Chapter #1

EEE8072

Subsea Control and Communication Systems

- **Introduction**
- **1^{st} order dynamics**
- **2^{nd} order dynamics**
- **Nonhomogeneous differential equations**

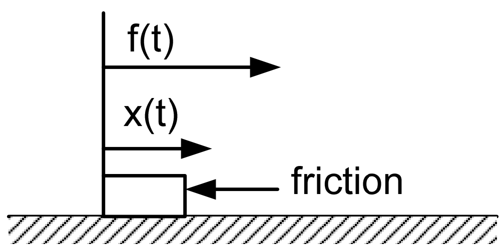
Introduction

System: is a set of objects/elements that are connected or related to each other in such a way that they create and hence define a unity that performs a certain objective.

Control: means regulate, guide or give a command**.**

Task: To study, analyse and ultimately to control the system to produce a "satisfactory" performance.

Model: Ordinary Differential Equations (ODE):



$$\Sigma F = ma \Leftrightarrow f - f_{friction} = ma \Leftrightarrow f - B\omega = ma \Leftrightarrow f - B\frac{dx}{dt} = m\frac{d^2x}{dt^2}$$

Dynamics: Properties of the system, we have to solve/study the ODE.

First order ODEs: $\frac{dx}{dt} = f(x,t)$

Analytical solution: Explicit formula for $x(t)$ (a solution which can be found using separate variables, integrating factor…) which satisfies $\frac{dx}{dt} = f(x,t)$

Example:

The ODE $\frac{dx}{dt} = a$ has as a solution $x(t) = at$.

First order Initial Value Problem : $\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$

Analytical solution: Explicit formula for $x(t)$ which satisfies $\frac{dx}{dt} = f(x,t)$ and passes through x_0 when $t = t_0$

Example:

$$\frac{dx}{dt} = a \Leftrightarrow \int dx = \int adt \Leftrightarrow x(t) = at + C \Rightarrow x(0) = C \Rightarrow x(t) = at + x_0$$

⇨ INFINITE curves (for all Initial Conditions (ICs)).

For that reason a different symbol is used for the above solution: $\phi(t, t_0, x_0)$.

You must be clear about the difference to an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

First order linear equations - (linear in *x* and *x'*)

General form: $\begin{cases} a(t)x'+b(t)x=c(t), & \textit{Non autonomous} \\ ax'+bx=c, & \textit{Autonomous} \end{cases}$

Example: $x'+kx=u$

<u>Analytical Solutions:</u>

Integrating factor (**NOT ASSESSED MATERIAL**):

$$e^{\int kdt} \overset{k=const}{=} e^{kt}$$

$$e^{kt}(x'+kx)=e^{kt}u \Rightarrow \left(e^{kt}x\right)'=e^{kt}u$$

$$\Rightarrow \int \left(e^{kt}x\right)' dt = \int e^{kt}udt$$

$$\Rightarrow e^{kt}x=\int e^{kt}udt+c \Rightarrow x=e^{-kt}\int e^{kt}udt+e^{-kt}c$$

or: $x=e^{-kt}x(0)+e^{-kt}\int_0^t e^{kt_1}udt_1$

The equation that we derived is:

$$x=e^{-kt}x(0)+e^{-kt}\int_0^t e^{kt_1}udt_1$$

Assuming that $k>0$ the first part is called transient and the second is called steady state solution.

Second order ODEs: $\frac{d^2x}{dt^2} = f(x', x, t)$

Second order linear ODEs with constant coefficients: $x'' + Ax' + Bx = u$

u=0 => Homogeneous ODE; I need two "representative solutions"

$x'' + Ax' + Bx = 0$, assume $x = e^{rt}$ => $x' = re^{rt}$ & $x'' = r^2 e^{rt}$ =>

$x'' + Ax' + Bx = 0 \Leftrightarrow r^2 e^{rt} + Are^{rt} + Be^{rt} = 0 \Leftrightarrow$

$r^2 + Ar + B = 0$; Characteristic or Eigenvalue equation => Check its roots.

$r = \frac{-A \pm \sqrt{A^2 - 4B}}{2}$, these are the Characteristic values or Eigenvalues.

- Roots are real and unequal: r_1 and r_2 ($A^2 > 4B$ =>Overdamped system)

$x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are solutions of the ODE =>

$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. If r_1 and r_2<0 then $x \to 0$.

