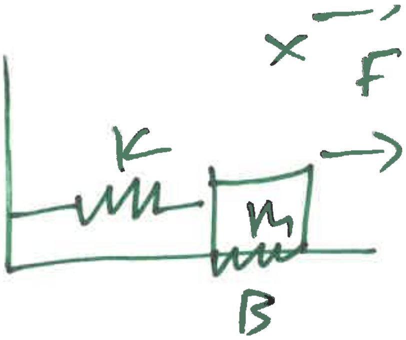
1



$$\boxed{m\ddot{x} = F - Kx - B\dot{x}}$$

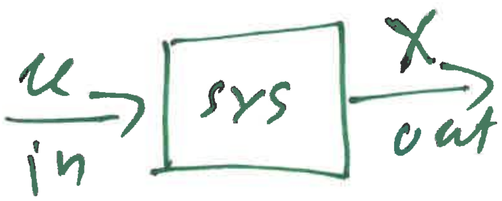
$X(t) = \ldots$

$$\dot{x} = 3x \Rightarrow \boxed{x(t) = x_0 \cdot e^{+3t}}$$

Analytical soln.

$$x_0 \cdot 3 \cdot e^{3t} = 3x_0 e^{3t}$$

F.O. D.E.



$\dot{x} + Kx = u$ Int Factor

$$x(t) = \underbrace{e^{-Kt} \cdot x_0}_{\text{I.C.}} + \underbrace{e^{-Kt} \int_0^t e^{Kt_1} u(t_1)\, dt_1}_{\text{Input}}$$

2

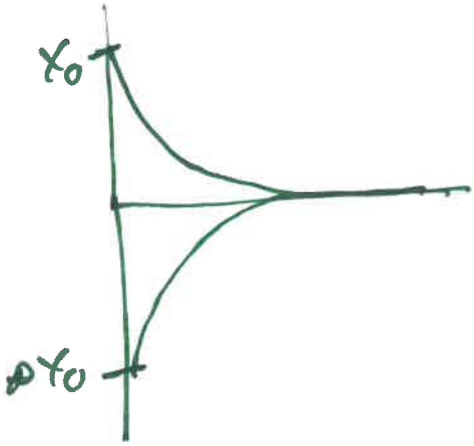
• $u = 0$

$x = e^{-kt} \cdot x_0$

$\boxed{a}$ $k > 0$, $-k < 0$ as $t \to \infty$

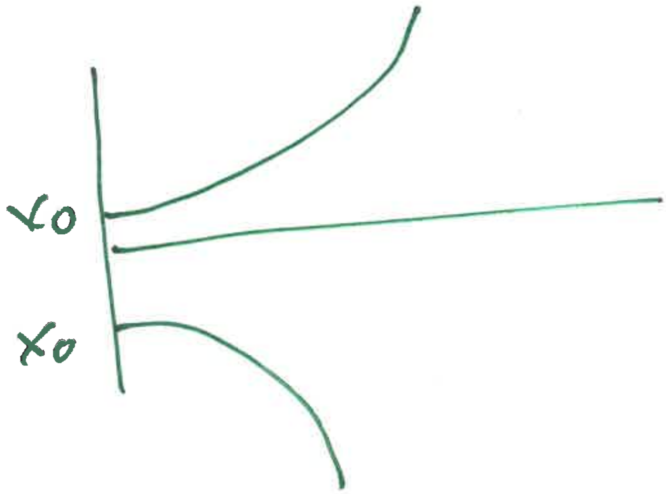
$-kt \to -\infty$

$e^{-kt} \to 0$



stable

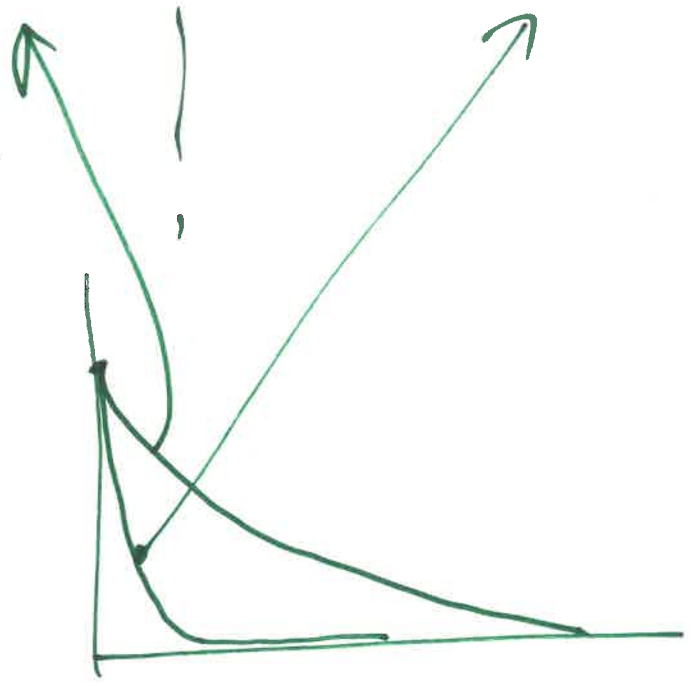
$\boxed{b}$ $k < 0$ e^{-kt} $e^{-kt} \to \pm \infty$



unstable

③

• $K = 2, x_0 = 1$ | $K = 2000000, x_0 = 1$

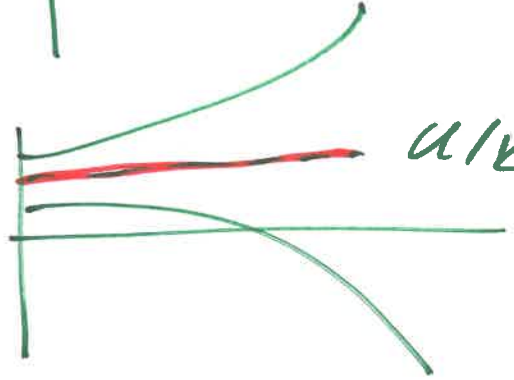


☒ $u \neq 0$, $u = const$

$$x(t) = e^{-K \cdot t} \cdot x_0 + \frac{u}{K}(1 - e^{-Kt})$$

• $K > 0$ s.s. $x(t) = \frac{u}{K}$

• $K < 0$



$$x = e^{-kt} \cdot x_0 + \frac{u}{k}\left(1 - e^{-kt}\right)$$

- $x_0 = \frac{u}{k}$

$$x(t) = \cancel{e^{-kt} \cdot \frac{u}{k}} + \frac{u}{k} \cancel{- \frac{u}{k} e^{-kt}} = \frac{u}{k}$$

e.g. $\dot{x} + 5x = 5$, $x_0 = 2$

- $k = 5 > 0 \rightarrow$ stable
- $u = 5 \neq 0$ $\quad x \rightarrow 1$

$\dot{x} = 3x + 0$ $\quad k = -3 \rightarrow U$

$\dot{x} = -3x + 0$ $\quad k = 3 \rightarrow S$

$\dot{x} = 3x + 5$ $\quad k = -3 \rightarrow u$

$\dot{x} = -3x + 5$ $\quad k = +3 \rightarrow S$

$\dot{x} + kx = 0$

$\ddot{x} + A\dot{x} + Bx = \cancel{u}$

$\downarrow u = 0$

$\ddot{x} + A\dot{x} + Bx = 0$

HOPE / PRAY

$x = e^{rt}$

$\dot{x} = r \cdot e^{rt}$

$\ddot{x} = r^2 e^{rt}$

$x = e^{rt}$

$\boxed{r = \text{Not known}}$

$r^2 \cancel{e^{rt}} + A \cdot r \cancel{e^{rt}} + B \cancel{e^{rt}} = 0$

$\boxed{r^2 + Ar + B = 0} \rightarrow$ C.E.

$$r_{1,2} = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$$

$x_1 = e^{r_1 t}$ $\qquad$ $x_2 = e^{r_2 t}$

For a 2nd order O.D.E. If you have 2. L.I. $\quad x = C_1 \cdot x_1 + C_2 x_2$

$\ddot{x} + 4\cdot\dot{x} + 3x = 0$

$x = e^{rt} \longrightarrow r^2 + 4\cdot r + 3 = 0$

$\Delta = 16 - 4\cdot 3 = 4 = 2^2$

$r_{1,2} = \dfrac{-4 \pm 2}{2}$ → $r_1 = -3$, $r_2 = -1$

$x_1 = e^{-3t}$

$x_2 = e^{-t}$

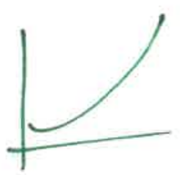
Gen soln $x = c_1 \cdot e^{-3t} + c_2 e^{-t}$

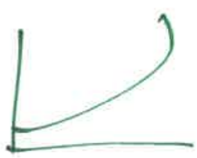
→ since r_1 and r_2 are negative → sys stable

C. V. → eigenvalues

$r_1 = 1$ $r_2 = 5$

↓ e^{t} ↓ e^{5t}





$x = c_1 e^{t} + c_2 e^{5}$

$\boxed{r = 1 \qquad r = -1}$ Saddle

$x = c_1 e^{t} + c_2 e^{-t}$

if $c_1 = 0$ $\quad x = c_2 \cdot e^{-t}$

if $c_1 = 0.000000000000000001$

$\ddot{x} + 4\dot{x} + 3x = 0$ $\qquad x_0 = 1$

$\qquad \dot{x}_0 = 0$

$x = c_1 \cdot e^{-3t} + c_2 e^{-t}$

$x(0) = c_1 \cdot 1 + c_2 \cdot 1$

$= \boxed{c_1 + c_2 = 1}$ ①

$\dot{x} = -3 c_1 e^{-3t} - c_2 e^{-t}$

$\dot{x}(0) = \boxed{-3c_1 - c_2 = 0}$ ②

$c_1 = -0.5$

$c_2 = 1.5$

$x = -0.5 e^{-3t} + 1.5 e^{-t}$ specific

$\ddot{x} + 7\dot{x} + 6x = 0$ $\qquad x_0 = 1$

$\qquad\qquad\qquad\qquad\quad \dot{x}_0 = 2$

1) Stability

2) Gen soln.

3) Spec. soln.

$r^2 + 7 \cdot r + 6 = 0 \qquad \Delta = 49 - 4 \cdot 6 = 25$

$$r_{1,2} = \frac{-7 \pm 5}{2} \begin{cases} \to r_1 = -1 \\ \to r_2 = -6 \end{cases}$$

Stable

$x = c_1 e^{-t} + c_2 e^{-6t}$ Gen soln.

$$\left.\begin{aligned} x_0 &= c_1 + c_2 = 1 \\ \dot{x} &= -c_1 e^{-t} - 6 \cdot c_2 \cdot e^{-6t} \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 8/5 \\ c_2 &= -3/5 \end{aligned}$$

$\dot{x}_0 = -c_1 - 6 \cdot c_2 = 2$

CHECK

$\ddot{x} + 2\dot{x} + x = 0 \qquad x_0 = 1,\ \dot{x}_0 = 0$ (9)

$\Downarrow$

$r^2 + 2 \cdot r + 1 = 0 \qquad \Delta = 4 - 4 = 0$

$r_{1,2} = \frac{-2 \pm 0}{2} = -1$

stable

$r = -1 \rightarrow x_1 = e^{-t}$

$x_2 = t\, e^{-t}$

$x = c_1 e^{-t} + c_2 t e^{-t}$

$x_0 = 1 \Leftrightarrow \quad c_1 + 0 = 1 \Rightarrow c_1 = 1$

$\dot{x} = -c_1 e^{-t} + c_2 e^{-t} - c_2 t e^{-t}$

$\dot{x}_0 = 0 \Leftrightarrow$

$-c_1 + c_2 + 0 = 0$

$c_2 = c_1 = 1$

$x = e^{-t} + t\, e^{-t}$

$(f \cdot g)' = f' g + f \cdot g'$

$\ddot{x} + \dot{x} + x = 0$ $\quad x_0 = 1 \quad \dot{x}_0 = 0$

C.E. $r^2 + r + 1 = 0 \quad \Delta = -3$

$$r_{1,2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

$$r_1 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \qquad r_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$$

$$r_1 = \overline{r_2}$$

$$x_1 = e^{\left(-1/2 + i\frac{\sqrt{3}}{2}\right)t}$$

$$x_2 = e^{\left(-1/2 - i\frac{\sqrt{3}}{2}\right)t}$$

$$x = c_1 \cdot x_1 + c_2 \cdot x_2$$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$x_1 = e^{(a+bi)t} = e^{at} \cdot e^{bit}$$

