = G_c \cdot E \\ E = R - C \end{array}\right\} \Rightarrow C = G \cdot G_c \cdot E \Rightarrow$$

$$C = G \cdot G_c \cdot (R - C)$$

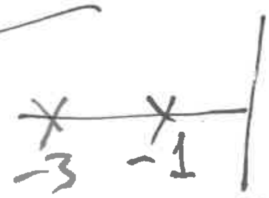
$$C = G \cdot G_c \cdot R - G \cdot G_c \cdot C$$

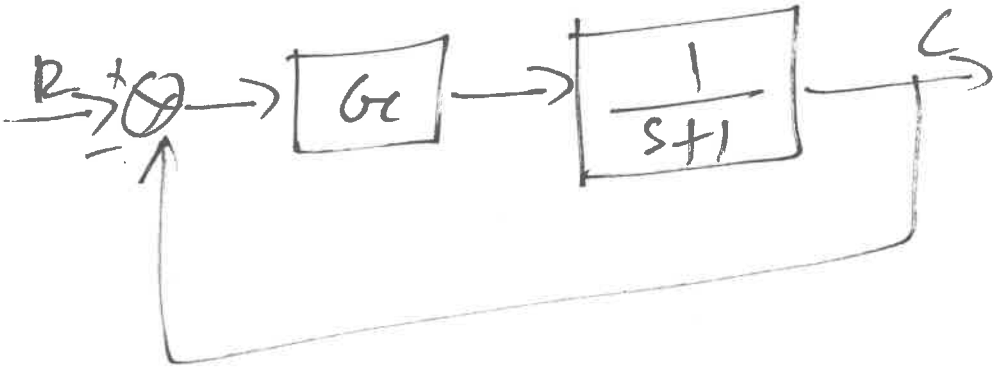
$$C \cdot (1 + G G_c) = R \cdot G \cdot G_c$$

$G(s) = \frac{1}{s+1}$ O.L.C.E $s+1=0$

pole $s=-1$

e^{-t}





choose $G_c(s) = 2$

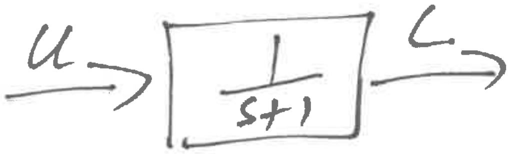
$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c} = \frac{\frac{1}{\cancel{s+1}} \cdot 2 \quad (\cancel{s+1})}{1 + \frac{1}{\cancel{s+1}} \cdot 2 \quad (\cancel{s+1})}$$

$$= \frac{2}{(s+1)+2} = \frac{2}{s+3}$$

C.L.C.E $s+3=0 \Rightarrow s=-3$

e^{-3t}

O.L. Sys



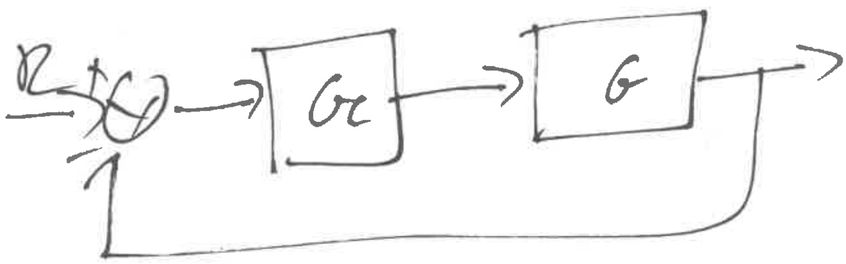
O.L.T.F. $\frac{c}{u} = \frac{1}{s+1}$

O.L. C.E $s+1=0$

O.L. poles $s=-1$

O.L. comp e^{-t}

C.L. Sys



C.L. T.F. $\frac{c}{R} = \frac{1}{s+3}$

C.L. C.E $s+3=0$

C.L. poles $s=-3$

C.L. comp e^{-3t}

$G(s) = \frac{1}{s+1}$ $\qquad G_c(s) = K$

C.L.T.F. $\quad G_{CL}(s) = \frac{K}{s+1+K} = \frac{C}{R}$

$\rightarrow$ pole at $s = -1-K$

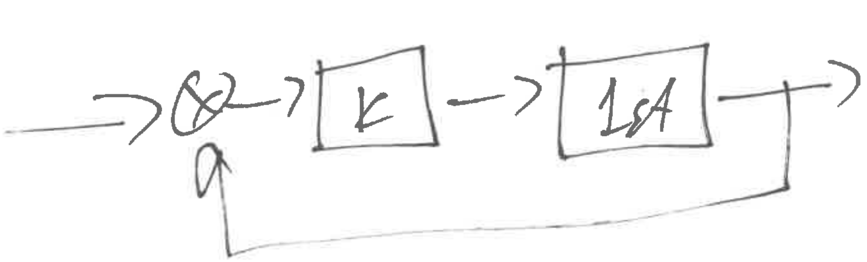
$C(s) = R \frac{K}{s+1+K}$

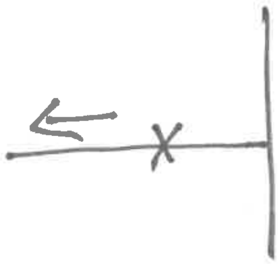
Assume $R(s) = 1/s$ $\rightarrow$ This means that desired output $= 1$

