Chapter #2

EEE 8072

Subsea Control and Communication Systems

- **Transfer functions**
- **Pole location and s-plane**
- **Time domain characteristics**
- **Extra poles and zeros**

Transfer functions

Laplace Transform

Used only on LTI systems

Differential expression => Polynomial expression

$$\frac{dy(t)}{dt} - y(t) = 10 \Rightarrow sY(s) - Y(s) = 10$$

The LT is transforming a DE from the time domain (domain is a set of values that describe a function, in that case the variable is the time) to another complex domain (i.e. the variable has a real and imaginary part).

$$F(s) = L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt \text{ , } s = \sigma \pm j\omega$$

Use formula tables => Easier

Properties:

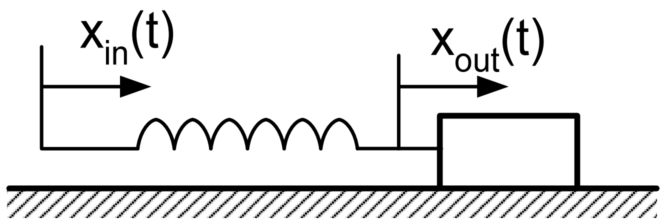
1. Differentiation
   a. $L\left\{\frac{df(dt)}{dt}\right\} = sF(s) - f(0)$
   b. $L\left\{\frac{d^2 f(dt)}{dt^2}\right\} = s^2 F(s) - sf(0) - \frac{df(0)}{dt}$

2. Final value theorem $f_{ss} = \lim_{s \to 0} sF(s)$, where f_{ss} is the value of $f(t)$ after infinite time.

Transfer functions

The ratio of the Laplace transform of the output over the Laplace transform of the input.

Example:



$$\sum f = ma \Leftrightarrow f_{Spring} = m\frac{d^2 x_{out}}{dt^2}$$

$$K(x_{in} - x_{out}) = m\frac{d^2 x_{out}}{dt^2}$$

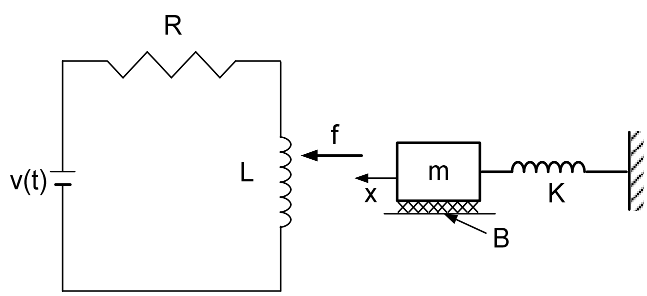
$$K(x_{in} - x_{out}) = m\frac{d^2 x_{out}}{dt^2} \underset{IC=0}{\overset{LT}{\Rightarrow}}$$
$$KX_{in}(s) - KX_{out}(s) = ms^2 X_{out}(s) \Leftrightarrow$$
$$KX_{in}(s) = X_{out}(s)\left(ms^2 + K\right) \Leftrightarrow$$
$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + K}$$

To find the characteristic equation of ODE:

$$Kx_{in} - Kx_{out} = m\frac{d^2x_{out}}{dt^2} \Rightarrow Kx_{in} = m\frac{d^2x_{out}}{dt^2} + Kx_{out}$$

The homogeneous system is: $0 = m\dfrac{d^2x_{out}}{dt^2} + Kx_{out}$ and therefore the **CE** is $mr^2 + K = 0$ i.e. exactly as the **denominator of TF**.

Example:



$$f = k_A\frac{i^2}{x^2} \Rightarrow f = k_A i$$

$$\left.\begin{aligned}\frac{di}{dt} &= \frac{1}{L}(v - iR)\\ k_A i - kx - B\dot{x} &= m\ddot{x}\end{aligned}\right\} \Rightarrow \left.\begin{aligned}sI(s) &= \frac{1}{L}(V(s) - I(s)R)\\ k_A I(s) - kX(s) - BsX(s) &= ms^2X(s)\end{aligned}\right\}$$

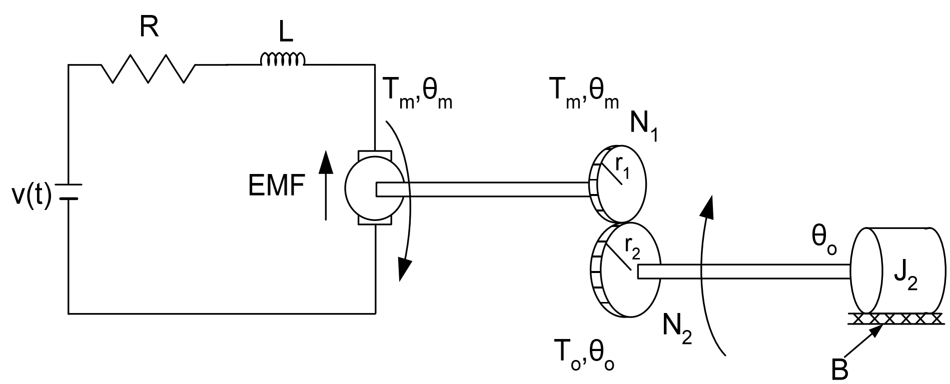
$$\left.\begin{aligned}sLI(s) + I(s)R &= V(s)\\ k_A I(s) - kX(s) - BsX(s) &= ms^2X(s)\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{array}{l} I(s) = \dfrac{V(s)}{sL+R} \\ k_A \dfrac{V(s)}{sL+R} - kX(s) - BsX(s) = ms^2 X(s) \end{array}\right\} \Rightarrow$$