$$= e^{at} \cdot (\cos bt + i \sin bt)$$

$$x_1 = e^{at} \cdot \cos bt + i\, e^{at} \sin bt$$

$$x_A = Re(x_1) = e^{at} \cdot \cos bt$$

$$x_B = Im(x_1) = e^{at} \cdot \sin bt$$

$$X = C_3 \cdot X_A + C_4 \cdot X_B$$

$$= C_3\, e^{at} \cdot \cos bt + C_4\, e^{at} \cdot \sin bt$$

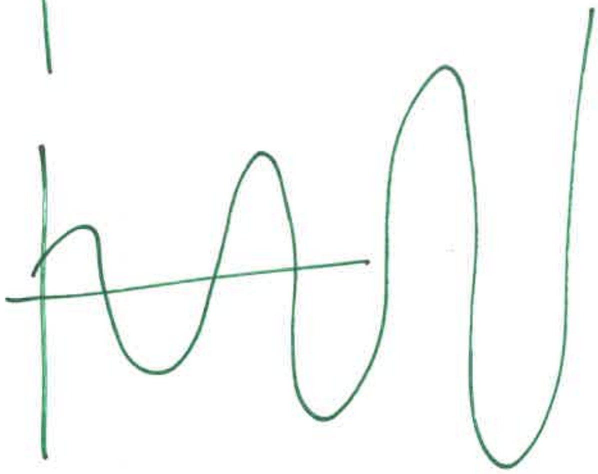
$$= e^{at} \cdot \left(C_3 \cdot \cos bt + C_4 \cdot \sin bt \right)$$

stable

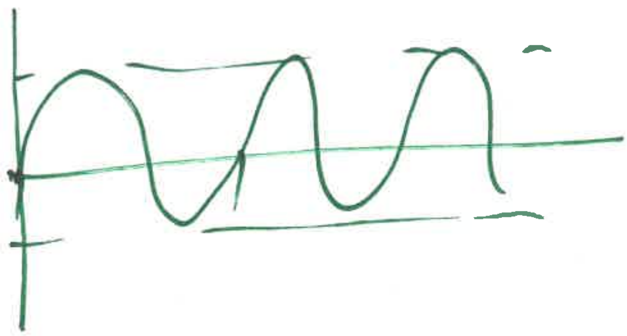
- $a < 0$

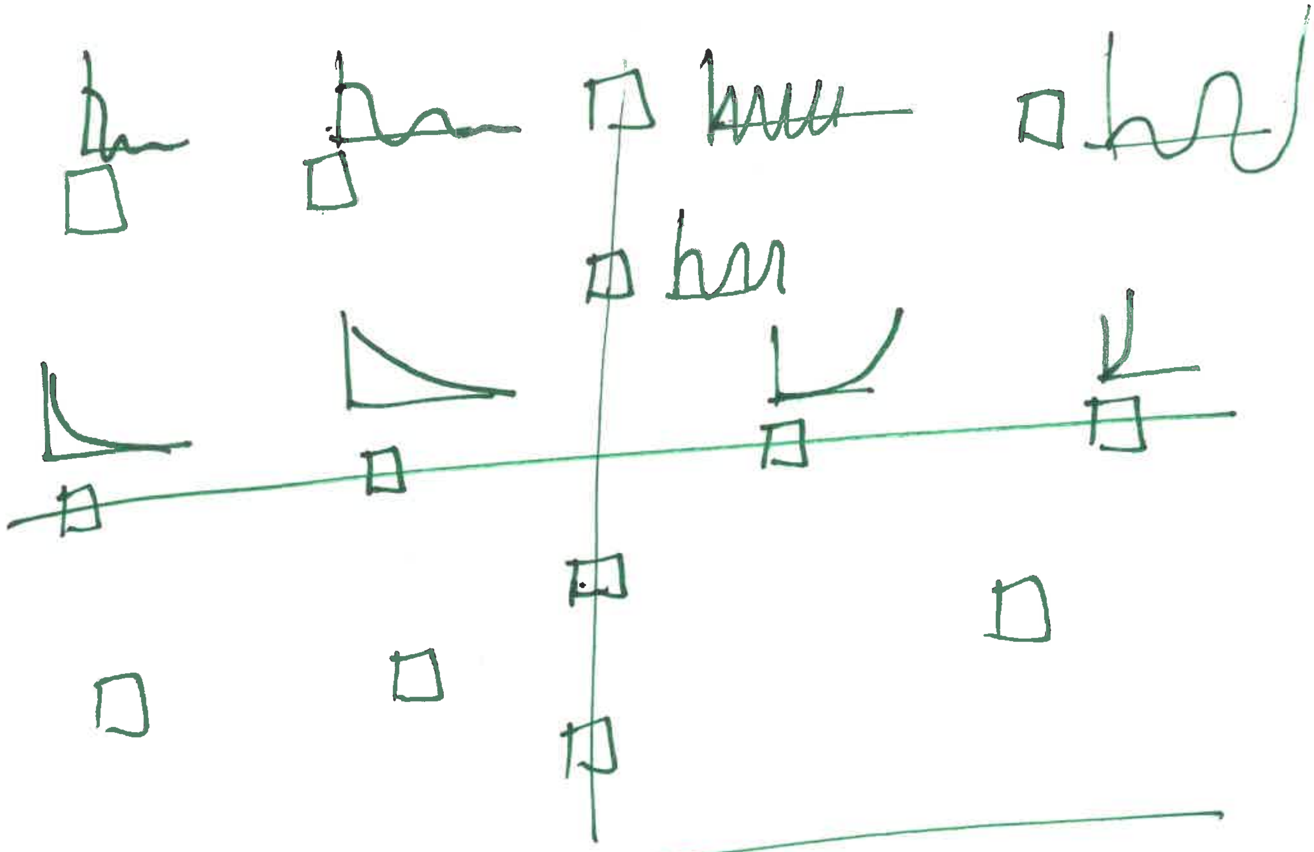
unstable

- $a > 0$



- $a = 0$





$$\ddot{x} + A\dot{x} + Bx = u$$

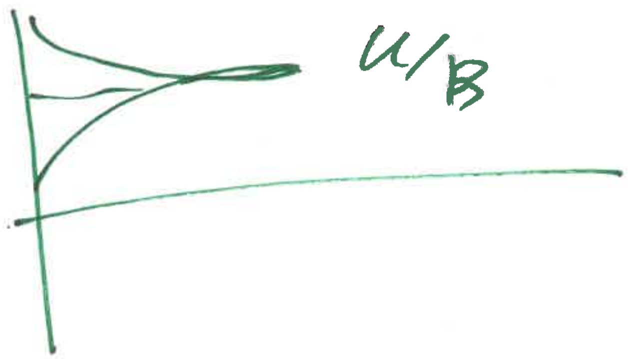
stable $\quad x_{ss} = \text{constant}$

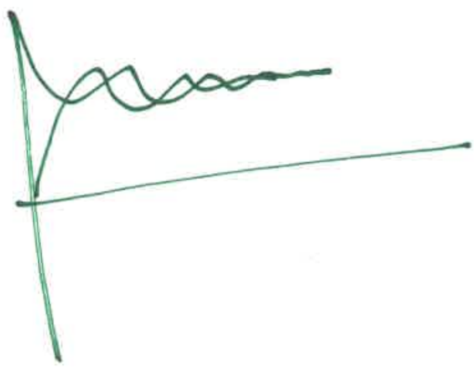
$$\dot{x}_{ss} = 0$$

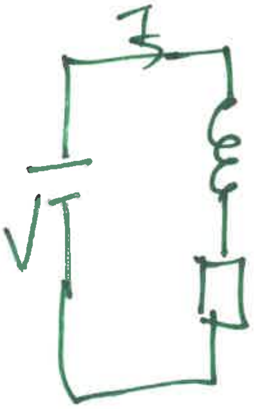
$$\ddot{x}_{ss} = 0$$

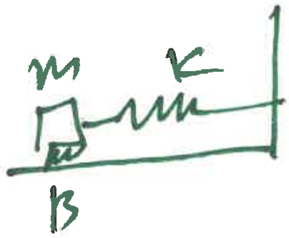
$$0 + A\cdot 0 + B\cdot x_{ss} = u$$

$$x_{ss} = \frac{u}{B}$$









$$V = IR + L\frac{dI}{dt}$$

$$F = K_A \cdot I$$

$$F - B\dot{x} - Kx = m\ddot{x}$$

change $V \rightarrow \dot{x}$? ? ? ? ?

$$\underbrace{\frac{dI}{dt}}_{x} + \underbrace{\frac{R}{L}}_{K}\underbrace{I}_{x} = \underbrace{\frac{V}{R}}_{u}$$

$$I = \frac{V}{R}\left(1 - e^{-R/L\,t}\right)$$

$$F = m\ddot{x} + B\dot{x} + Kx$$

$$\ddot{x} + \frac{B}{m}\dot{x} + \frac{K}{m}x = F/m = \frac{K_A \cdot I}{m}$$

Laplace Transform

$$f(t) \to F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$s = a + bi$$

$$f(t) \longrightarrow F(s)$$

$$1 \to 1/s$$

$$t \to 1/s^2$$

$$\dot{f}(t) = s \cdot F(s) \quad + \cancel{I.C.} \to 0$$

$$f''(t) = s^2 F(s)$$

$$\vdots$$

$$f^{(k)}(t) = s^k F(s)$$

$\dot{x} + 5x = u \xrightarrow[x=e^{rt}]{u=0} r+5=0$

C.E.

$u(t) \longrightarrow U(s)$

$x(t) \longrightarrow X(s)$

$\dot{x}(t) \longrightarrow s\,X(s)$

$s \cdot X(s) + 5 \cdot X(s) = U(s)$

$X(s)\,(s+5) = U(s)$

$$\frac{X(s)}{U(s)} = \frac{1}{s+5}$$

T.F. between U(s) and X(s)

$\hookrightarrow s+5=0$ C.E.

$$V = iR + L\frac{di}{dt}$$

$$F = m\ddot{x} + B\dot{x} + Kx = K_A \cdot i$$

$V(t) \rightarrow V(s)$

$i(t) \rightarrow I(s)$

$i'(t) \rightarrow sI(s)$

$$V(s) = I(s) \cdot R + LsI(s)$$

$$\frac{I(s)}{V(s)} = \frac{1}{Ls+R} \quad \text{or} \quad I(s) = V(s)\frac{1}{Ls+R}.$$

$x \rightarrow X(s)$

$\dot{x} \rightarrow sX(s)$

$\ddot{x} \rightarrow s^2X(s)$

$$K_A I(s) = m \cdot s^2 X(s) + B \cdot sX(s) + KX(s)$$

$$= (ms^2 + Bs + K) \cdot X(s)$$

$$\frac{V(s) \cdot K_A}{Ls+R} = X(s) \cdot (ms^2 + Bs + K)$$

$$\frac{X(s)}{V(s)} = \frac{K_A}{(Ls+R)(ms^2+Bs+K)}$$

C.E. $(Ls+K)(ms^2+Bs+K)=0$ (17)

$s_1 = \ldots$

$s_2 = \ldots$

$\dddot{x} + 5\ddot{x} + 3\dot{x} + x = u(t) + 3\dot{u}(t)$

$x \rightarrow X(s)$

$\dot{x} \rightarrow sX(s)$

$\ddot{x} \rightarrow s^2 X(s)$

$\dddot{x} \rightarrow s^3 X(s)$

$u \rightarrow U(s)$

$\dot{u} \rightarrow sU(s)$

$s^3 \cdot X(s) + 5s^2 X(s) + 3 \cdot sX(s) + X(s) = U(s) + 3sU(s)$

$X(s)(s^3 + 5s^2 + 3s + 1) = U(s)(s+3)$

$$\frac{X(s)}{U(s)} = \frac{s+3}{s^3+5s^2+3s+1}$$

$$f(t) \rightarrow F(s).$$

$$\lim_{t \to \infty} f(t)$$

$$F(s) \rightarrow F_{ss} = \lim_{s \to 0} s \cdot F(s)$$

Revision

$$\dot{x} + kx = u \Rightarrow x = e^{-kt}x_0 + e^{-kt}\int_0^t e^{kt_1}u(t_1)\,dt_1$$

$u = \text{const.}$

$$\Rightarrow x = e^{-kt}x_0 + \frac{u}{k}\left(1 - e^{-kt}\right)$$

- $k > 0$ $\quad x \to u/k$
- $k < 0$ $\quad x \to \pm\infty$

$k \to$ stability
$\;\;\to$ speed.

$x_0 \to$ I.C.

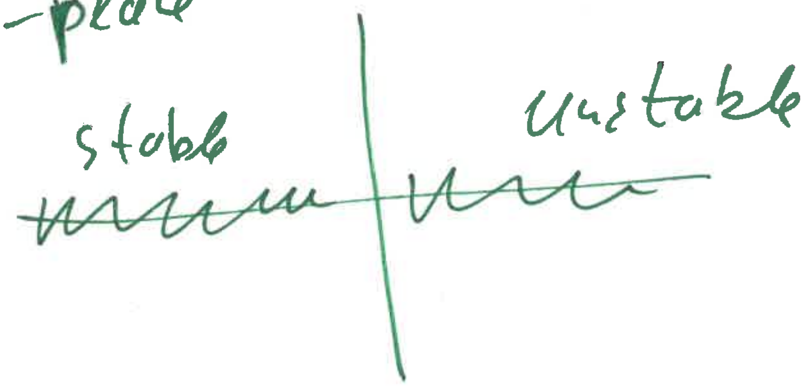
$u \to$ s.s.

$u = 0$

Try $x = e^{rt}$ $\quad r \to$ eigenvalue.