$C(s) = \frac{1}{s} \frac{K}{s+1+K}$

$C_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{K}{s+1+K} = \frac{K}{1+K}$

- $K = 1 \quad C_{ss} = \frac{1}{2} = 0.5$, pole at -2
- $K = 10 \quad C_{ss} = \frac{10}{11} = 0.9$, pole at -11
- $K = 1000 \quad C_{ss} = \frac{1000}{1001} = 0.999$, pole at -1001

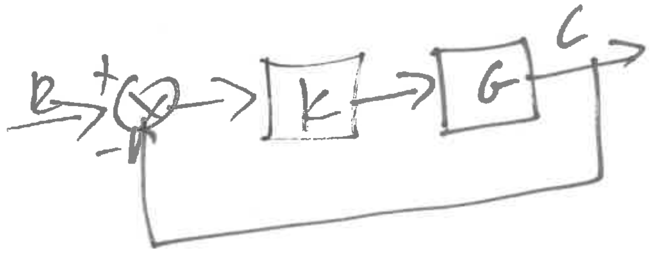


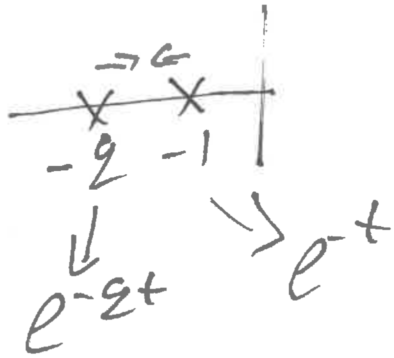


$K \uparrow$ $\rightarrow$ Faster

$\rightarrow$ smaller S.S. Error

$$G(s) = \frac{1}{(s+1)(s+2)}$$





C.L.T.F. $\quad \frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c}$

$$= \frac{K}{(s+1)(s+2) + K}$$

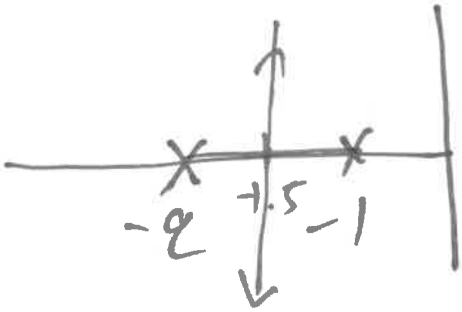
C.L.C.E $\quad (s+1)(s+2) + K = 0$

$$s^2 + 3s + 2 + K = 0$$

- $K = 0 \quad s^2 + 3s + 2 = 0 \Rightarrow s_1 = -1, \; s_2 = -2$
- $K = 0.1 \quad s^2 + 3s + 2.1 = 0 \rightarrow s_1 = -1.11, \; s_2 = -1.88$

⋮

- $K = 0.25 \quad s^2 + 3s + 2.25 = 0 \Rightarrow s_1 = s_2 = -1.5$
- $K = 1 \quad \ldots \quad \ldots \quad s = -1.5 \pm 0.86i$
- $K = K \quad \Delta = 1 - 4 \cdot K \Rightarrow s_{1,2} = \frac{-3 \pm \sqrt{1 - 4K}}{2}$



K↑ : faster sys
+ osc.
Always stable

$R = 1/s$ I want $C_{ss} \to 1$

$$C(s) = R(s)\,\frac{K}{(s+1)(s+2)+K}$$

$$C_{ss} = \lim_{s\to 0} \cancel{s}\,\frac{1}{\cancel{s}}\,\frac{K}{(s+1)(s+2)+K} = \frac{K}{K+2}$$