$$k_A \frac{V(s)}{sL+R} = \left(ms^2 + Bs + k\right) X(s) \Rightarrow$$

$$\frac{X(s)}{V(s)} = \frac{k_A}{(sL+R)\left(ms^2 + Bs + k\right)}$$

Example:



$$\left.\begin{array}{l} \left.\begin{array}{l} J\ddot{\theta}_0 = T_0 - B\dot{\theta}_0 \\ T_0 = \dfrac{n_2}{n_1} T_m \\ T_m = K_T \varphi i_a \end{array}\right\} \Rightarrow J\ddot{\theta}_0 = \dfrac{n_2}{n_1} K_T \varphi i_a - B\dot{\theta}_0 \\ v_a = i_a R_a + L_a \dfrac{di_a}{dt} + K_T\, \varphi \dot{\theta}_m \Leftrightarrow v_a = i_a R_a + L_a \dfrac{di_a}{dt} + K_T\, \varphi \dfrac{n_2}{n_1} \dot{\theta}_0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{aligned} J\ddot{\theta}_0 &= \frac{n_1}{n_2}K_T\varphi i_a - B\dot{\theta}_0 \\ v_a &= i_a R_a + L_a\frac{di_a}{dt} + K_T\varphi\frac{n_2}{n_1}\dot{\theta}_0 \end{aligned}\right\}$$

$$\overset{LT}{\underset{K_1=K_T\varphi\frac{n_2}{n_1},\ K_2=K_T\varphi\frac{n_1}{n_2}}{\Rightarrow}} \left.\begin{aligned} Js^2\Theta_0(s) &= K_2 I_a(s) - B\Theta_0(s) \\ V_a(s) &= I_a(s)R_a + L_a s I_a(s) + K_1\Theta_0(s) \end{aligned}\right\}$$

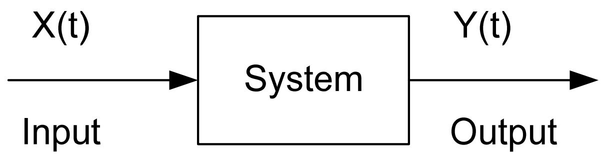
$$\left.\begin{aligned} I_a(s) &= \frac{(Js^2+B)}{K_2}\Theta_0(s) \\ V_a(s) &= (R_a + L_a s)I_a(s) + K_1\Theta_0(s) \end{aligned}\right\}$$

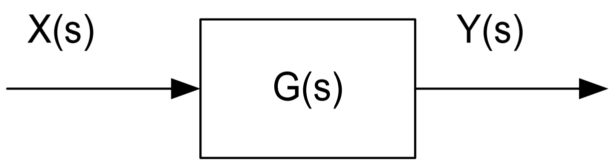
$$\Rightarrow V_a(s) = (R_a + L_a s)\frac{(Js^2+B)}{K_2}\Theta_0(s) + K_1\Theta_0(s)$$

$$\frac{\Theta_0(s)}{V_a(s)} = \frac{1}{\left((R_a + L_a s)\frac{(Js^2+B)}{K_2} + K_1\right)} =$$

$$\frac{K_2}{(R_a + L_a s)(Js^2+B) + K_1 K_2}$$

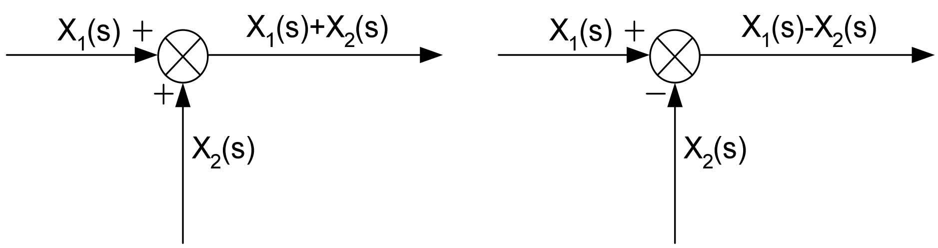
Block diagrams



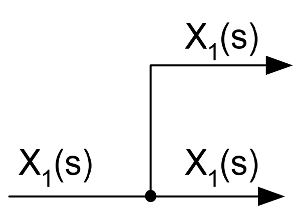


Block Diagram Algebra

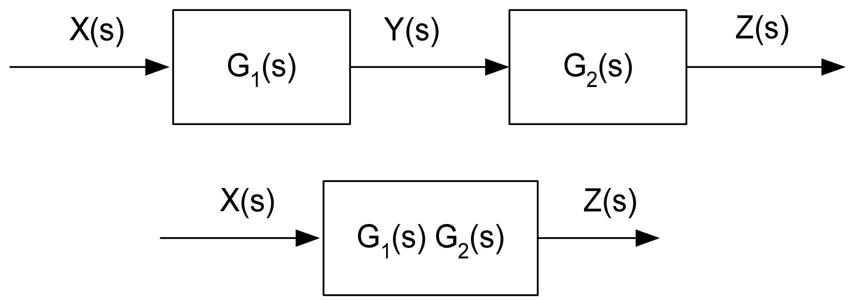
1. To sum (subtract) two signals, we use a summing point:



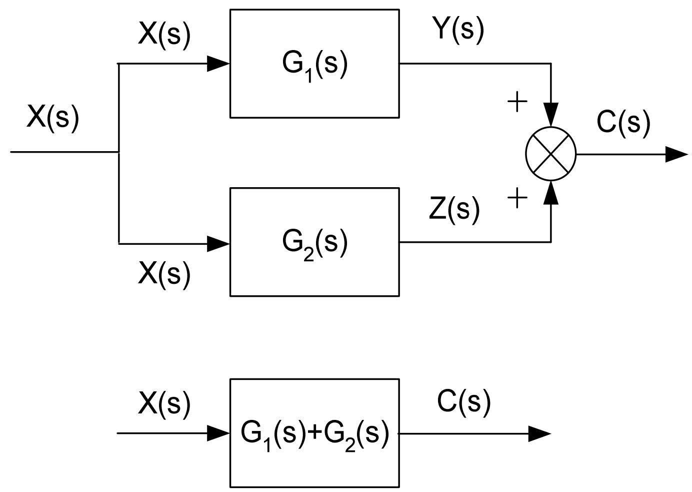
2. To "distribute" a signal, we use a branch point:



3. Series connection:



4. Parallel connection



Pole location / s-plane

In previous example

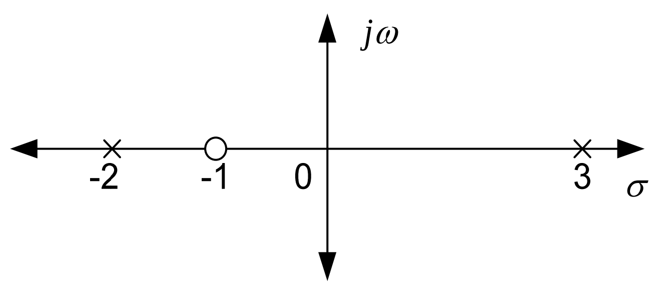
$$K\left(x_{in} - x_{out}\right) = m\frac{d^2 x_{out}}{dt^2} \underset{IC=0}{\overset{LT}{\Rightarrow}} \frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + K}$$

The order of the ODE is 2 = order of the denominator = order of the system.

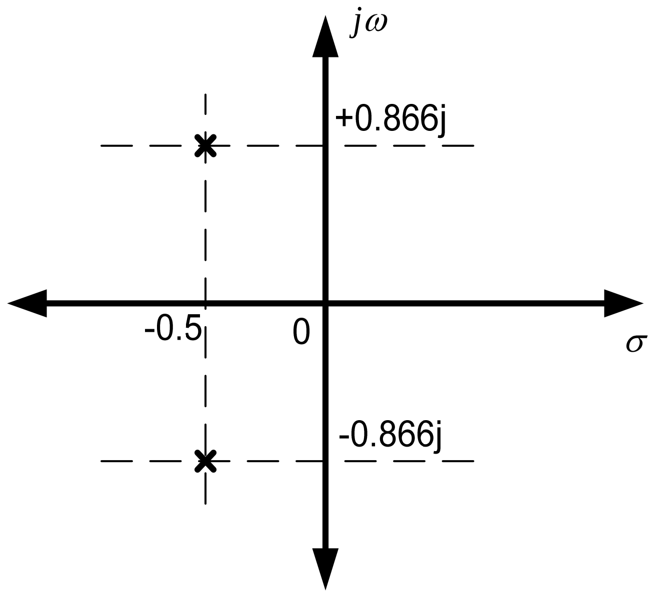
Example $\frac{Z(s)}{Y(s)} = \frac{K_1 K_2}{(m_1 s^2 + K_1 + K_2)(K_2 + m_2 s^2) - K_2^2}$ => order = 4.

$G(s) = \frac{N(s)}{D(s)}$: roots of the numerator are called zeros, while the roots of the denominator are called poles.

$\frac{s+1}{(s+2)(s-3)}$ one zero at s=-1 and two poles at s=-2 and s=+3:



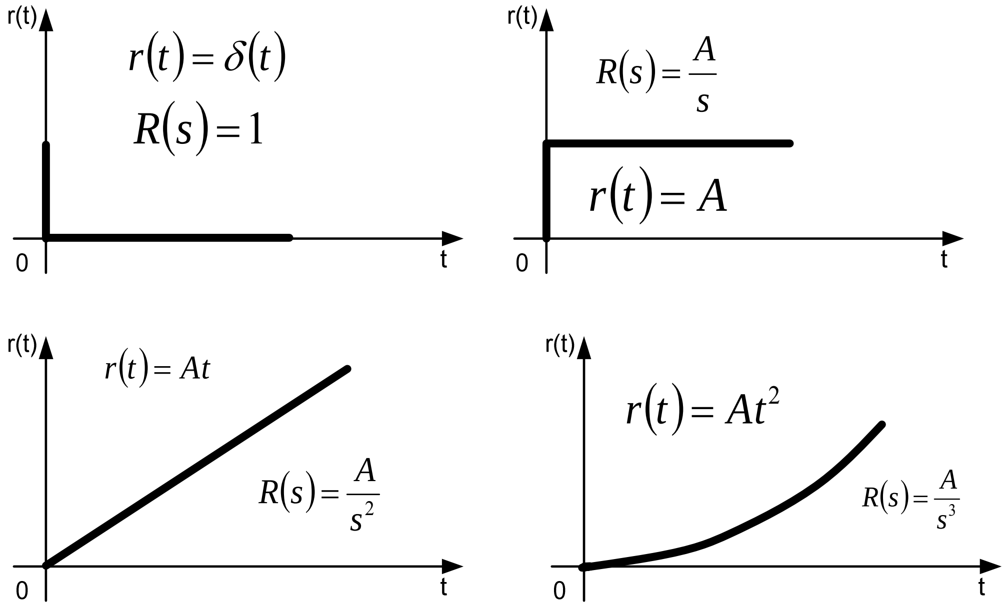
$$G(s) = \frac{1}{s^2 + s + 1}$$



Time domain characteristics

Typical input signals

1. The Dirac function
2. The step or the pulse function (!!!)
3. The Ramp function
4. Parabolic function

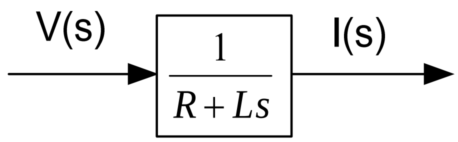


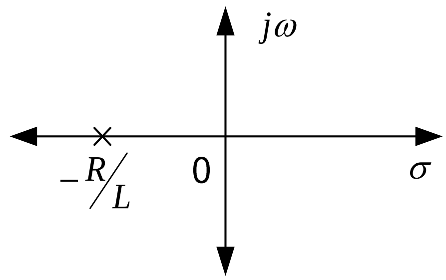
First order systems

Remember: $x'+ax=b$; HE: $x'+ax=0$, CE: $m+a=0$ a solution of HE: $x=C_1e^{at}$ hence stable solution if a<0 or the pole is at the LHS.