$r + k = 0 \Rightarrow r = -k$

r-plane



$\ddot{x} + A\dot{x} + Bx = u.$

$\downarrow u = 0$

$\ddot{x} + A\dot{x} + Bx = 0$, Try $x = e^{rt}$

$r^2 + Ar + B = 0 \Rightarrow \Delta = A^2 - 4B$

$r_1 \to x_1$, $r_2 \to x_2$

$x = c_1 x_1 + c_2 x_2.$

- $\Delta > 0$ $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$

  $x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$

- $\Delta = 0$ $r_1 = r_2 = r \in \mathbb{R}$

  $x_1 = e^{rt}$, $x_2 = t e^{rt}$

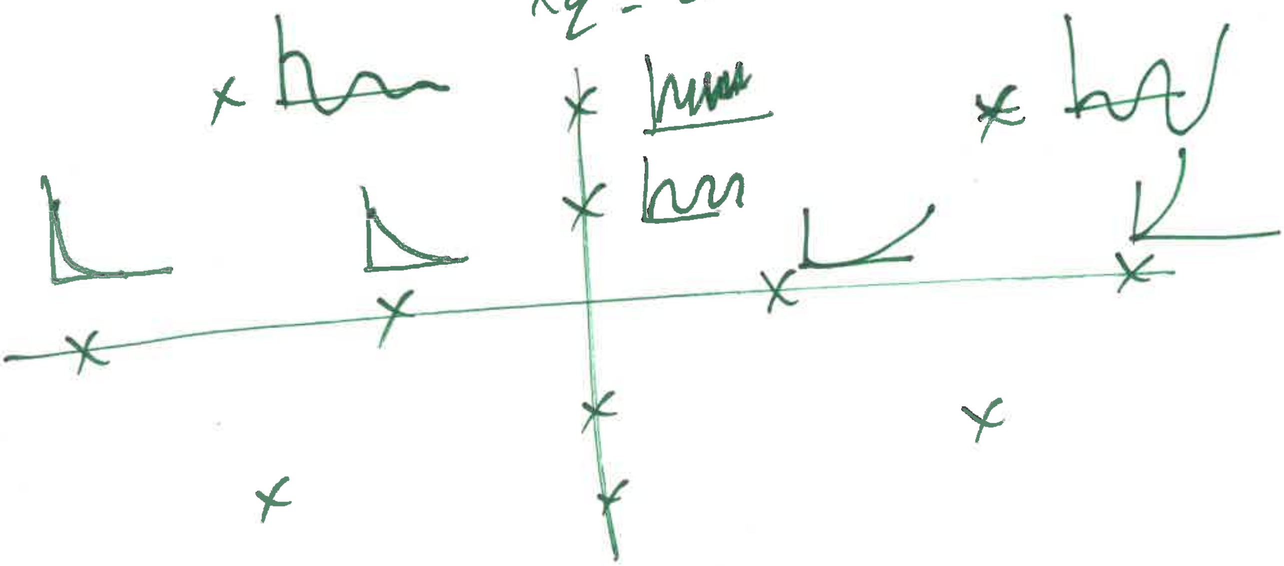
- $\Delta < 0$ $r_1 = a + bi$

  $r_2 = a - bi$

  ⊙ $x_1 = e^{r_1 t}$ $x_2 = e^{r_2 t}$

  ⊙ $x_1 = e^{at} \cdot \cos bt$

  $x_2 = e^{at} \cdot \sin bt$



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$\dddot{x} + A\ddot{x} + B\dot{x} + Cx = 0$

C.E $\quad \downarrow e^{rt}$

$r^3 + Ar^2 + Br + C = 0 \Rightarrow r_1 = \ldots$

$r_2 = \ldots$

$r_3 = \ldots$

e.g. $r_1 = -1 \quad r_2 = -2 \quad r_3 = 3$

$x = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{3t}$

L.T.

$f(t) \rightarrow F(s)$

$\dot{f}(t) \rightarrow s \cdot F(s)$

$\ddot{f}(t) \rightarrow s^2 \cdot F(s)$

$\vdots \qquad \vdots$

$f_{ss} = \lim_{t \to \infty} f(t)$

$F_{ss} = \lim_{s \to 0} s F(s)$

$\downarrow$

T.F. $\dfrac{Out(s)}{In(s)} = \dfrac{P(s)}{\boxed{Q(s)}}$

$\hookrightarrow Q(s) = 0$ C.E.

$P(s) = 0 \Rightarrow s = \ldots$ zeros

$Q(s) = 0 \Rightarrow s = \ldots$ poles

$$\frac{X(s)}{V(s)} = \frac{KA}{(Ls+R)(ms^2+Bs+K)}$$

C.E. $(Ls+R)(ms^2+Bs+K)=0$

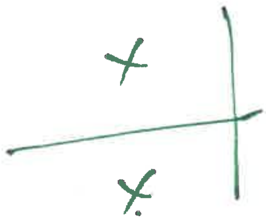
- $Ls+R=0 \Rightarrow s=-R/L$

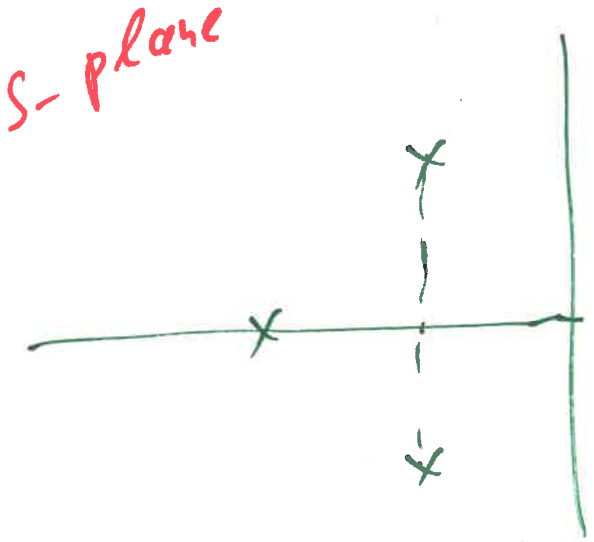
- $ms^2+Bs+K=0$

$m=B=K=1$

$s^2+s+1=0$

$\Delta=-3 \qquad s_{1,2}=\frac{-1\pm i\sqrt{3}}{2}$





if $v(t)=$ const $\qquad X_{ss}=?$

$v(t)=A \rightarrow V(s)=\frac{A}{s}$

$$X(s)=V(s)\frac{KA}{(Ls+R)(ms^2+Bs+K)}$$

$$X_{ss} = \lim_{s\to 0} s\, X(s)$$

$$= \lim_{s\to 0} \cancel{s} \cdot \frac{A}{\cancel{s}} \; \frac{KA}{(\underbrace{sL}_{0}+R)(\underbrace{ms^2+Bs}_{0}+K)}$$

$$= \frac{A \cdot KA}{R \cdot K}$$

$$\dddot{x}(t) + \ddot{x}(t)\cdot 5 + 6\cdot \dot{x}(t) + 7\cdot x(t) = u(t) + 3\dot{u}(t)$$

$\downarrow$ solve ODE

$u(t) = A$

$U(s) = \frac{A}{s}$

$r_1 = \dots \rightarrow x(t) = \dots$

$\lim_{t\to\infty} x(t)$

$x(t) \rightarrow X(s)$

$\dot{x}(t) \rightarrow X(s)\cdot s$

$\ddot{x} \rightarrow s^2 X(s)$

$\dddot{x} \rightarrow s^3 X(s)$

$u(t) \rightarrow U(s)$

$\dot{u} \rightarrow s\cdot U(s)$

$$s^3\cdot X(s) + 5\cdot s^2 \cdot X(s) + 6\cdot s\, X(s) + 7\, X(s) = U(s) + 3\cdot s\, U(s)$$

$$X(s)\cdot(s^3+5\cdot s^2+6\cdot s+7) = U(s)\cdot(3s+1)$$

$$\frac{X(s)}{U(s)} = \frac{3s+1}{s^3+5s^2+6s+7}$$

$$X(s) = U(s)\cdot$$

$$X_{ss} = \lim_{s\to 0} \cancel{s}\, \frac{A}{\cancel{s}}\, \frac{3s+1}{s^3+5s^2+6s+7} = A/7$$

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$$\ddot{x} + 5\dot{x} + x = u(t) + 3\cdot\dot{u}(t)$$

$u(t) = 5$ $\qquad$ $x_{ss} = \ldots$

$$U(s) = \frac{5}{s}$$

$x(t) \rightarrow X(s)$

$\dot{x} \rightarrow s\,X(s)$

$\ddot{x} \rightarrow s^2\cdot X(s)$

$u(t) \rightarrow U(s)$

$\dot{u} \rightarrow s\cdot U(s)$

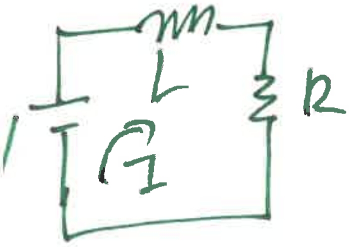
$$s^2\cdot \underline{X(s)} + 5\cdot \underline{X(s)}\cdot s + \underline{X(s)} = \underline{U(s)} + 3\cdot s\,\underline{U(s)}$$

$$X(s)\,(s^2 + 5\cdot s + 1) = U(s)\,(3s+1)$$

$$\frac{X(s)}{U(s)} = \frac{3s+1}{s^2+5s+1}$$

$$x_{ss} = \lim_{s\to 0} s\; U(s)\,\frac{3s+1}{s^2+5s+1}$$

$$= \lim_{s\to 0} \cancel{s}\,\frac{5}{\cancel{s}}\,\frac{\cancel{3s}+1}{\cancel{s^2}+\cancel{5s}+1} = 5\,\frac{1}{1} = 5$$



$$V = iR + L\,\frac{di}{dt}$$

$$\frac{di}{dt} + i\underbrace{\left(\frac{R}{L}\right)}_{K} = \underbrace{\frac{V}{L}}_{u}$$

$$i_{ss} = \frac{u}{K} = \frac{V/\cancel{L}}{R/\cancel{L}} = \frac{V}{R}$$

$v = \text{cost}$

$v \to$ step function

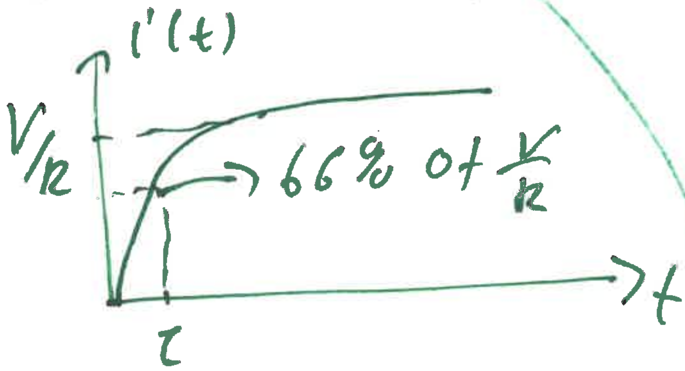
$v = V$

$$i(t) = \frac{V}{R}\left(1 - e^{-R/L\,t}\right)$$

$t = 0 \qquad e^{0} = 1 \Rightarrow i(0) = 0$

$t = \infty \qquad e^{-\infty} = 0 \Rightarrow i(\infty) = \frac{V}{R}$

$$\tau = \frac{L}{R}$$



$$V(s) = I(s)\cdot R + s\cdot L\,I(s) = I(s)\cdot(sL+R)$$

$$\frac{I(s)}{V(s)} = \frac{1}{SL+R} \Rightarrow I(s) = V(s)\,\frac{1}{LS+R}$$

C.E. $SL + R = 0$

$\Rightarrow s = -R/L$

$$I_{ss} = \lim_{s\to 0} \cancel{s}\,\frac{V}{\cancel{s}}\cdot\frac{1}{\cancel{LS}+R} = \frac{V}{R}$$

2nd order sys

$$m\ddot{x} + Kx + B\dot{x} = F$$

$$m \cdot s^2 \cdot X(s) + B \cdot s\, X(s) + K\, X(s) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + Bs + K} = \frac{1/m}{s^2 + \frac{B}{m}s + \frac{K}{m}}$$

C.E. $s^2 + \frac{B}{m}s + \frac{K}{m} = 0$

$\Downarrow$

$$s^2 + 2 \cdot Z \cdot w_n s + w_n^2 = 0$$

Gen form of 2nd order C.E.

$Z \rightarrow$ damping factor.