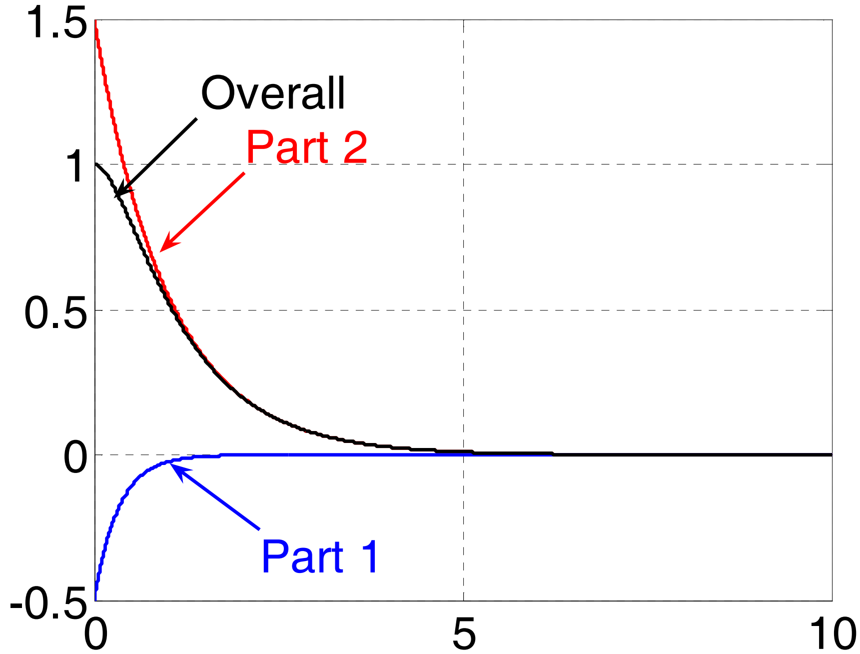
Example:

$x'' + 4x' + 3x = 0 \Leftrightarrow r^2 + 4r + 3 = 0 \Leftrightarrow (r+3)(r+1) = 0$

$x = C_1 e^{-3t} + C_2 e^{-t}$. Assume that $x(0) = 1$ and $x'(0) = 0$:

$x(0) = C_1 + C_2 = 1$ and $x' = -3C_1 e^{-3t} - C_2 e^{-t} \Rightarrow$ $x'(0) = -3C_1 - C_2 = 0$ =>
C_1=-0.5, C_2=3/2 => $x = -0.5e^{-3t} + \frac{3}{2} e^{-t}$:

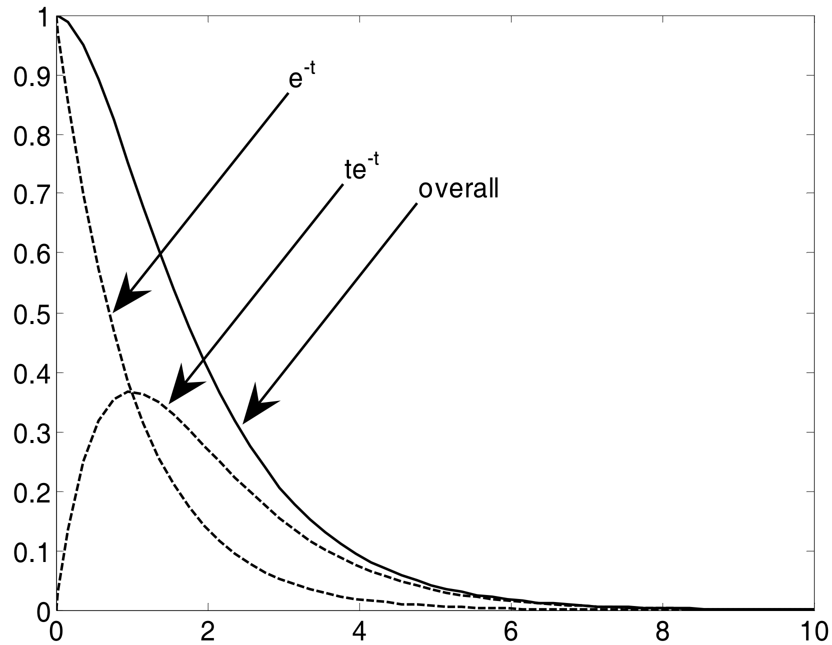
Analytical Solution:



- Roots are real and equal: $r_1=r_2$ ($A^2 = 4B$ Critically damped system)

$x_1 = e^{rt}$ and $x_2 = te^{rt}$ => $x = C_1x_1 + C_2x_2 = C_1e^{rt} + C_2te^{rt}$

Example: A=2, B=1, x(0)=1, x'(0)=0 => $c_1=c_2=1$



- Roots are complex: r=a+bj $A^2 < 4B$
  - Underdamped system $A \neq 0$

So $x = e^{rt} = e^{(a+bj)t} = e^{at+bjt} = e^{at}e^{jbt} = e^{at}(\cos(bt) + j\sin(bt))$=Re+jIm.