The same at s-plane:

$$\frac{I(s)}{V(s)}=\frac{1}{R+Ls}$$

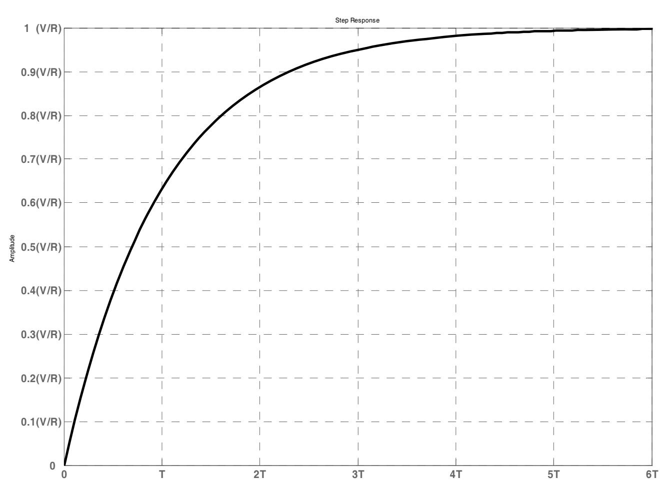


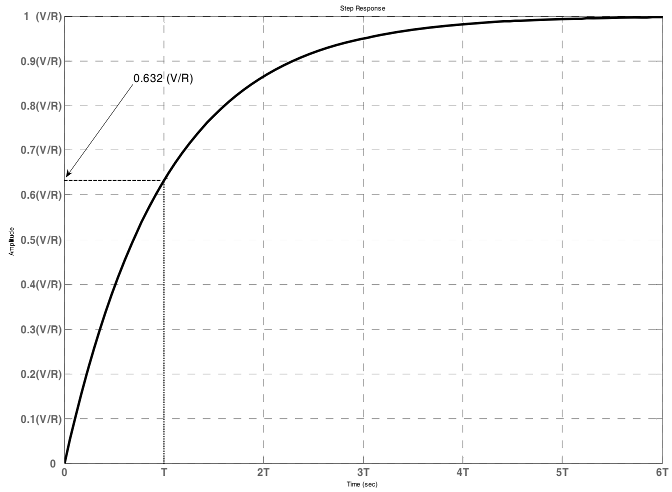


$$\frac{I(s)}{V(s)} = \frac{K}{1+\tau s}$$, K=1/R and τ=L/R. => $I(s) = V(s)\frac{K}{\tau s + 1}$

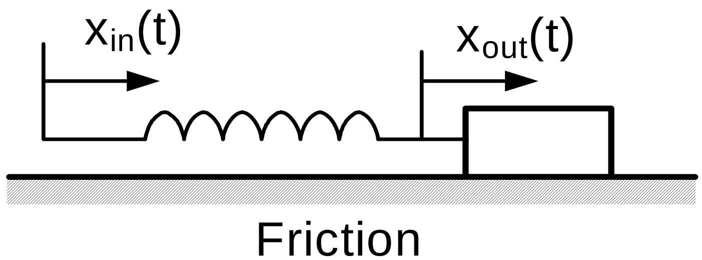
- Step response: $I(s) = K\frac{V}{s}\frac{1}{\tau s + 1}$ => $i(t) = \frac{V}{R}\left(1 - e^{-t/\tau}\right)$
- $i_F = \lim_{t\to\infty}\frac{V}{R}\left(1 - e^{-t/\tau}\right) = \frac{V}{R} - 0 = \frac{V}{R}$ or

$$I_{ss} = \lim_{s\to 0} s\frac{V}{s}\frac{K}{\tau s + 1} = VK = \frac{V}{R}$$





Second order systems



$$K\left(x_{in} - x_{out}\right) - B\frac{dx_{out}}{dt} = m\frac{d^2 x_{out}}{dt^2} \underset{IC=0}{\overset{LT}{\Rightarrow}}$$

$$KX_{in}(s) - KX_{out}(s) - BsX_{out}(s) = ms^2 X_{out}(s)$$

$$KX_{in}(s) = \left(K + Bs + ms^2\right)X_{out}(s)$$

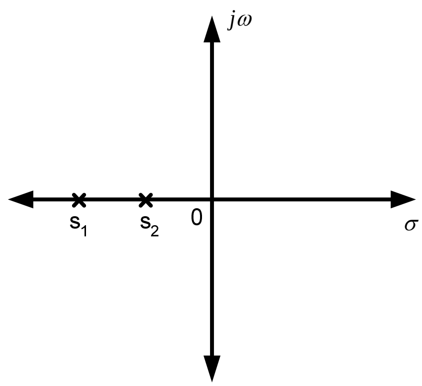
$$\frac{X_{out}(s)}{X_{in}(s)} = \frac{K}{ms^2 + Bs + K} \Rightarrow \frac{X_{out}(s)}{X_{in}(s)} = \frac{K/m}{s^2 + B/m\, s + K/m}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2s\zeta\omega_n + \omega_n^2},\ 2\zeta\omega = B/m, \quad \omega_n^2 = K/m$$