$w_n \rightarrow$ natural freq.

$$s^2 + 2\cdot Z\cdot w_n\cdot s + w_n^2 = 0$$

$$\Delta = 4\cdot Z^2\cdot w_n^2 - 4\cdot w_n^2$$

$$= 4\cdot w_n^2\cdot(Z^2-1)$$

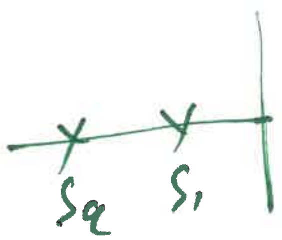
- $Z > 1$ $\quad \Delta > 0$

$$S_{1,2} = \frac{-2\cdot Z\cdot w_n \pm \sqrt{4\cdot w_n^2 (Z^2-1)}}{2}$$

$$= \frac{-\cancel{2}\cdot Z\cdot w_n \pm \cancel{2}\cdot w_n \sqrt{Z^2-1}}{\cancel{2}}$$

$$= -Z\cdot w_n \pm w_n\sqrt{Z^2-1}$$

$S_1 < 0 \qquad S_2 < 0$



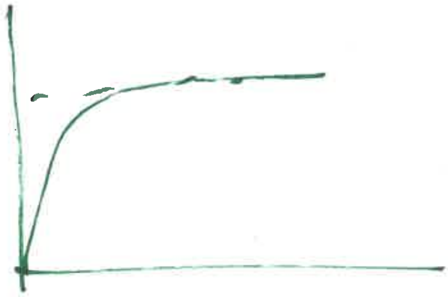
over damped

$\Delta = 0 \quad z = 1$

$$\Delta = 4 \cdot w_n^2 (z^2 - 1)$$

$$s_{1,2} = \frac{-2 \cdot z \cdot w_n \pm 0}{2} = -z \cdot w_n = -w_n$$

as $z = 1$



critically

$$\Delta = 4 \cdot w_n^2 (z^2 - 1)$$

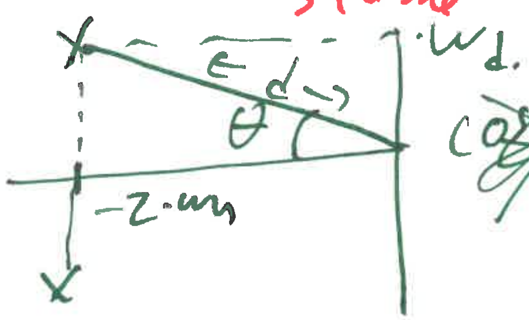
$z \in (0, 1)$

$$s_{1,2} = -z \cdot w_n \pm w_n \sqrt{z^2 - 1}$$

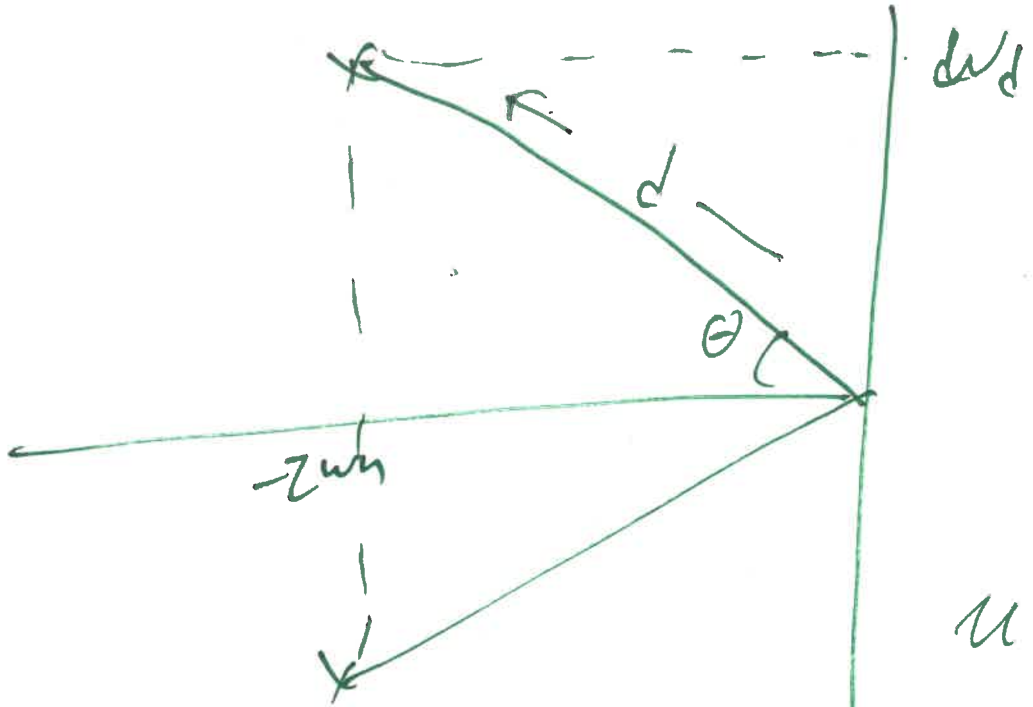
$$= \underbrace{-z \cdot w_n}_{Re} \pm \underbrace{w_n \cdot i \sqrt{1 - z^2}}_{Im}$$

$$= \underbrace{-z \cdot w_n}_{\text{stable}} \pm w_d \cdot i$$

$\rightarrow$ damping freq.



$\cancel{\cos \frac{w_d}{d}}$



underdamped

$$\cos\theta = \frac{\zeta\cdot\omega_n}{d}$$

$$d^2 = \zeta^2\cdot\omega_n^2 + \omega_d^2$$

$$= \zeta^2\omega_n^2 + \omega_n^2\cdot(1-\zeta^2)$$

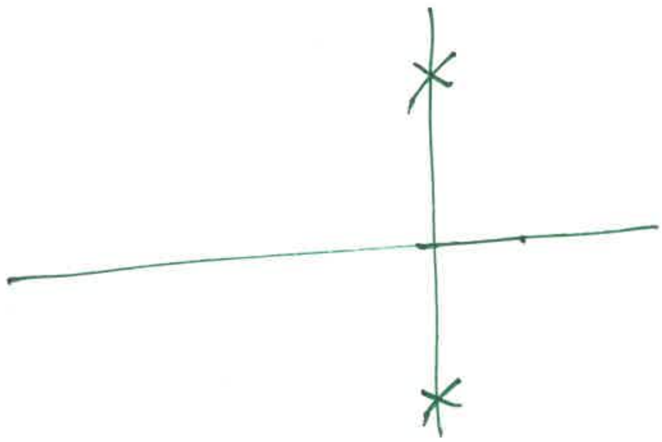
$$= \cancel{\zeta^2\cdot\omega_n^2} + \omega_n^2 - \cancel{\zeta^2\cdot\omega_n^2}$$

$$d = \omega_n$$

$$\cos\theta = \frac{\zeta\cdot\omega_n}{\omega_n} = \zeta.$$

$Z=0$ $\quad s_{1,2} = -Z\cdot\omega_n \pm \omega_n \cdot i\sqrt{1-Z^2}$

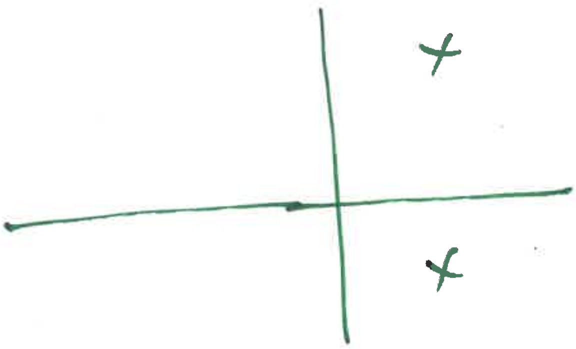
$= 0 \pm \omega_n i$

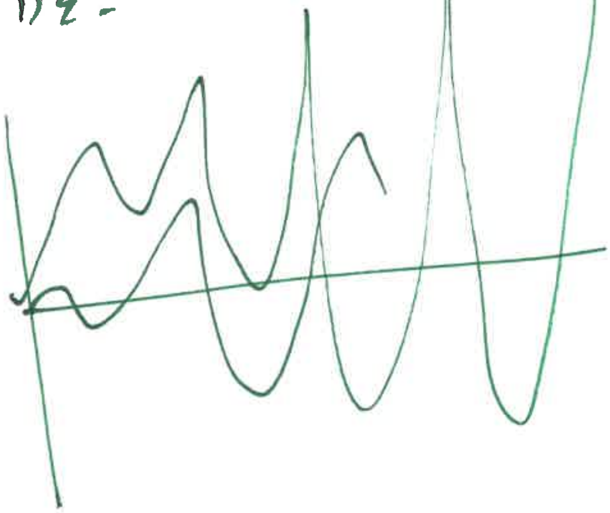


marginally stable

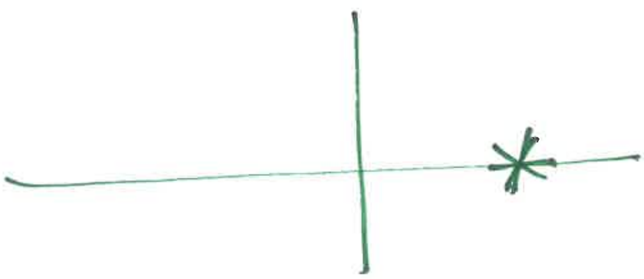
$Z \in (-1, 0)$

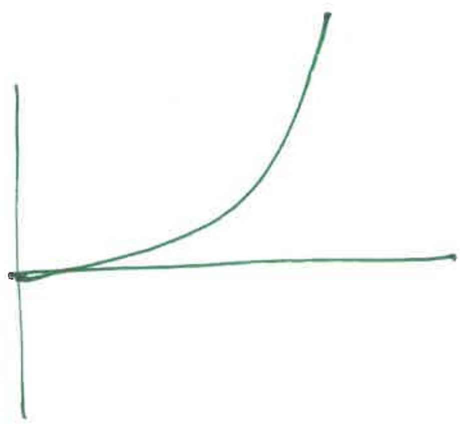
$s_{1,2} = -Z\cdot\omega_n \pm \omega_n i\sqrt{1-Z^2}$





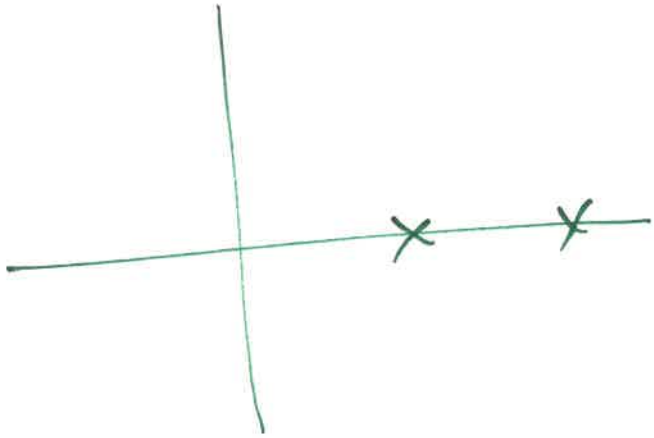
$Z = -1.$ $\quad s_{1,2} = \omega_n$

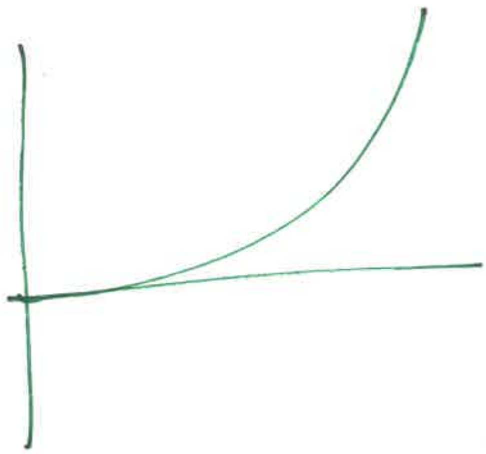




$\zeta > 1$

$$s_{1,2} = -\zeta \cdot \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$





C.E. $s^2 + 2 \cdot \zeta \cdot \omega_n s + \omega_n^2 = 0$

$$s_{1,2} = \frac{-2 \cdot \zeta \omega_n \pm \sqrt{4\zeta^2 \cdot \omega_n^2 - 4\omega_n^2}}{2}$$

$$= -\zeta \omega_n \pm \sqrt{\omega_n^2 (\zeta^2 - 1)}$$

$$= -\zeta \cdot \omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$$

(32)

$$G(s) = \frac{3s+1}{s^2+s+5} \qquad u=1 \qquad U(s) = 1/s$$

$t_p = ?$ $\quad t_r = ?$

$M_p = ?$ $\quad t_{s_{2\%}} = ?$

S.S. out.