$K\uparrow$ $C_{ss} \to 1$

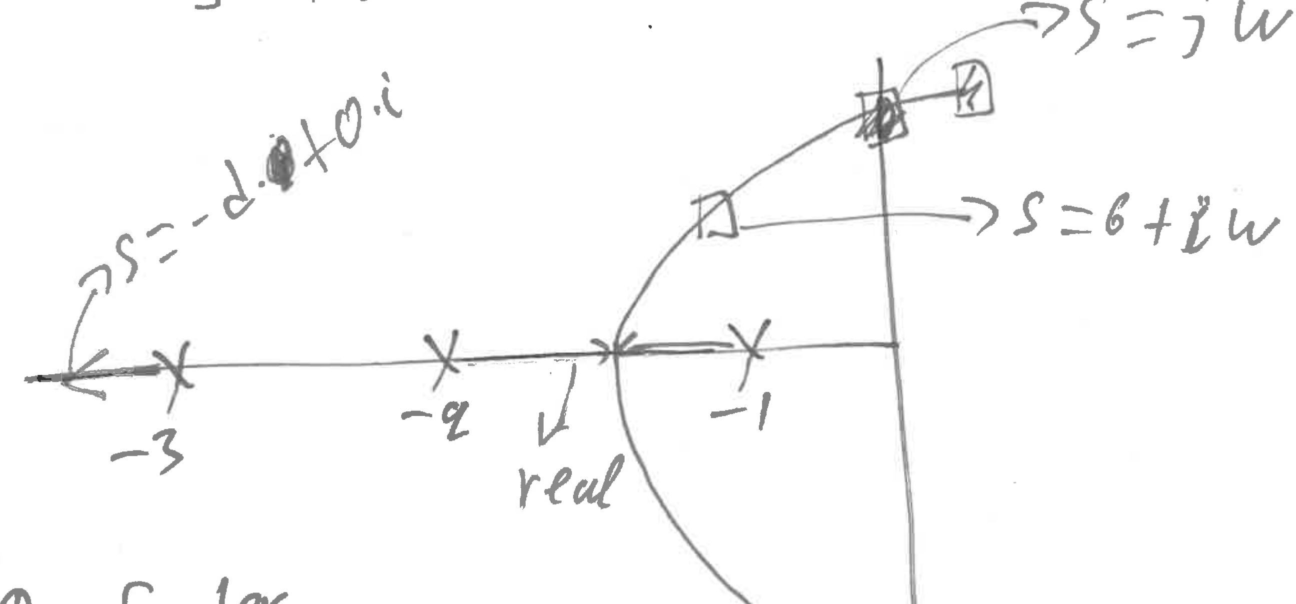
or $E_{ss} \to 0$

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)} \Rightarrow$$

$$G_{CL}(s) = \frac{K}{(s+1)(s+2)(s+3)+K}$$

C.E: $(s+1)(s+2)(s+3)+K=0$

$s^3+6s^2+11s+6+K=0$



$K\uparrow$ faster

+ osc.

+ inst.

$\cancel{C.E.}$ $s = j\omega$, $s^2 = -\omega^2$, $s^3 = -j\omega^3$

$-j\omega^3 + 6\cdot(-\omega^2) + 11\cdot j\omega + 6 + \cancel{K} = 0$

$-j\omega^3 + 11\cdot j\omega - 6\omega^2 + 6 + K = 0$

$(-\omega^3 + 11\cdot\omega)\cdot j + (-6\omega^2 + 6 + K) = 0 \Rightarrow$

$-\omega^3 + 11\cdot\omega = 0 \xrightarrow{\omega \neq 0} \omega^2 = 11$

$\omega = 3.31$ rad/s

$-6\cdot\omega^2 + 6 + K = 0$

$\Rightarrow K = 60$

$G_{OL}(s) = \frac{1}{(s+1)(s+2)}$ $\quad G_c(s) = K$

$K = ?$ $\quad \omega_n = \sqrt{12}$ rad/s

step 1: C.L.T.F.

$$\frac{C}{R} = \frac{G_{OL}\cdot K}{1 + G_{OL}\cdot K} = \frac{K}{(s+1)(s+2)+K}$$

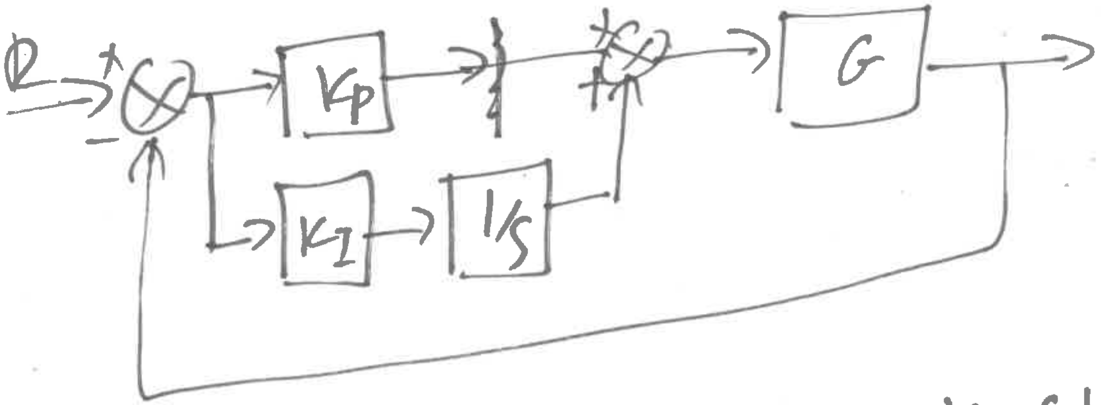
step 2: C.L.C.E $\quad s^2 + 3s + 2 + K = 0$

step 3: Gen C.E. $\quad s^2 + 2\cdot z\cdot\omega_n s + \omega_n^2 = 0$

step 4: eq. coef. $\quad 3 = 2\cdot z\cdot\omega_n$

$2 + K = \omega_n^2 = 12 \Rightarrow \boxed{K = 10}$

$3 = 2\cdot z\cdot\sqrt{12} \Rightarrow z = 0.43$



PI cont.

$$G_c(s) = K_p + K_I \frac{1}{s} = \frac{K_p s + K_I}{s}$$

$$G(s) = \frac{1}{s+1}$$

$$\frac{C}{R} = \frac{K_p s + K_I}{s(s+1) + K_p s + K_I}$$

$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c} = \frac{\frac{1}{s+1} \cdot \frac{K_p s + K_I}{s} \; s(s+1)}{1 + \frac{1}{s+1} \frac{K_p s + K_I}{s} \; s(s+1)}$$