Roots of CE: $\Rightarrow s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

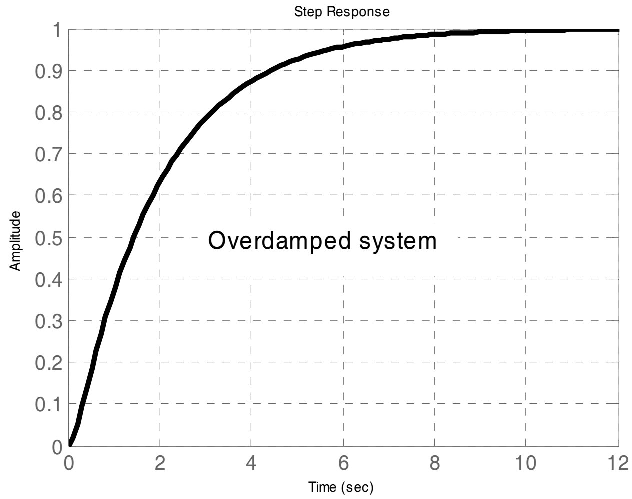
Case 1: $\zeta > 1$

Then the system has two negative real roots and is called overdamped:



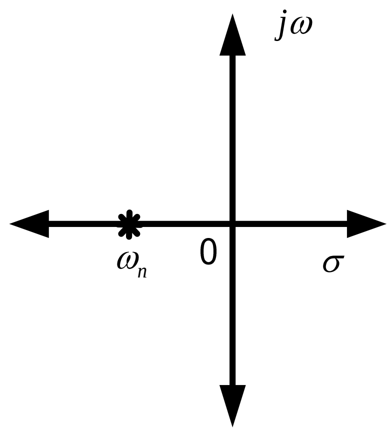
$$c(t) = 1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{+s_1 t}}{s_1} - \frac{e^{+s_2 t}}{s_2}\right)$$

$\zeta >> 1 \Rightarrow c(t) = 1 - e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t}$ => Overdamped system will be like a very slow response of a first order system

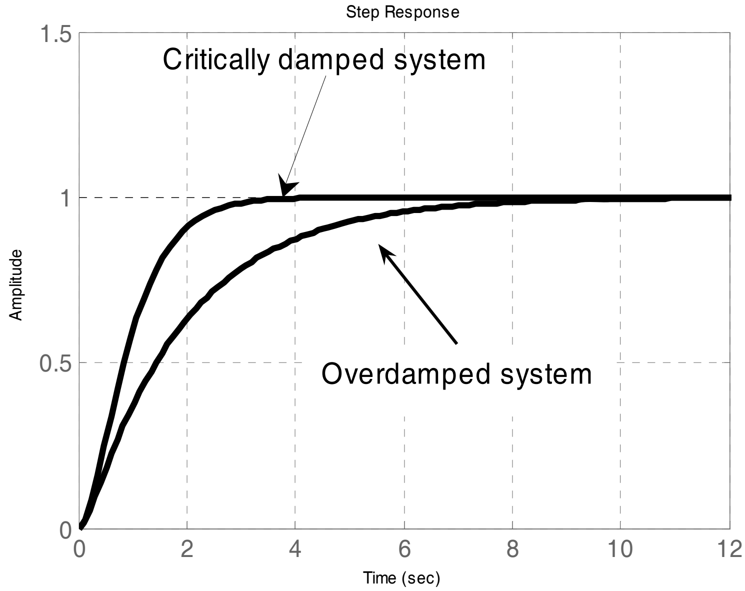


Case 2: $\zeta = 1$

The system has two equal real roots at $s = \omega_n$:

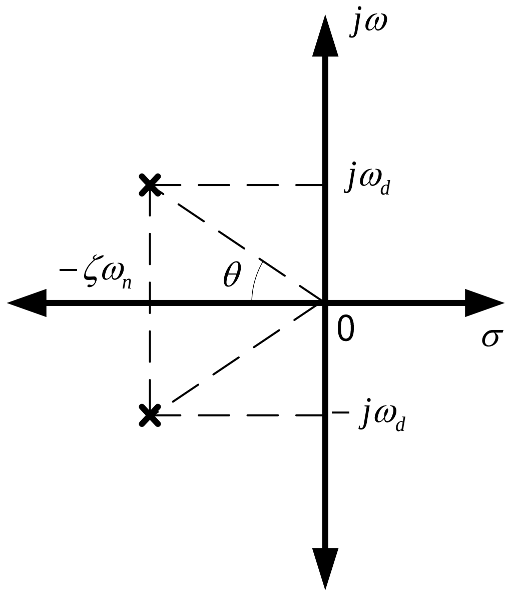


$c(t) = 1 - e^{\omega_n t}(1 + \omega_n t)$:



Case 3: $0 < \zeta < 1$

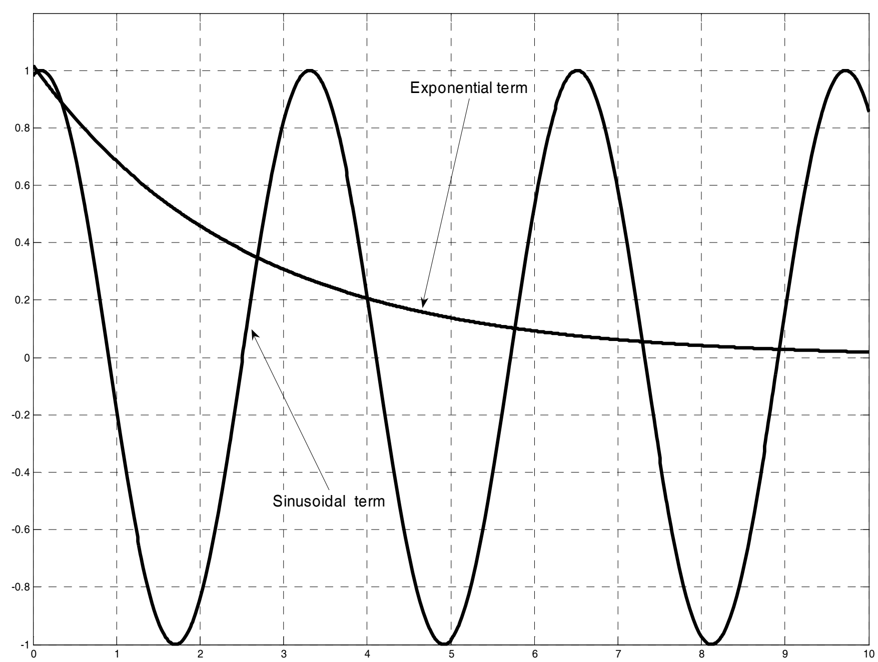
$$\Rightarrow s = \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \text{ or } \Rightarrow s = \zeta\omega_n \pm j\omega_d$$



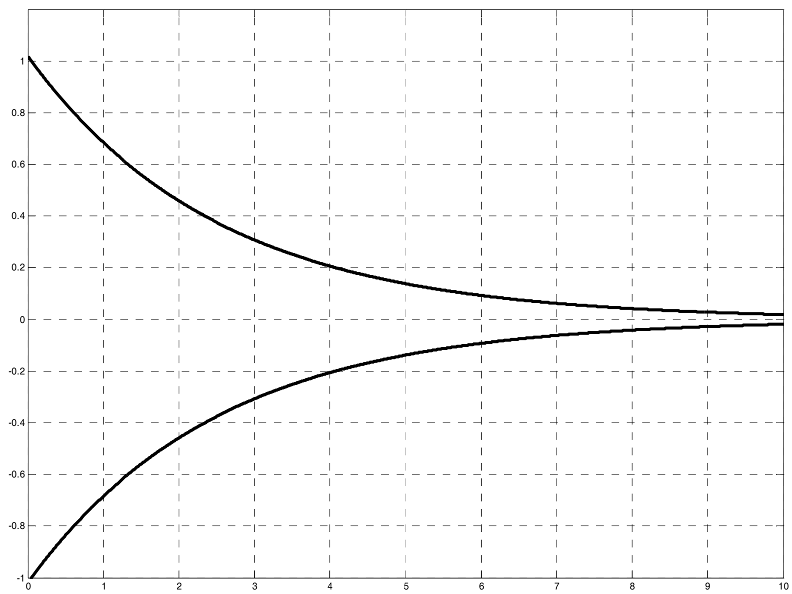
The line between the origin and the pole is: $d = \sqrt{\omega_d^2 + \zeta^2 \omega_n^2} = \sqrt{\omega_n^2 \left(1 - \zeta^2\right) + \zeta^2 \omega_n^2} = \omega_n$

The angle is $\cos\theta = \dfrac{\omega_n \zeta}{\omega_n} \Leftrightarrow \theta = \cos^{-1}(\zeta)$.

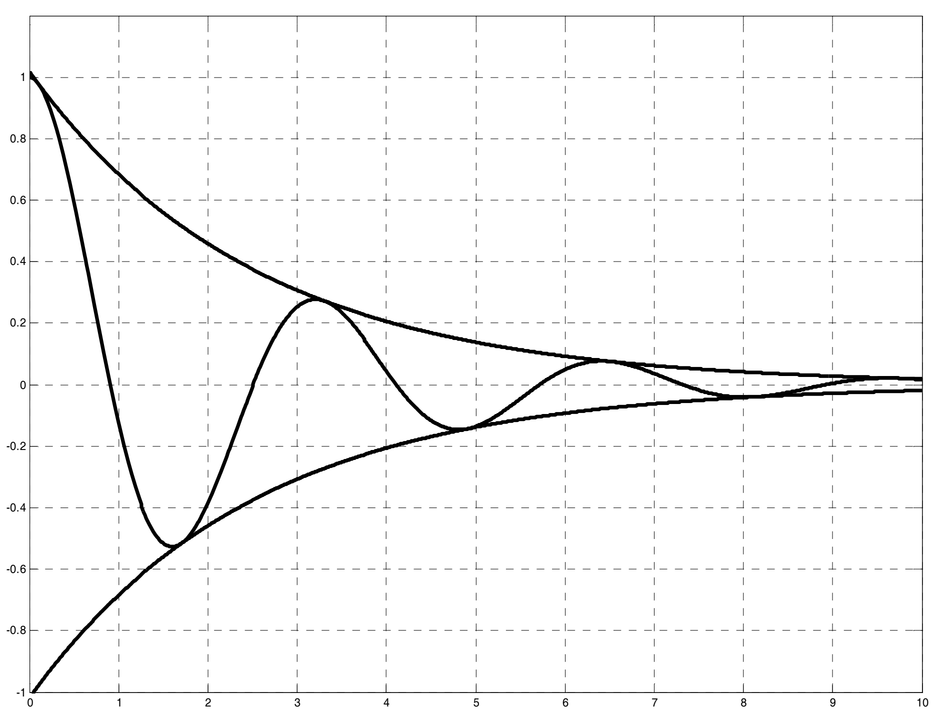
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