Draw

1st Find Z, ω_n

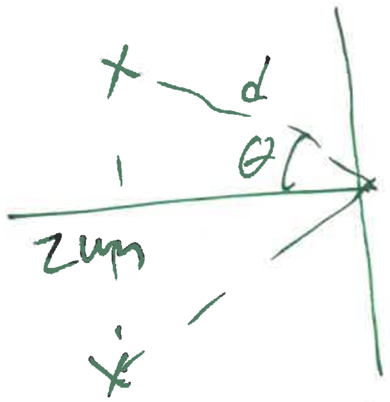
C.E. $s^2+s+5=0$

$s^2+2 \cdot Z \cdot \omega_n \cdot s + \omega_n^2 = 0$ $\Big\} = 1$

$2 \cdot Z \cdot \omega_n = 1$

$\omega_n^2 = 5 \Rightarrow \omega_n = 2.23$ rad/s

$\Rightarrow Z = 0.223$



$\cos\theta = Z$

$\theta = \cos^{-1} Z = \ldots$

$\omega_d = \omega_n \sqrt{1-Z^2} = 2.179$ rad/s

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{2.179} = 1.44\,s$$

$$t_s = \frac{4}{Z \cdot \omega_n} = 8.04\,s$$

$$M_p = \exp\left(\frac{-Z\pi}{\sqrt{1-Z^2}}\right) = 48.6\%$$

0.486

$$G(s) = \frac{3s+1}{s^2+s+5}$$

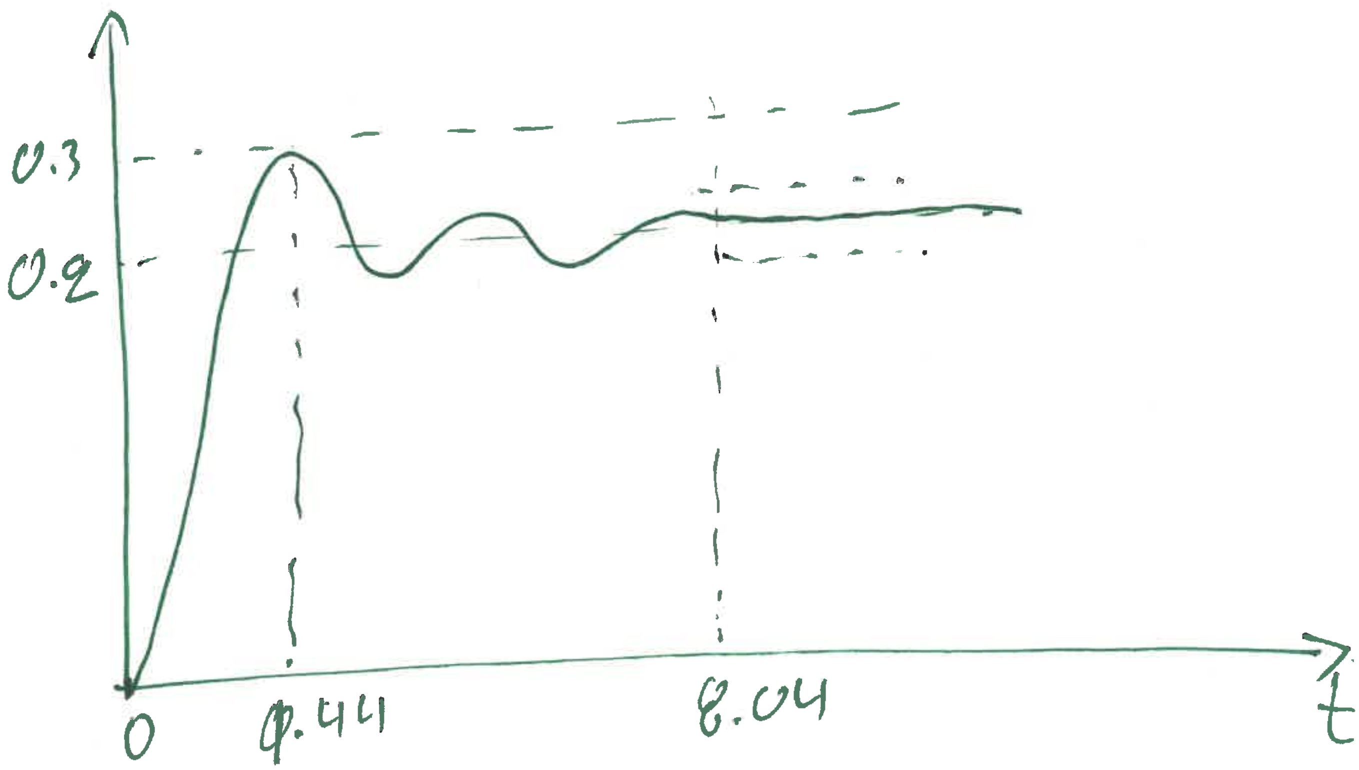
34

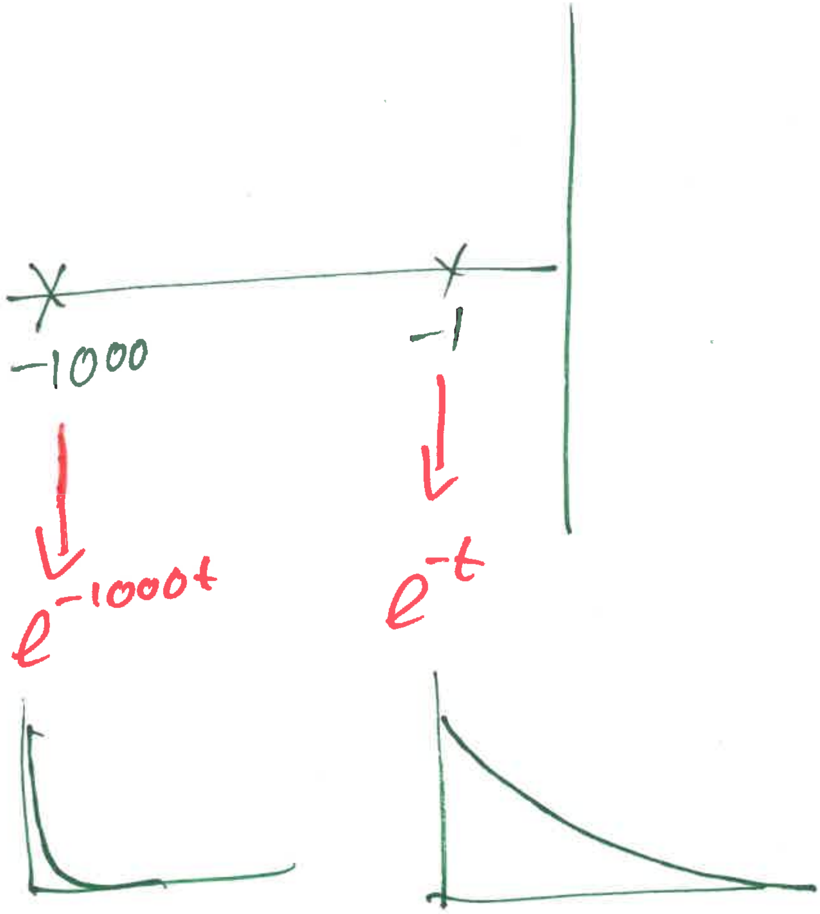
$$\frac{Out}{In} = \frac{3s+1}{s^2+s+5} = \frac{C(s)}{U(s)}$$

$$C(s) = \frac{1}{s} \cdot \frac{3s+1}{s^2+s+5}$$

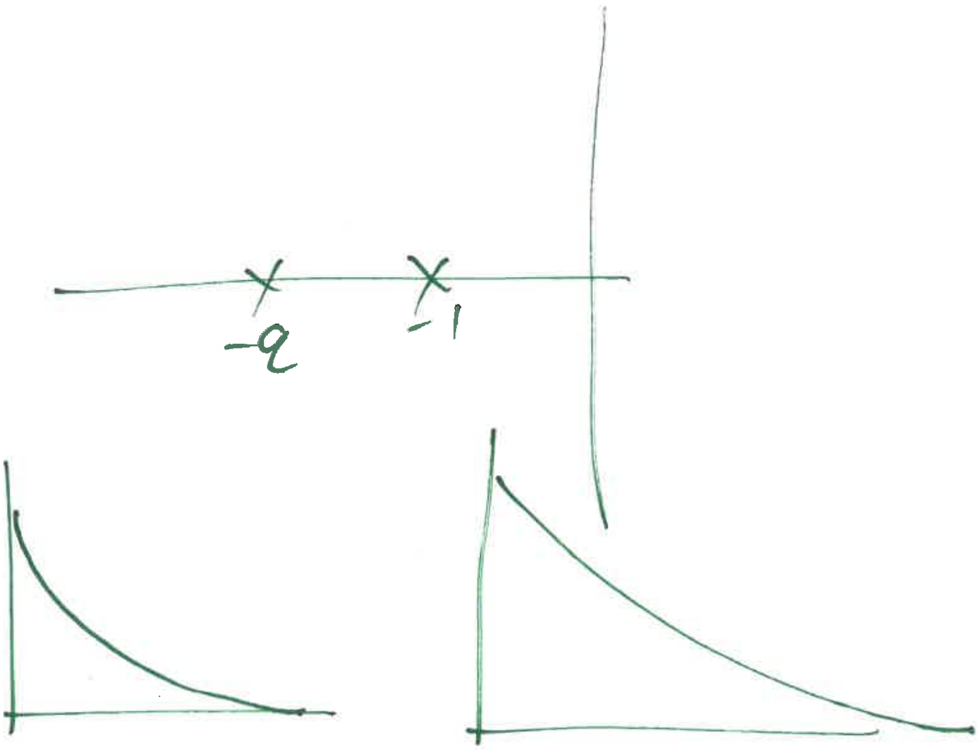
$$C_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{\cancel{3s}+1}{\cancel{s^2}+\cancel{s}+5} = 1/5 = 0.2$$