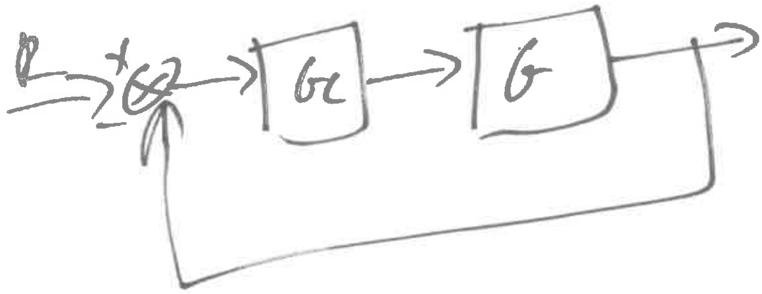
$$= \frac{K_p s + K_I}{s(s+1) + K_p s + K_I}.$$

$$R(s) = 1/s$$

$$C(s) = \frac{1}{s} \frac{K_p s + K_I}{s(s+1) + K_p s + K_I}.$$

$$C_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{\cancel{K_p s} + K_I}{s(s+1) + \cancel{K_p s} + K_I} = \frac{K_I}{K_I} = 1$$

$$G(s) = \frac{1}{s^2+11s-34}$$



$$G_c(s) = \frac{K_p s + K_I}{s}$$

$K_p = ?$ $K_I = ?$

$w_n = 6$ rad/s

$Z = 0.5$

$M_p =$
$t_p =$
instead

$$\frac{C}{R} = \frac{G\cdot G_c}{1+G\cdot G_c} = \frac{K_p s + K_I}{s(s^2+11s-34) + K_p s + K_I}$$

C.E. $s^3 + 11s^2 + s(-34+K_p) + K_I = 0$

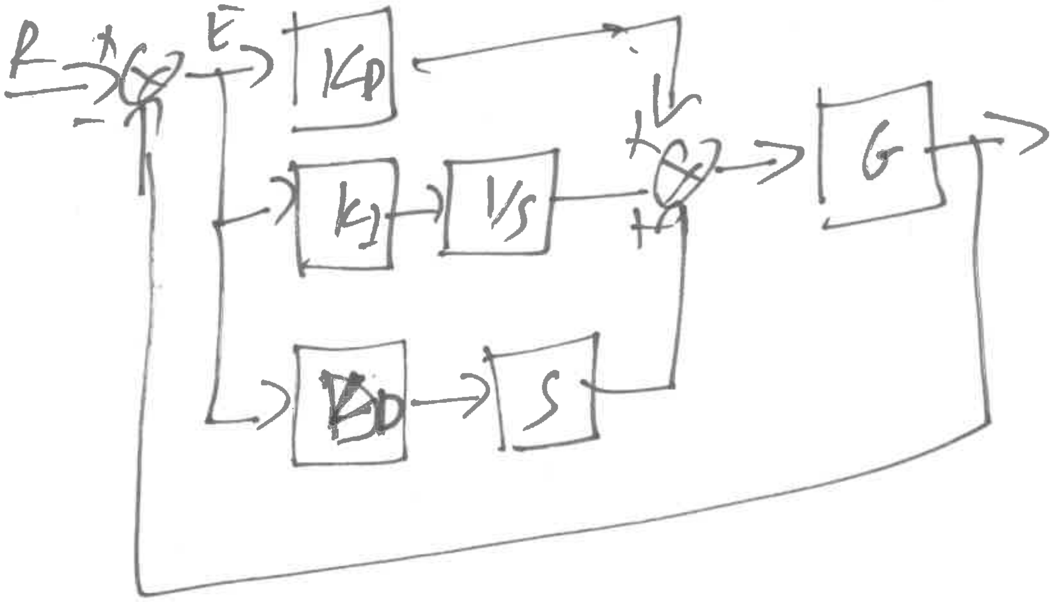
G.C.E $s^3 + (2\cdot Z\cdot w_n + a)s^2 + (2Zw_n a + w_n^2)\cdot s$

$+\, a\cdot w_n^2$

$11 = 2\cdot Z\cdot w_n + a \longrightarrow a = 5$

$-34 + K_p = 2Zw_n a + w_n^2 \longrightarrow K_p = 100$

$K_I = a\cdot w_n^2 \longrightarrow K_I = 180$


R
+
-
E
Kp
KI
1/s
KD
s
+
+
+
G

Revision

T.F. $G(s) = \frac{OUT(s)}{In(s)} = \frac{N(s)}{D(s)}$

$N(s) \rightarrow$ zeros

$D(s) \rightarrow$ poles

$In \rightarrow \boxed{G} \rightarrow out$

C.E. $D(s) = 0$

$OUT(s) = G(s) \cdot In(s)$

- A/s step
- A/s^2 ramp

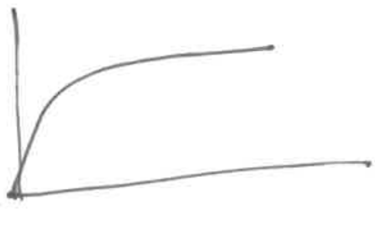
F.V.T.