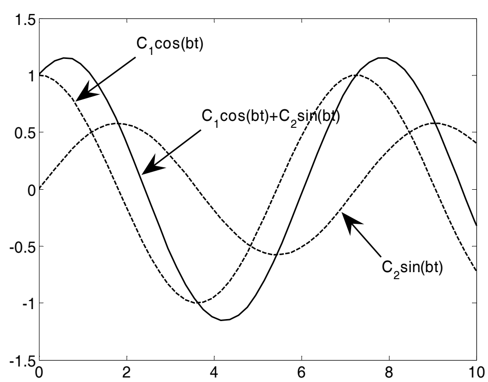
Theorem: If x is a complex solution to a real ODE then Re(x) and Im(x) are the real solutions of the ODE:

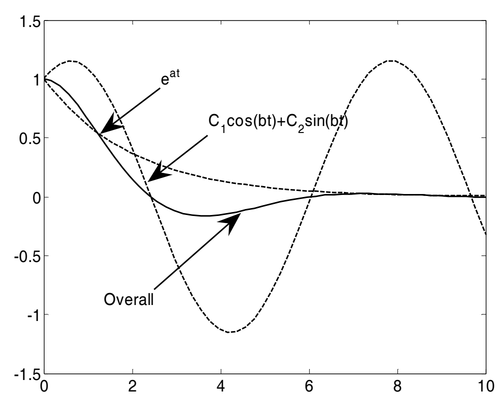
$x_1 = e^{at}\cos(bt), x_2 = e^{at}\sin(bt)$ =>

$$x = c_1x_1 + c_2x_2 = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt)$$
$$= e^{at}(c_1\cos(bt) + c_2\sin(bt)) = e^{at}G\cos(bt - \phi)$$

where $G = \dfrac{c_1}{\cos\left(\tan^{-1}\left(c_2/c_1\right)\right)}$, & $\phi = \tan^{-1}\left(c_2/c_1\right)$

A=1, B=1, x(0)=1, x'(0)=0 => c_1=1, c_2=1/sqrt(3)



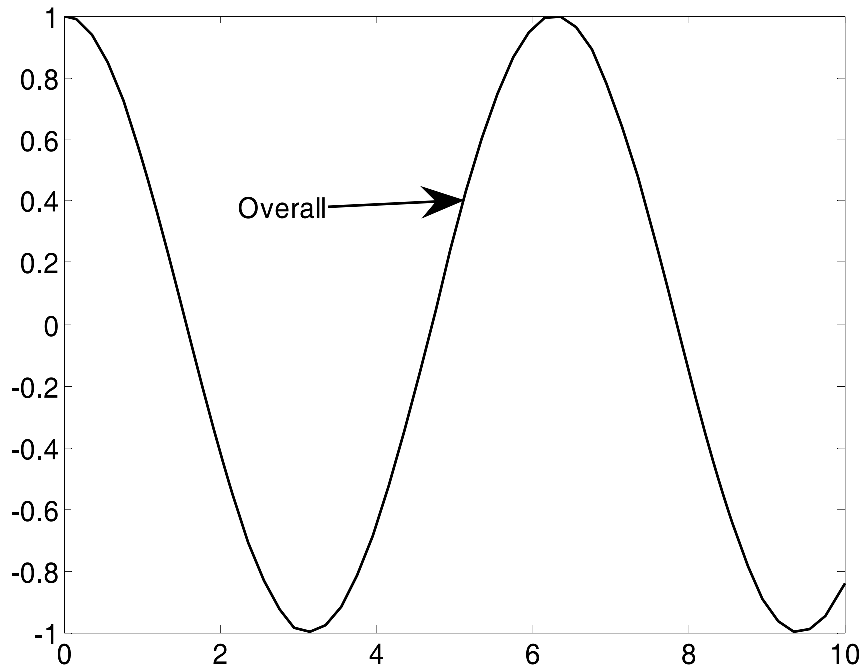


- Undamped system $A = 0$

$x'' + 0 + Bx = 0 \Leftrightarrow r^2 e^{rt} + 0 + Be^{rt} = 0 \Rightarrow r^2 = -B$ => Imaginary roots (If B<0 then I would have two equal real roots).