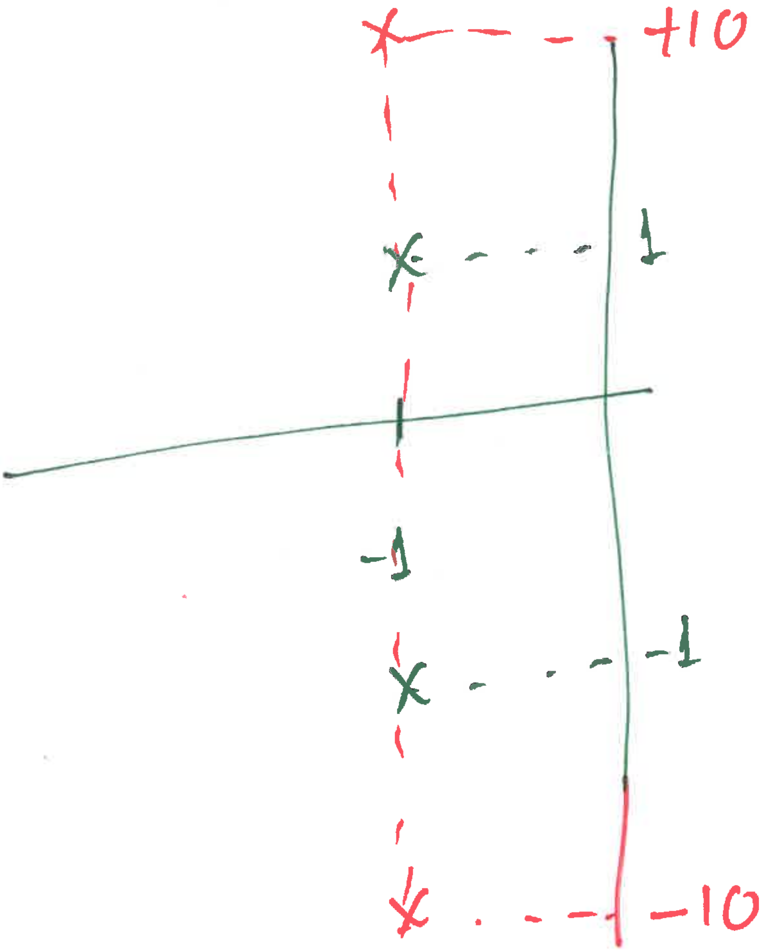
$0.2 \cdot 0.486 \simeq 0.3$

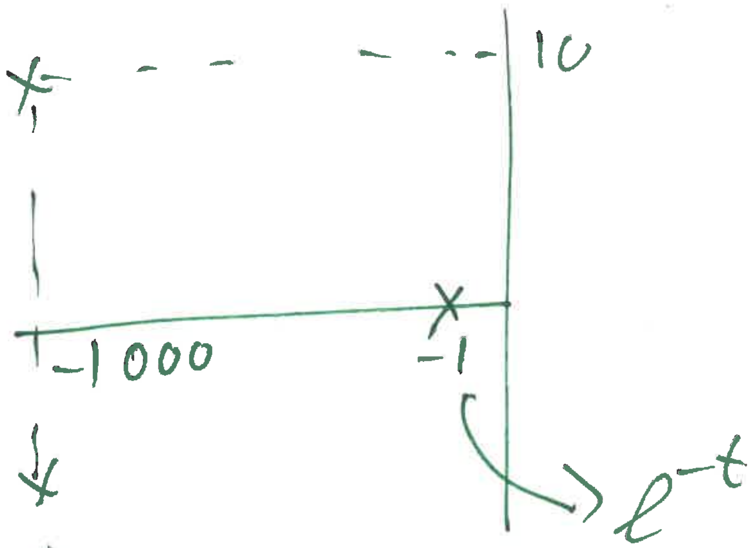




Dominant pole

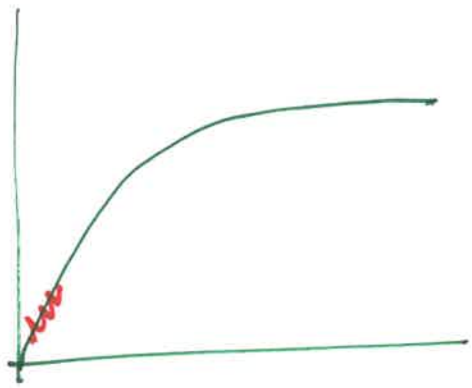


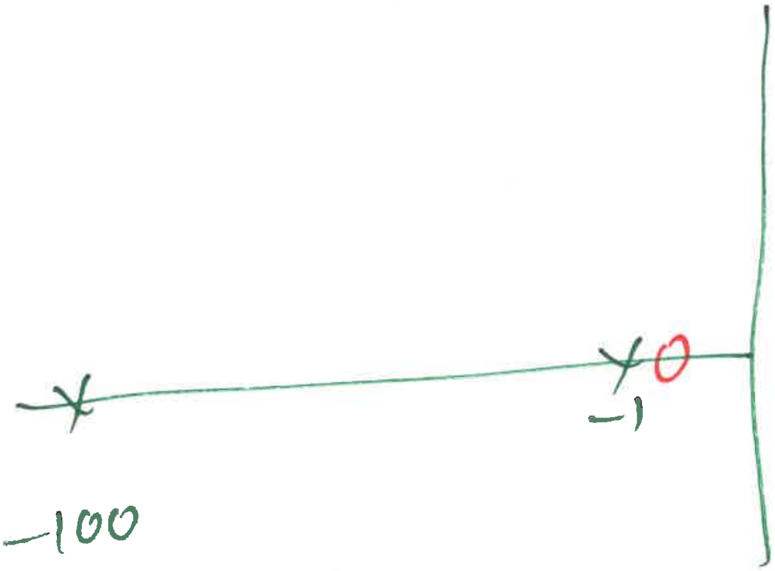




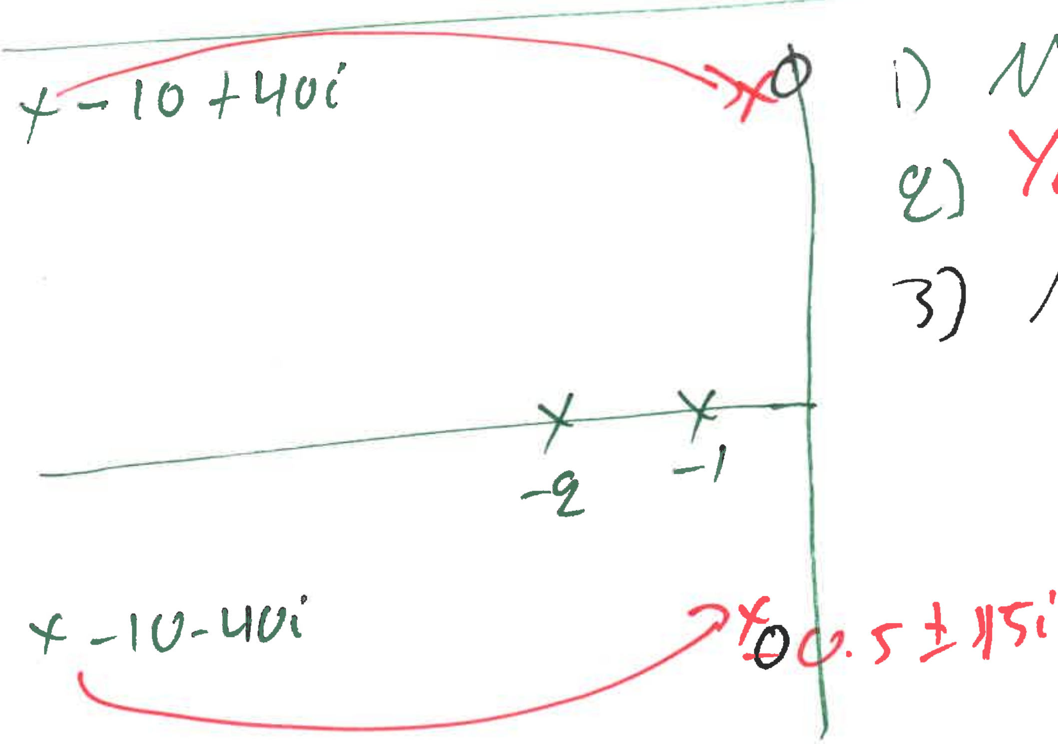
$\rightarrow e^{-t}$

$\rightarrow e^{-1000t} \cdot \cos 10t$

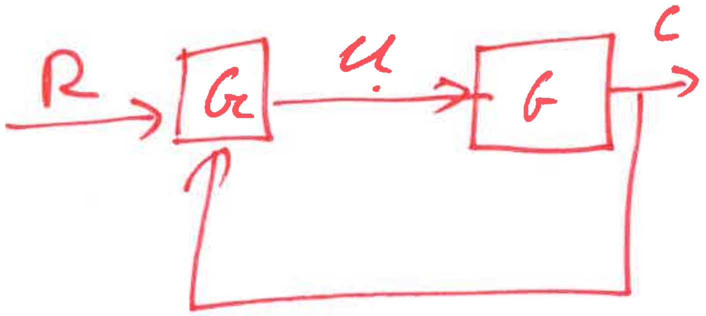


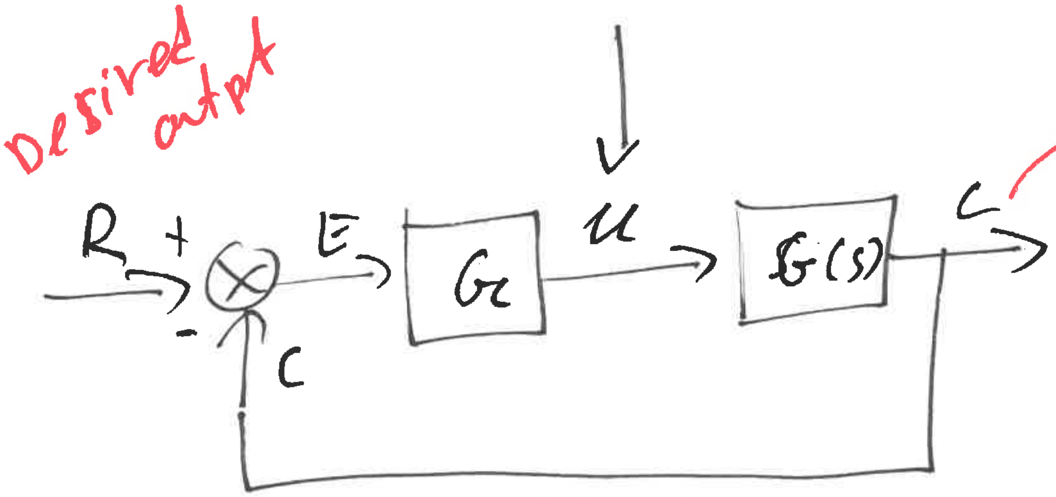


$$\frac{\cancel{s+0.99999}}{(s+100)\cancel{(s+1)}} = \frac{1}{s+100}$$



1) No osc.
2) Yes
3) No.





$$\boxed{\begin{array}{lll} u = ? & : & C \to R \\ u = ? & : & E \to 0 \end{array}}$$

$G(s) \to$ T.F. of sys

$Gc(s) \to$ T.F. of cont.

$$\left.\begin{array}{l} C = G \cdot u \\ u = Gc \cdot E \end{array}\right\}$$

$$E = R - C$$

$$C = G \cdot Gc \cdot E$$

$$C = G \cdot Gc\,(R - C)$$

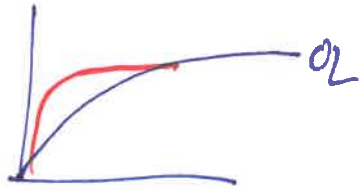
$$C = G \cdot Gc\, R - G\, Gc\, C$$

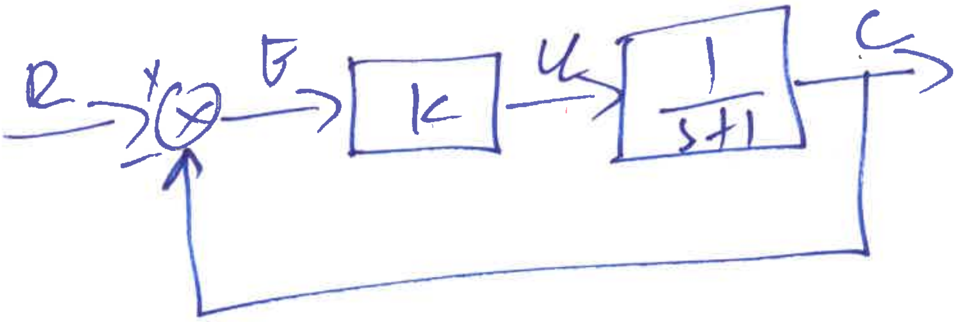
$$C(1 + G\,Gc) = G \cdot Gc \cdot R \Rightarrow$$

$$\frac{C}{R} = \frac{G \cdot Gc}{1 + G\,Gc}$$ C.L.T.F.

$G(s) = \frac{1}{s+1}$ $\qquad G_c(s) = K = \text{const}$

eig. or pole at $-1 \to e^{-t}$





assume $K = 2$

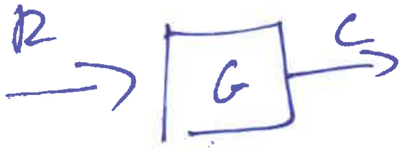
$$\frac{C}{R} = \frac{G\cdot G_c}{1+G\cdot G_c} = \frac{2\cdot\cancel{\frac{1}{s+1}}}{1+2\cancel{\frac{1}{s+1}}} \quad \frac{\cancel{(s+1)}}{\cancel{(s+1)}}$$

$$= \frac{2}{s+3}$$

C.E. $\quad s+3=0 \Rightarrow s=-3 \to e^{-3t}$

$G(s) = \frac{1}{s+1}$ $\quad$ $G_c(s) =$

$R(s) = \frac{1}{s}$ $\quad$ $C_{ss} = ?$



$$C_{ss} = \lim_{s \to 0} s \cdot \frac{1}{s} \cdot \frac{1}{s+1} = 1.$$

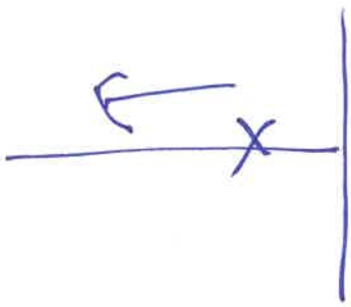
$$\frac{C}{R} = \frac{K}{s+1+K}$$

$$C_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s}} \frac{K}{s+1+K} = \frac{K}{K+1}$$

$K = 2$ $\quad$ $C_{ss} = \frac{2}{3} < 1$

• $K = 1000$ $\quad$ $C_{ss} = \frac{1000}{1001} \simeq 1.$

↳ pole $-1001 \rightarrow e^{-1001t}$

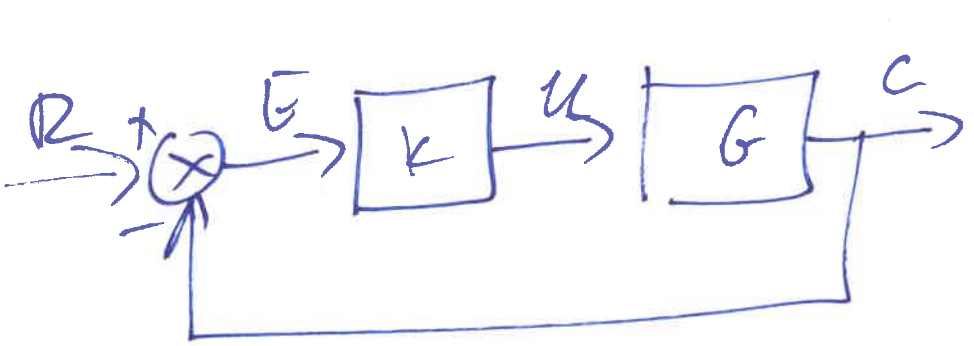


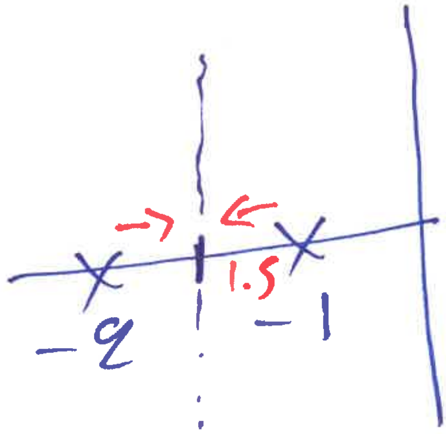
Faster

↑K ↓E_{ss}

41

proportional controller





$$G(s) = \frac{1}{(s+1)(s+2)}$$

$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c}$$

$$= \frac{\frac{1}{(s+1)(s+2)} \cdot K}{1 + \frac{1}{(s+1)(s+2)} \cdot K} \cdot \frac{(s+1)(s+2)}{(s+1)(s+2)}$$

$$= \frac{K}{(s+1)(s+2) + K}$$

C.E. $(s+1)(s+2) + K = 0$

$s^2 + 3s + 2 + K = 0$

• $K = 0$ $(s+1)(s+2) = 0$ → -1, → -2

• $K = 0.1$ $s^2 + 3s + 2.1 = 0$ → -1.11, → -1.8

$K = 0.25$ $s^2 + 3s + 2.25 = 0$ → -1.5, → -1.5

42

$\cdot K = 1 \quad s^2 + 3s + 3 = 0 \to -1.5 + 0.866i$

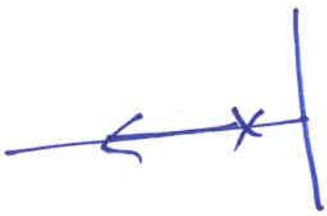
$\to -1.5 - 0.866i$

$K = K \quad s^2 + 3s + 2 + K = 0$

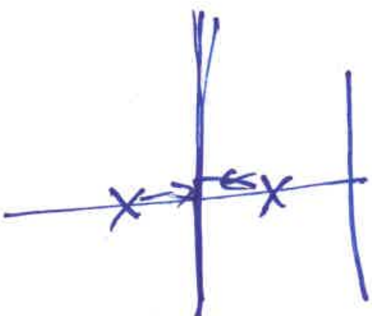
$$\Delta = 9 - 4(2+K) = 9 - 8 - 4K$$
$$= 1 - 4K$$

$$s_{1,2} = \frac{-3 \pm \sqrt{1-4K}}{2}$$

1st



2nd



$C_{ss} = \frac{K}{K+2}$

$K\uparrow$ Faster

+ osc.

always stable

C.E. $s^3+6s^2+11s+6+K=0$

$s=j\omega$

$-j\omega^3+6(-\omega^2)+11j\omega+6+K=0+0j$

$(-\omega^3+11\omega)\cdot j+6(-\omega^2)+6+K=0$

$-\omega^3+11\cdot\omega=0$ → $\omega\neq0$ → $11=\omega^2 \Rightarrow \omega=3.31$ rad/s

$-6\omega^2+6+K=0$

→ $K=60$

$G(s)=\frac{1}{(s+1)(s+2)}$ $\quad G_c(s)=K$

$K=?$: $\omega_n=\sqrt{12}$ rad/s

$Z=?$

1st Find the C.L.T.F

$\frac{C}{R}=\frac{K}{(s+1)(s+2)+K}$

2nd Find the C.L.C.E.