$$Out_{ss} = \lim_{s \to 0} s \cdot OUT(s)$$

$$\frac{E}{R} = \frac{1}{1+G} \Rightarrow E = R \frac{1}{1+G}$$

$$E_{ss} = \lim_{s \to 0} s\, E(s)$$

1st order sys $G(s) = \frac{K}{\tau s + 1}$



2nd order sys

$$G(s) = \frac{\dots}{C.E}$$

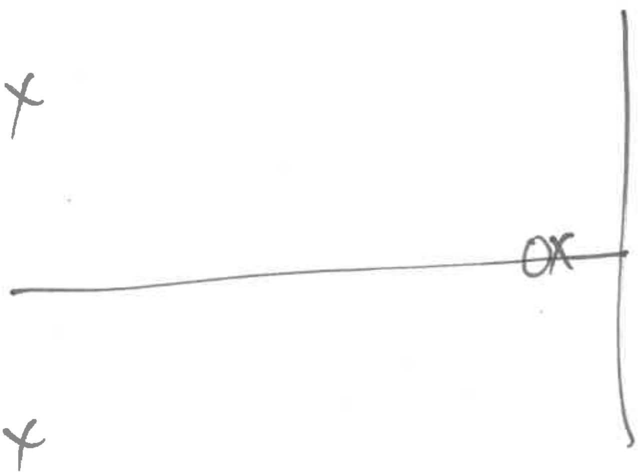
C.E. $s^2 + 2 \cdot Z \cdot w_n \cdot s + w_n^2 = 0$

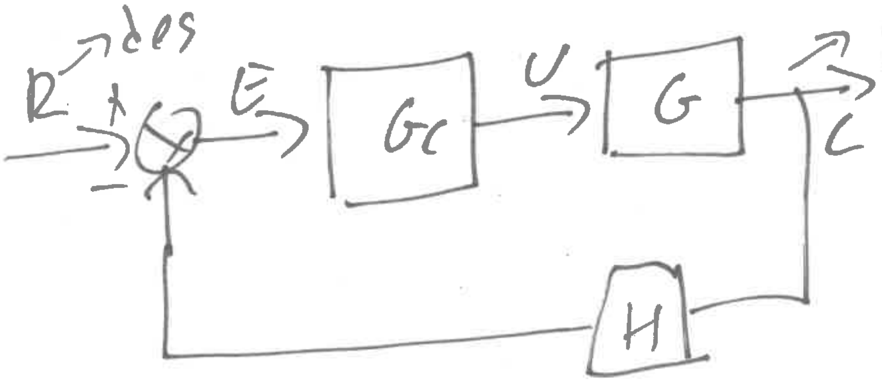
- $Z > 1$

- $Z = 1$

- $Z < 1$

- $Z = 0$





H = 1
unity feedback

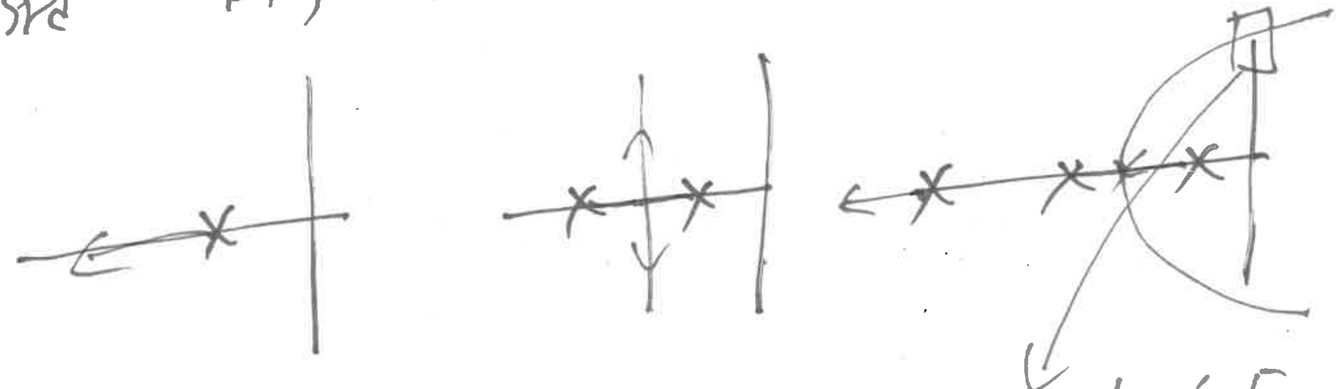
$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c \cdot H}$$