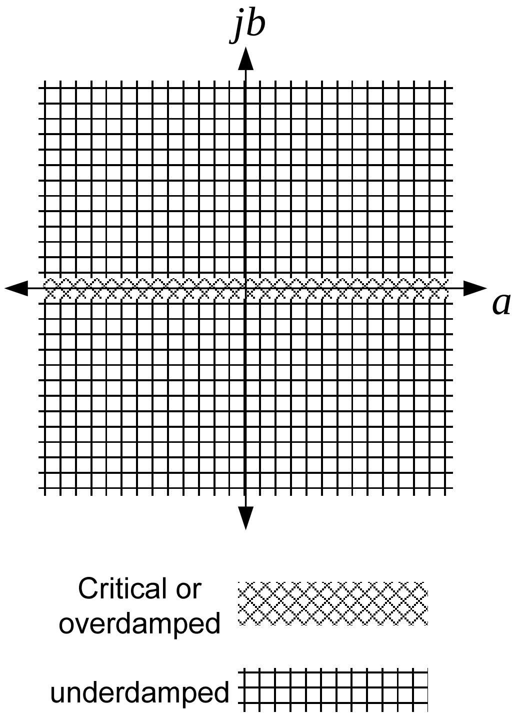
So $r = jb$ => $x = c_1 \cos(bt) + c_2 \sin(bt) = G\cos(bt - \phi)$

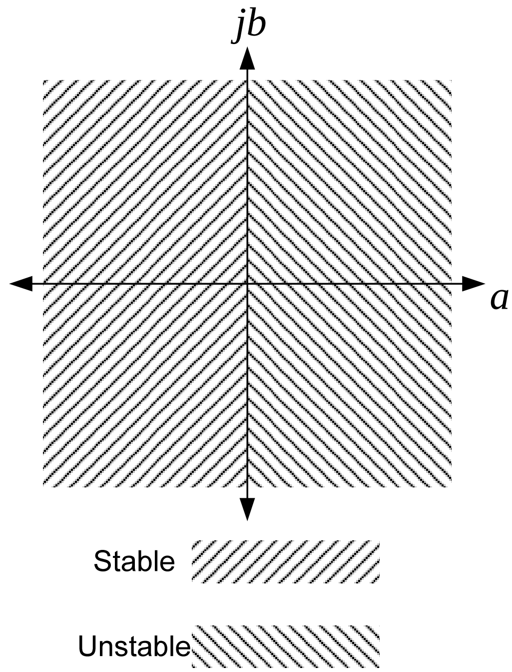
A=0, B=1, x(0)=1, x'(0)=0 =>c_1=1, c_2=0:



In all previous cases if the real part is positive then the solution will diverge to infinity and the ODE (and hence the system) is called unstable.

Root Space





Name	Oscillations?	Components of solution
Overdamped	No	Two exponentials: $e^{k_1 t}, e^{k_2 t},\ k_1, k_2 < 0$
Critically damped	No	Two exponentials: $e^{kt}, te^{kt},\ k < 0$
Underdamped	Yes	One exponential and one cosine $e^{kt}, \cos(\omega t),\ k < 0$
Undamped	Yes	one cosine $\cos(\omega t)$

NonHomogeneous (NH) differential equations

$x''+Ax'+Bx=u$

- u=0 => Homogeneous => x_1 & x_2.
- Assume a particular solution of the nonhomogeneous ODE: x_p
    - If u(t)=R=cosnt => $x_P = \frac{R}{B}$
- Then all the solutions of the NHODE are $x = x_P + c_1 x_1 + c_2 x_2$
- So we have all the previous cases for under/over/un/critically damped systems plus a constant R/B.
- If complementary solution is stable then the particular solution is called steady state.

Example:

$x''+x'+x=2 \Rightarrow x_P = 2,\ x = 2 + c_1 x_1 + c_2 x_2 = 2 + e^{at}\left(c_1 \cos(bt) + c_2 \sin(bt)\right)$

x(0)=1, x'(0)=0 => c_1=-1, c_2=-1/sqrt(3)