$(s+1)(s+2)+K=0$

$s^2+3s+2+K=0$

3rd Generic C.E.

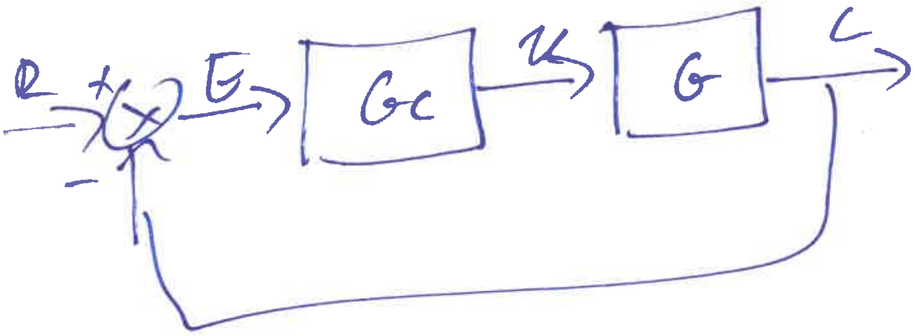
$s^2+2\cdot z\cdot\omega_n s+\omega_n^2=0$

$3=2\cdot z\cdot\omega_n$

$\omega_n^2=2+K$

$12=2+K \Rightarrow K=10$

$3=2\cdot z\cdot\sqrt{12} \Rightarrow$ $z=0.43$



P.I. cont.

$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s}$$

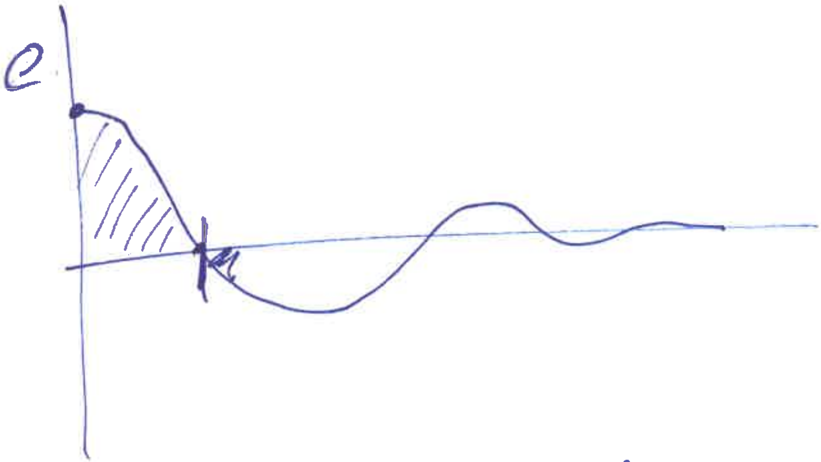
$$G(s) = \frac{1}{s+1}$$

$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G G_c} = \frac{\frac{1}{s+1} \cdot \frac{K_p s + K_I}{s}}{1 + \frac{1}{s+1} \, \frac{K_p s + K_I}{s}} \; \frac{s(s+1)}{s(s+1)}$$

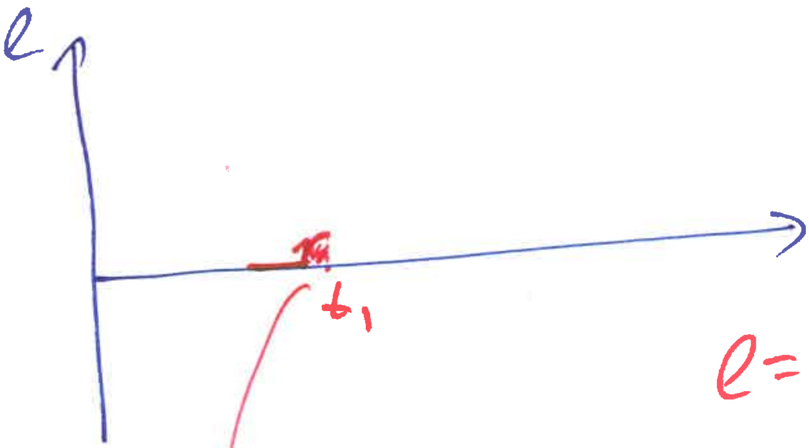
$$= \frac{K_p s + K_I}{s(s+1) + K_p s + K_I} = G_{CL}(s)$$

$$C(s) = R(s) \cdot G_{CL}(s) \quad , R(s) = 1/s$$

$$C_{ss} = \lim_{s \to 0} \cancel{s} \, \frac{1}{\cancel{s}} \, \frac{\cancel{K_p s} + K_I}{\cancel{s(s+1)} + \cancel{K_p s} + K_I} = 1$$



$$K_P + \frac{K_I}{s} + K_D s \to \frac{d}{dt}$$



$e = 0.001$

$K_P = 10 \quad 10 \cdot 0.001 = 001$

$\int \Longrightarrow 0 \quad e = 0$ at 10:00

$e = 0.001$ at 10:00:01

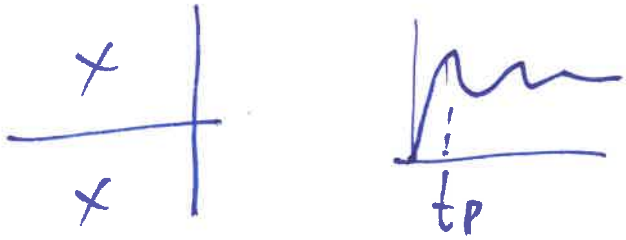
2nd order $\dfrac{\boxed{\cdots}}{C.E.}$

C.E. $s^2+2\cdot\zeta\cdot\omega_n s+\omega_n^2=0$

• $\zeta>1$

• $\zeta=1$

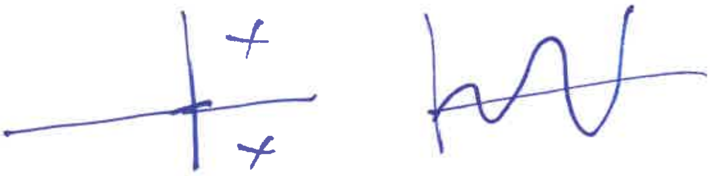
• $\zeta<1$

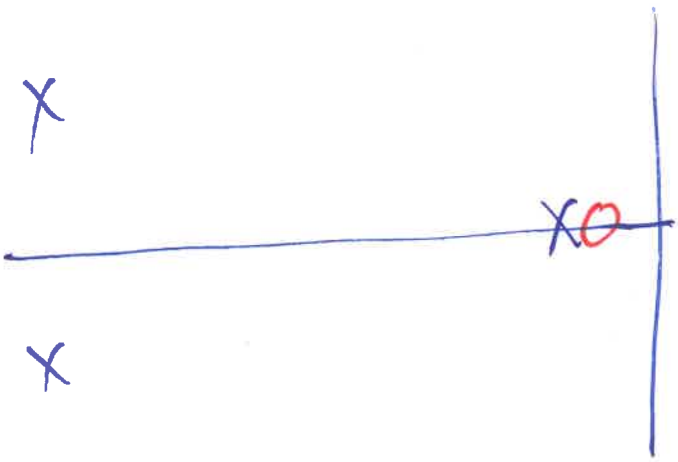


t_s , M_p

• $\zeta=0$

• $\zeta<0$





P.I.D.

$$G_c(s) = K_p + K_I \frac{1}{s} + K_D s$$

$$= K_p\left(1 + \frac{K_I}{K_p}\frac{1}{s} + \frac{K_D}{K_p} s\right)$$

$$= K_p\left(1 + \frac{1}{T_i}\frac{1}{s} + T_d s\right)$$

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

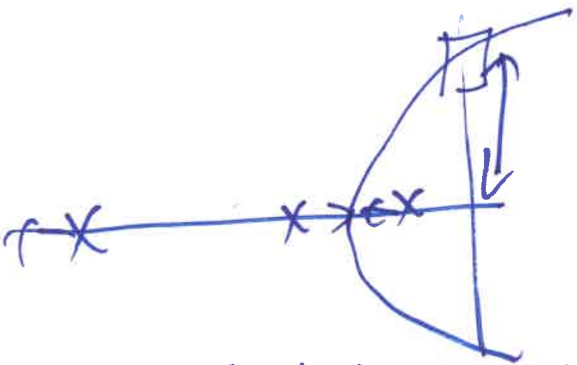
$H = 1$

$K_p = ?$

$K_I = ?$

$K_D = ?$

$$G_c(s) = K_p + \underbrace{\cancel{K_I \frac{1}{s}}}_{\to 0} + \underbrace{\cancel{K_D s}}_{\to 0}$$



$K_{cr} = 60$

$\omega = 3.31$ rad/s

$\omega = \frac{2\pi}{T}$

$T = P_{cr} = \frac{2\pi}{\omega} = 1.89$ s

$K_p = 0.6 \cdot K_{cr} = 0.6 \cdot 60 = 36$

$T_i = 0.5 \cdot P_{cr} = 0.5 \cdot \cancel{3.31}$

$0.5 \cdot 1.89 = 0.95$ sec.

$T_d = 0.125\, P_c = 0.11$ sec.

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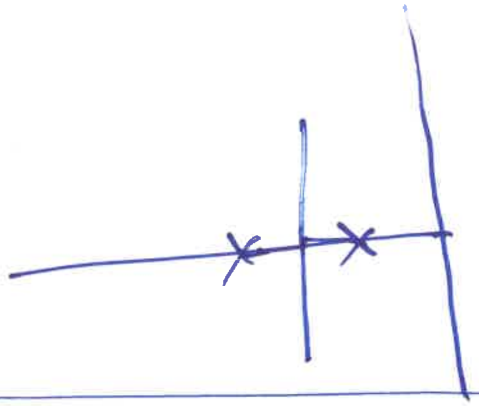
$$G(s) = \frac{1}{(s+1)(s+2)}$$



$$G(s) = \frac{1}{s^2+11s-34}$$

$$G_c(s) = K_p + K_I \frac{1}{s}$$

$M_p = 0.163$

$t_{s\,2\%} = 1.33$ sec.

C.L.T.F.