$G_c = K$

1st $K\uparrow$, faster, $E_{ss}\downarrow$

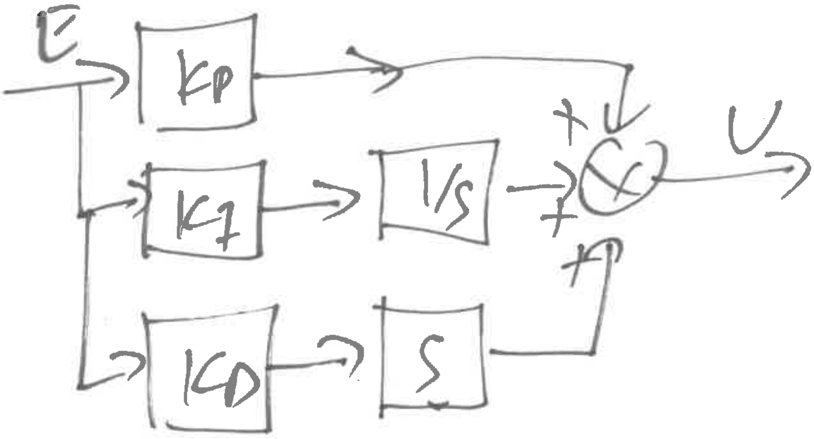
2nd $K\uparrow$, faster, $E_{ss}\downarrow$, + oscillations

3rd $K\uparrow$, faster, $E_{ss}\downarrow$, + osc + inst.



Find C.E.
$s = j\omega$

P.I.D.



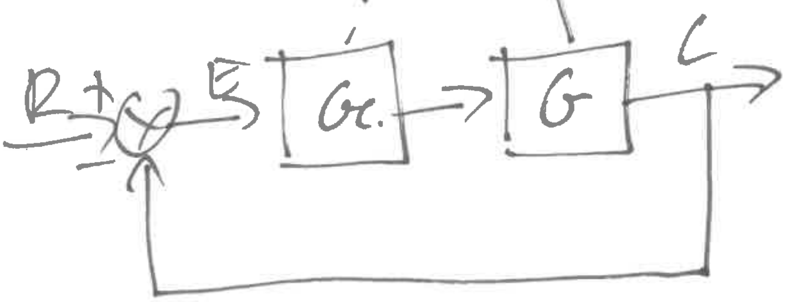
$$G_c(s) = K_P + K_I \frac{1}{s} + K_D \cdot s$$

$$= K_P\left(1 + \frac{K_I}{K_P}\frac{1}{s} + \frac{K_D}{K_P} s\right)$$

$$= K_P\left(1 + \frac{1}{T_i}\frac{1}{s} + T_d \cdot s\right)$$

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

In a unity feedback control str. Tune P.I.D.



$K_I = 0 \quad K_d = 0 \quad G_c = K_P$

$$\frac{C}{R} = \frac{K_P}{(s+1)(s+2)(s+3) + K_P}$$

C.E. $s^3+6s^2+11s+6+K=0$

$s=j\omega \Rightarrow \ldots \quad K_{cr}=60$

$\omega=3.31$ rad/s

$P_{cr}=T=\frac{2\pi}{\omega}=1.89\,s$

$K_p=0.6\cdot K_{cr}=0.6\cdot 60=36$

$T_i=0.5\cdot P_{cr}\cong 0.95$ sec

$T_d=0.125\cdot P_{cr}=0.11$ sec

$G(s)=\frac{K_1}{s^2} \qquad H(s)=s+1 \qquad K_1=?$

$M_p=0.05$

C.L.T.F. $\frac{C}{R}=\frac{G}{1+G\cdot H}=\frac{K_1/s^2}{\left(1+\frac{K_1}{s^2}\cdot(s+1)\right)}\cdot\frac{s^2}{s^2}$

$$=\frac{K_1}{s^2+K_1s+K_1}$$

C.E. $s^2+K_1s+K_1=0$

$s^2+2\cdot z\cdot \omega_n\cdot s+\omega_n^2=0$

$K_1=2\cdot z\cdot \omega_n$

$K_1=\omega_n^2$

$$M_p = \ln \exp\left(\frac{-z\cdot\pi}{\sqrt{1-z^2}}\right) = \ln(0.05)$$

$$\left(\frac{-z\cdot\pi}{\sqrt{1-z^2}}\right)^2 = (-3)^2$$

$$\frac{z^2\cdot\pi^2}{1-z^2} = 9$$

$$z^2\cdot\pi^2 = 9 - 9z^2$$

$$(\pi^2+9)\cdot z^2 = 9$$

$$z^2 = \frac{9}{\pi^2+9}$$

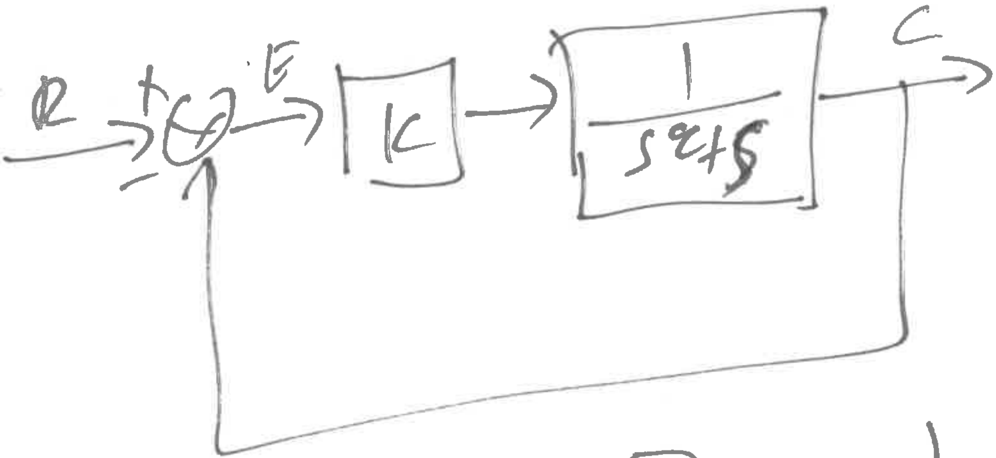
$$z = \sqrt{\frac{9}{\pi^2+9}} = 0.7 \Rightarrow K_1 = 1.96$$

$G(s) = \dfrac{1}{s^2+s} \rightarrow$ 7 am es

1) B.D. $G_c(s) = K$, unity feedback

2) Find K, $E_{ss} < 0.1$ $r(t) = t$

$R(s) = 1/s^2$



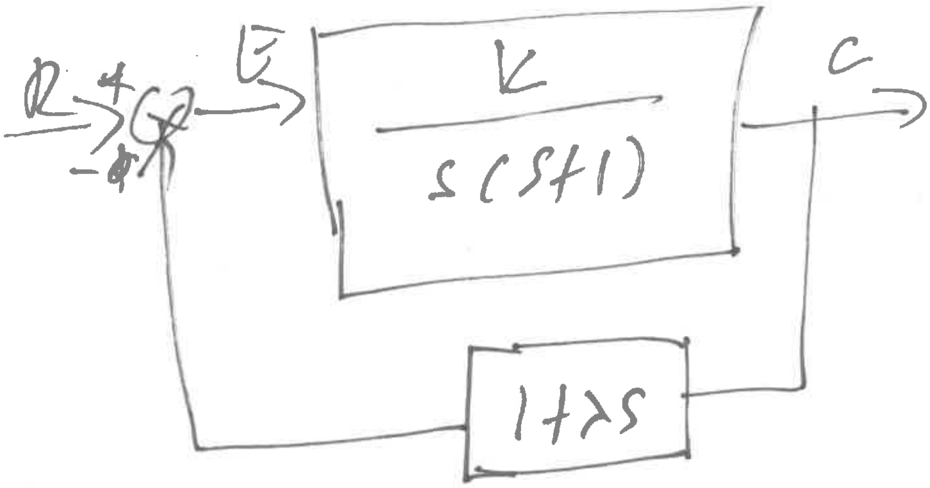
$G(s) = \dfrac{K}{s^2+s}$ $\dfrac{E}{R} = \dfrac{1}{G+1}$

$= \dfrac{1}{\left(1 + \dfrac{K}{s^2+s}\right)} \dfrac{\cdot(s^2+s)}{(s^2+s)} = \dfrac{s^2+s}{s^2+s+K}$

$E(s) = R(s) \cdot \dfrac{s^2+s}{s^2+s+K} = \dfrac{1}{s^2} \dfrac{s(s+1)}{s^2+s+K}$

$E_{ss} = \lim_{s \to 0} \cancel{s} \dfrac{1}{\cancel{s^2}} \dfrac{\cancel{s}(s+1)}{s^2+s+K} = \dfrac{0+1}{0+K} < 0.1$

$\Rightarrow \boxed{K > 10}$



$K = ?\quad \lambda = ?$

$M_p = 0.4$

$t_p = 1s$

O.L.T.F. $G(s) = \dfrac{K}{s(s+1)}$, $H(s) = 1+\lambda s$

C.L.T.F. $\dfrac{C}{R} = \dfrac{G}{1+G\cdot H} = \dfrac{\frac{K}{s(s+1)}}{1+\frac{K}{s(s+1)}(1+\lambda s)}$

$$= \frac{K}{s(s+1)+K(1+\lambda s)} = \frac{K}{s^2+(1+K\lambda)s+K}$$

C.E. $s^2 + (1+K\lambda)s + K = 0$

$s^2 + 2\cdot z\cdot w_n\cdot s + w_n^2 = 0$

$1+K\lambda = 2\cdot z\cdot w_n$ I need z, w_n

$K = w_n^2$

$M_p = 0.4 = \exp\left(\dfrac{-z\pi}{\sqrt{1-z^2}}\right) \Rightarrow \ldots\ z = 0.279$