(1) $$G_c(s) = \frac{K_p s + K_I}{s(s^2+11s-34) + K_p s + K_I}$$

(2) C.E. $\cancel{s^3 + 11s^2 + (-34 + K_p)s + K_I = 0}$

$$s^3 + (2\zeta\omega_n + a)s^2 + (2\zeta\omega_n a + \omega_n^2)s + a\,\omega_n^2 = 0$$

$$s^3 + 11s^2 + s(K_p - 34) + K_I = 0$$

$$\left.\begin{aligned} 11 &= 2\zeta\omega_n + a \\ K_p - 34 &= 2\zeta\omega_n a + \omega_n^2 \\ K_I &= a\,\omega_n^2 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= 5 \\ K_p &= 100 \\ K_I &= 180 \end{aligned}$$

$$\%M_p = \ln\left(\exp\left(\frac{-z\pi}{\sqrt{1-z^2}}\right)\right) = \ln(0.163)$$

$$\left(\frac{-z\pi}{\sqrt{1-z^2}}\right)^2 = (-1.8)^2$$

$$\frac{z^2\pi^2}{1-z^2} = 3.24$$

$$\frac{z^2\, 9.86}{1-z^2} = 3.24.$$

$$9.86\, z^2 = 3.24 - 3.24 z^2 \Rightarrow \ldots$$

$$\Rightarrow \ldots \quad z = 0.5$$

$$t_s = \frac{4}{z\cdot\omega_n} = 1.33 \Rightarrow \omega_n = 6 \text{ rad/s}$$

$$G(s) = \frac{K_1}{s^2} \qquad K_1 = ?$$

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$$H(s) = s+1 \qquad M_p = 0.05$$

$$\frac{C}{R} = \frac{G}{1+G \cdot H} = \frac{K_1/s^2}{1+\frac{K_1}{s^2}(s+1)} \quad \frac{s^2}{s^2}$$

(1)
$$= \frac{K_1}{s^2+K_1 s+K_1}$$

(2)
$$s^2+K_1 s+K_1=0$$

$$s^2+2 \cdot Z \cdot \omega_n s+\omega_n^2=0$$

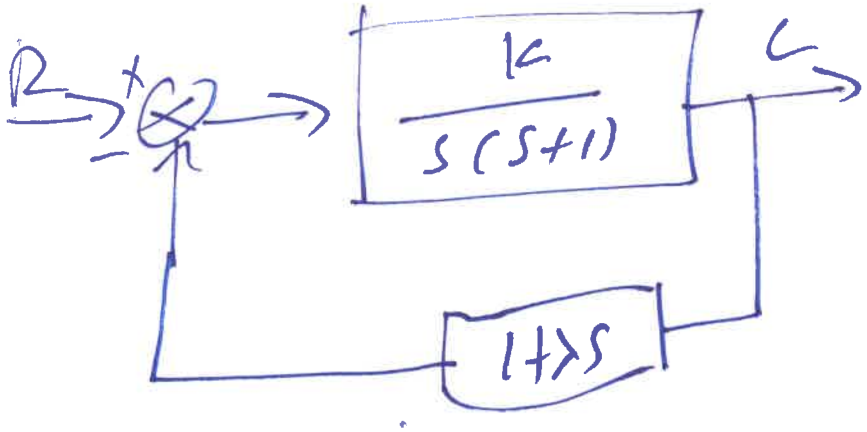
$K_1 = 1.96$

(3)
$$\left.\begin{array}{l} K_1 = 2 \cdot Z \cdot \omega_n \\ K_1 = \omega_n^2 \end{array}\right\} \quad \begin{array}{l} 2 \cdot Z \cdot \omega_n = \omega_n^2 \\ 2 \cdot Z = \omega_n \Rightarrow \omega_n = \ldots \end{array}$$

$$M_p = e^{\left(\frac{-Z\pi}{\sqrt{1-Z^2}}\right)} = 0.05$$

$$\frac{-Z\pi}{\sqrt{1-Z^2}} = -2.99$$

$$\frac{Z^2}{1-Z^2} = 0.96 \Rightarrow \ldots Z = 0.7$$



$K = ?$

$\lambda = ?$

$M_p = 0.4$

$t_p = 1s$

$$M_p = \exp\left(\frac{-z\pi}{\sqrt{1-z^2}}\right)$$

$$t_p = \frac{\pi}{\omega_d}$$

$$\frac{C}{R} = \frac{G}{1+GH} = \frac{\frac{K}{s(s+1)}}{1+\frac{K}{s(s+1)}\cdot(1+\lambda s)} \quad \frac{s(s+1)}{s(s+1)}$$

$$= \frac{K}{s(s+1) + K(1+\lambda s)}$$

C.E. $s^2 + (1+K\lambda)\cdot s + K = 0$

$s^2 + 2\cdot z\cdot\omega_n s + \omega_n^2 = 0$

$1 + K\lambda = 2\cdot z\cdot\omega_n$

$K = \omega_n^2$

$M_p = \ldots \Rightarrow z = 0.279$

$\omega_d = \pi = \omega_n\sqrt{1-z^2} \Rightarrow \omega_n = 3.27$ rad/s

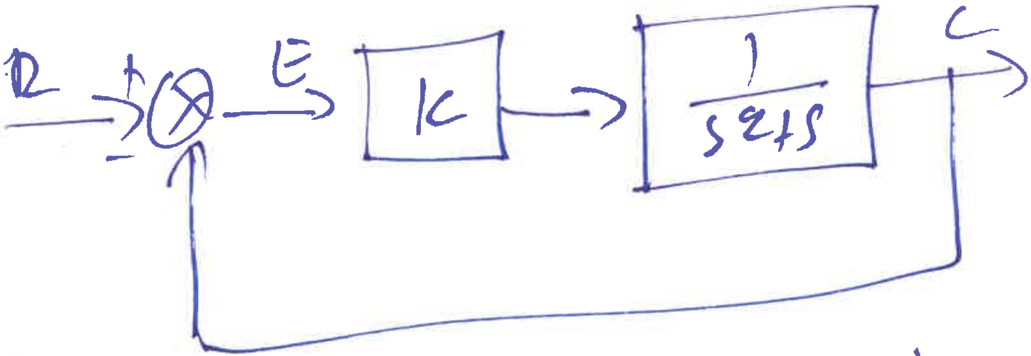
$K = 10.69, \quad \lambda = 0.08$

$$G(s) = \frac{1}{s^2+s}$$

① B.D. $G_c(s) = K$, unity feedback

② Find K_1, $E_{ss} < 0.1$, $r(t) = t$

$R(s) = 1/s^2$



$$\frac{C}{R} = \frac{G \cdot G_c}{1 + G \cdot G_c} \qquad \frac{E}{R} = \frac{1}{1 + G \cdot G_c}$$

$$\frac{E}{R} = \frac{1}{1 + K \frac{1}{s^2+s}} \quad \frac{s^2+s}{s^2+s} = \frac{s^2+s}{s^2+s+K}$$

$$E(s) = \frac{1}{s^2} \, \frac{s^2+s}{s^2+s+K}$$

$$E_{ss} = \lim_{s \to 0} \cancel{s} \frac{1}{\cancel{s^2}} \frac{\cancel{s}(s+1)}{s^2+s+K} = \frac{1}{K} < 0.1$$

$$K > 10$$

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$$G_1 = \frac{1}{s+1} \rightarrow -1$$

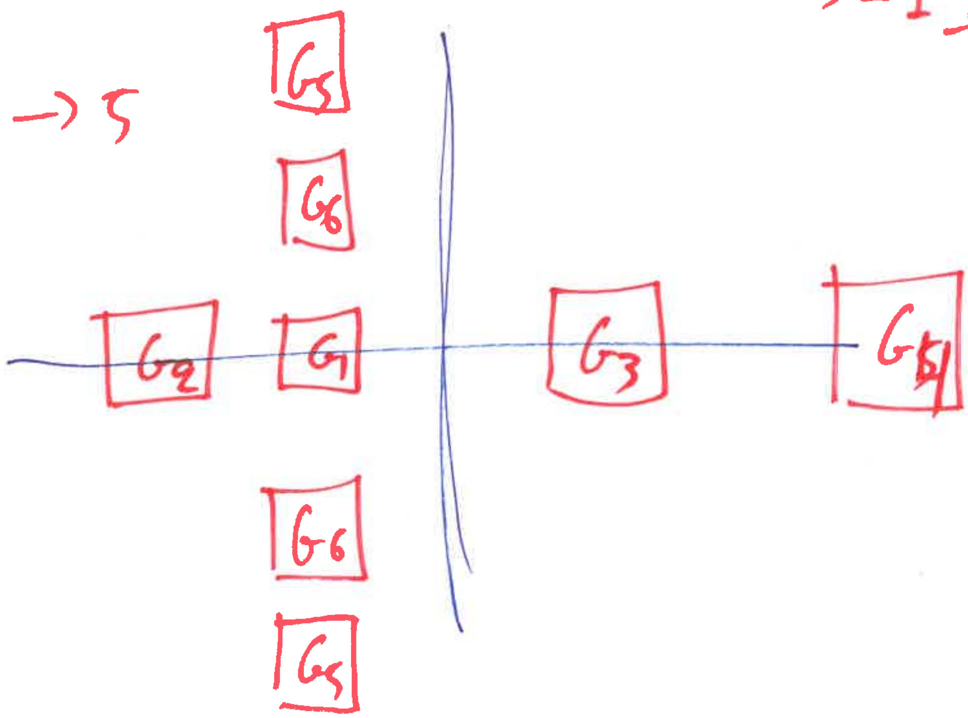
$$G_2 = \frac{1}{s+2} \rightarrow -2$$

$$G_3 = \frac{1}{s-1} \rightarrow 1$$

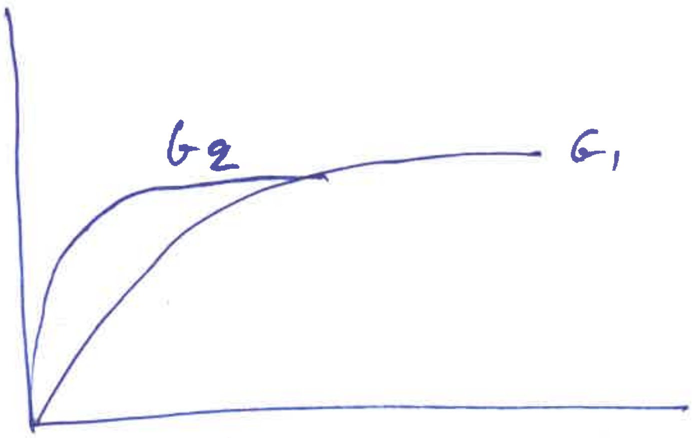
$$G_4 = \frac{1}{s-5} \rightarrow 5$$

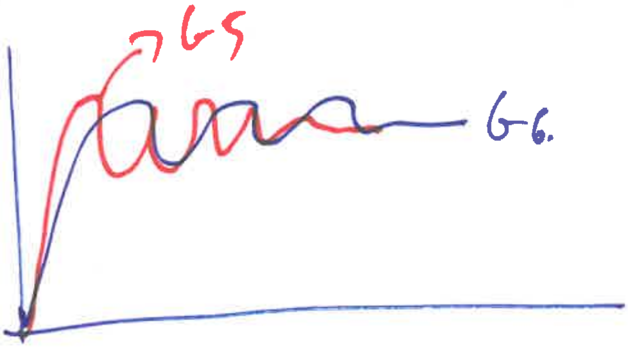
$$G_5 = \frac{1}{s^2+2s+101} \rightarrow -1 \pm 10i$$

$$G_6 = \frac{1}{s^2+2s+26} \rightarrow -1 \pm 5i$$



G_1:





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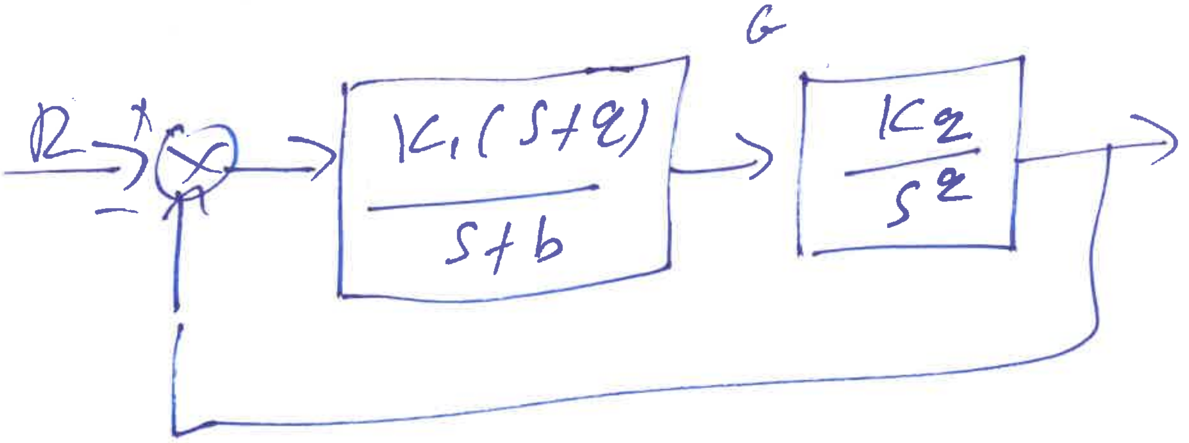

$K_1 \cdot K_2 = ?$ $\quad b = ?$ $\quad \omega_n = 6$ rad/s

$\theta = 60^\circ$

$$\frac{C}{R} = \frac{G}{1+G} = \frac{\dfrac{K_1(s+2)\cdot K_2}{s^2(s+b)}}{1+\dfrac{K_1(s+2)\cdot K_2}{s^2(s+b)}} \quad \frac{s^2(s+b)}{s^2(s+b)}$$

$K = K_1 K_2$

$$= \frac{K(s+2)}{s^2(s+b) + K(s+2)}$$

C.E. $\quad s^3 + b s^2 + K s + 2K = 0$

$$s^3 + (2 \cdot z \cdot \omega_n + a) \cdot s^2 + (a z \omega_n 2 + \omega_n^2) \cdot s + a \omega_n^2 = 0$$

$b = 2 \cdot z \omega_n + a$

$K = a z \omega_n 2 + \omega_n^2 \leftarrow z = \cos\theta = 1/2$

$2K = a \omega_n^2 \leftarrow \omega_n = 6$

$b = 9$

$a = 3$ $\quad K = 54$