$T_p = \dfrac{\pi}{w_d} = 1 \Rightarrow w_d = \pi$ rad/s

$w_n\cdot\sqrt{1-z^2} = \pi \Rightarrow w_n = 3.27$ rad/s

$K = 10.69,\ \lambda = 0.08$

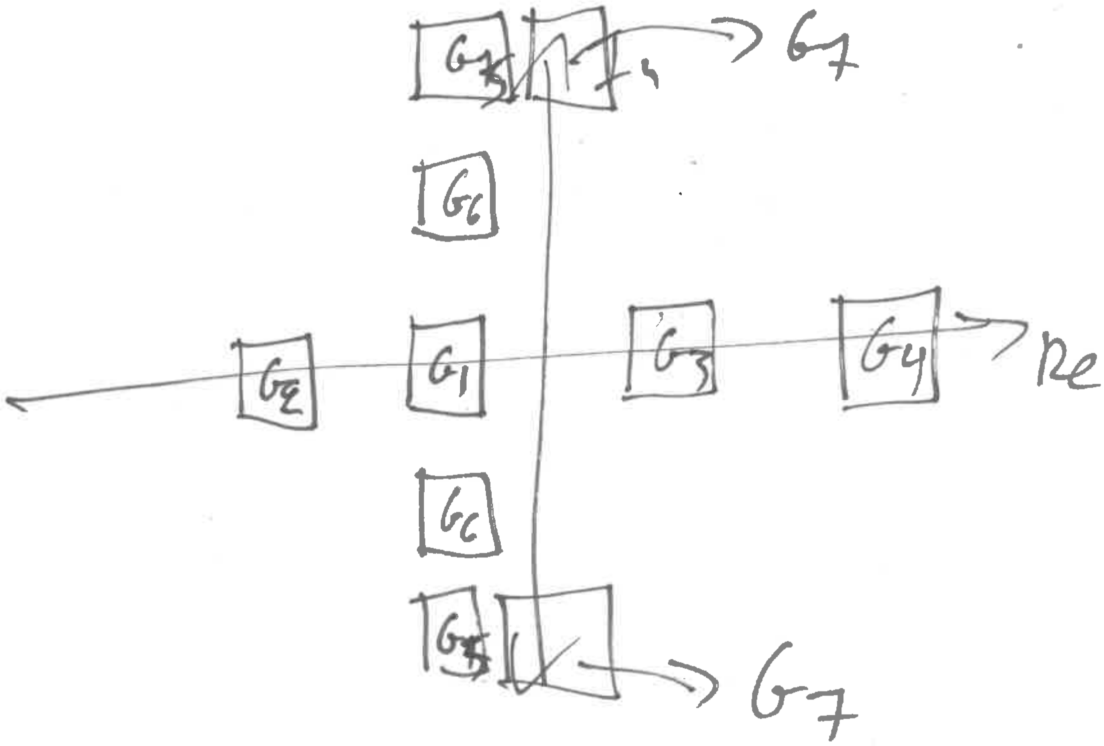
$G_1 = \frac{1}{s+1} \to -1$, $G_2 = \frac{1}{s+2} \to -2$

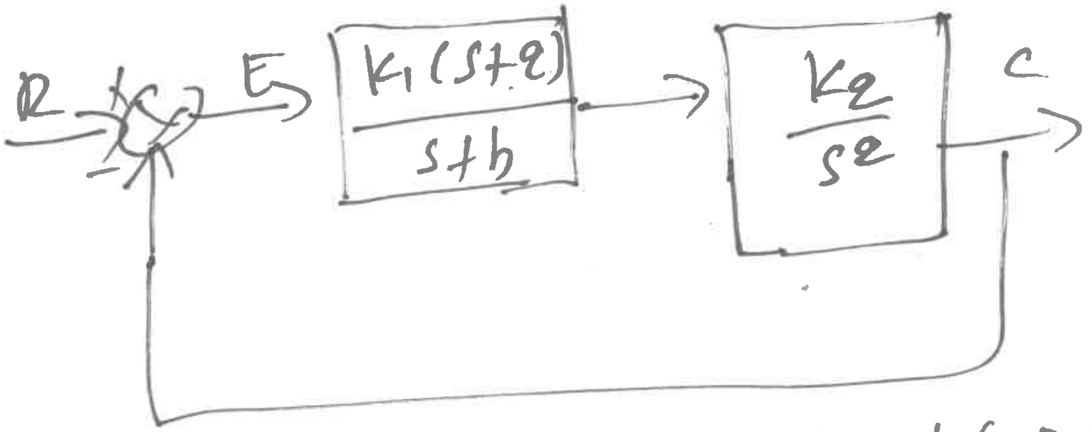
$G_3 = \frac{1}{s-1} \to 1$, $G_4 = \frac{1}{s-5} \to 5$

$G_5 = \frac{1}{s^2+2s+101}$ $\to -1 \pm 10i$

$G_6 = \frac{1}{s^2+2s+26}$ $\to -1 \pm 5i$

$G_7 = \frac{1}{s^2+100} \to 10i$





$K_1 \cdot K_2 = ?$ $b = ?$ $w_n = 6$ rad/s
$\theta = 60°$

O.L.T.F. $G(s) = \frac{K_1(s+2) \cdot K_2}{s^2(s+b)}$ $K = K_1 K_2$

$$= \frac{K \cdot (s+2)}{s^2 (s+b)}$$

C.L.T.F. $$\frac{C}{R} = \frac{K(s+2)}{s^2(s+b) + K(s+2)}$$

C.E. $s^3 + b s^2 + K s + 2K = 0$

$$s^3 + (2 \cdot z \cdot w_n + a) \cdot s^2 + (a z \cdot w_n 2 + w_n^2) \cdot s + a \cdot w_n^2 = 0$$

$b = 2 \cdot z \cdot w_n + a$

$K = a \cdot z \cdot w_n 2 + w_n^2$

$2K = a \cdot w_n^2$

$w_n = 6 \leftarrow$ given
$\theta = 60°$
$z = \cos 60 = 1/2$

$\rightarrow b = 9$
$a = 3$
$K = 54$