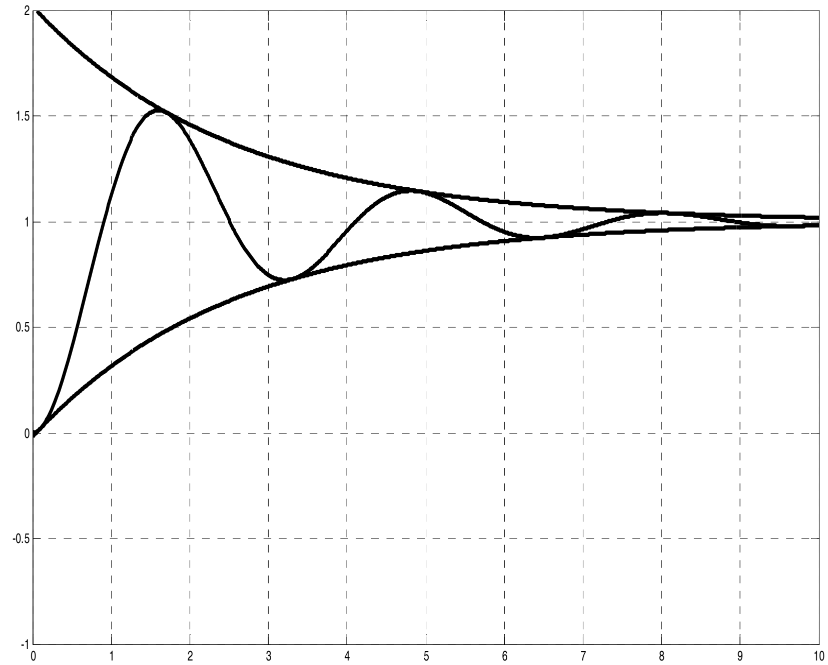
The envelope that will be created from the exponential terms is:

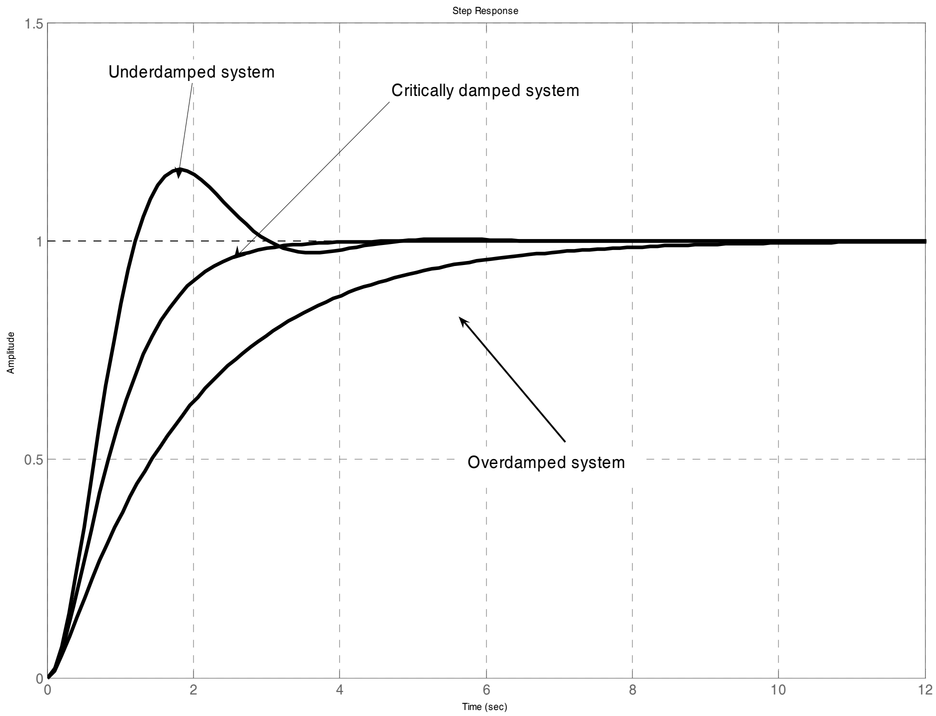


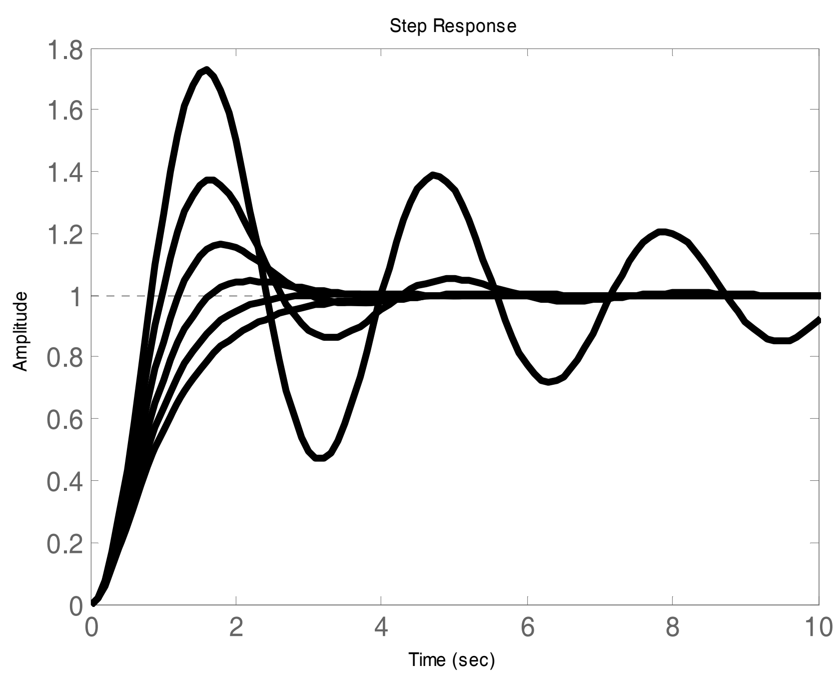
And their product:



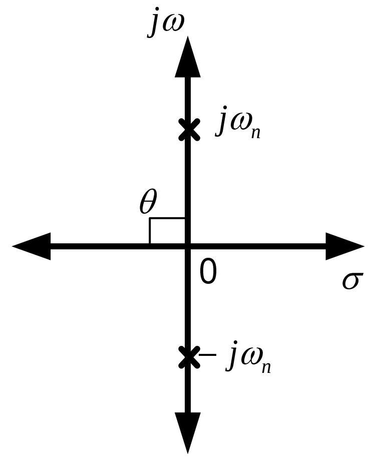
And by adding the constant factor:


2
1.5
1
0.5
0
-0.5
-1
0
1
2
3
4
5
6
7
8
9
10

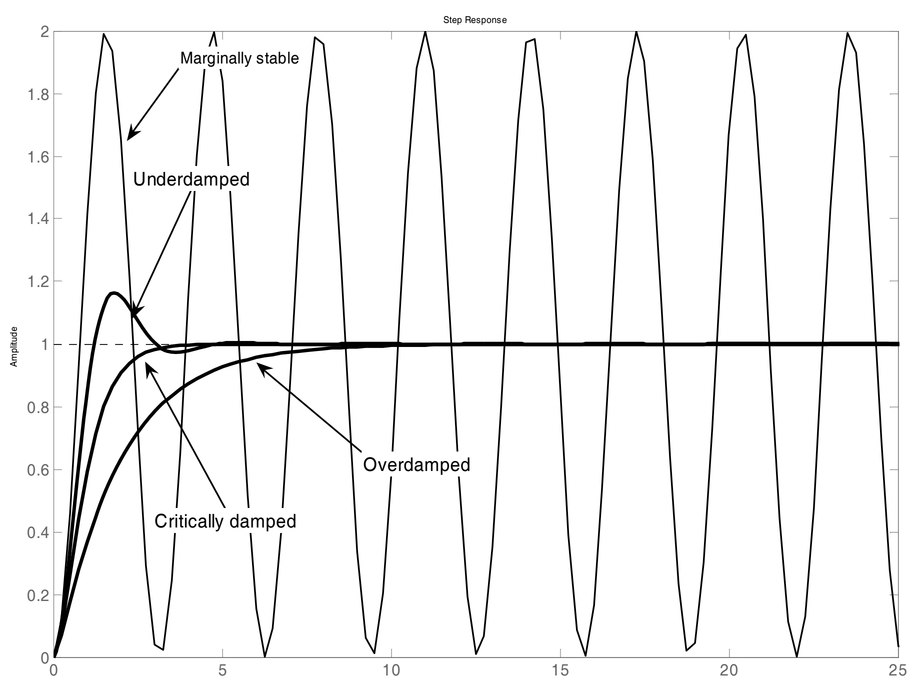

Step Response
Underdamped system
Critically damped system
Overdamped system
Amplitude
Time (sec)
1.5
1
0.5
0
0
2
4
6
8
10
12


Step Response
Amplitude
Time (sec)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
2
4
6
8
10

Case 4: $\zeta = 0$

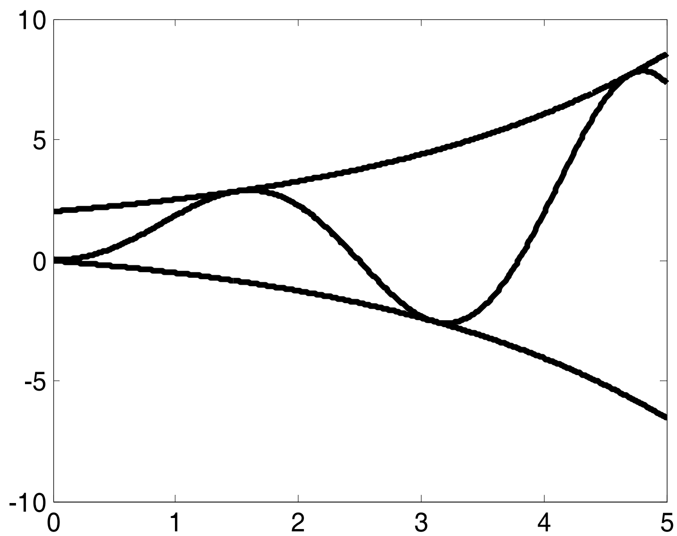


Note: The system is called marginally stable because the solutions do not diverge to infinity. Hence if the previous four cases are combined to one graph:

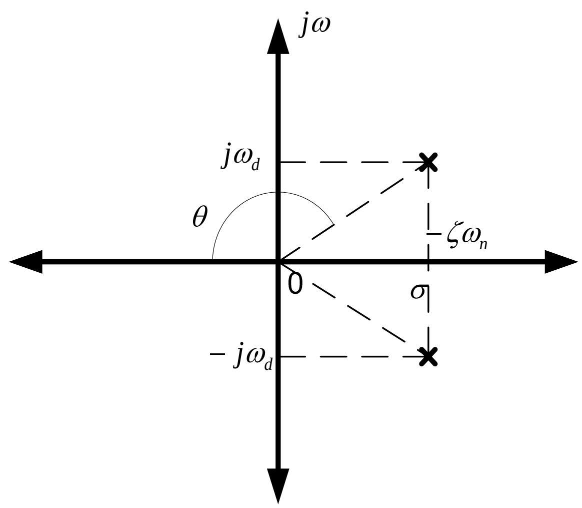


Case 5: $\zeta < 0$

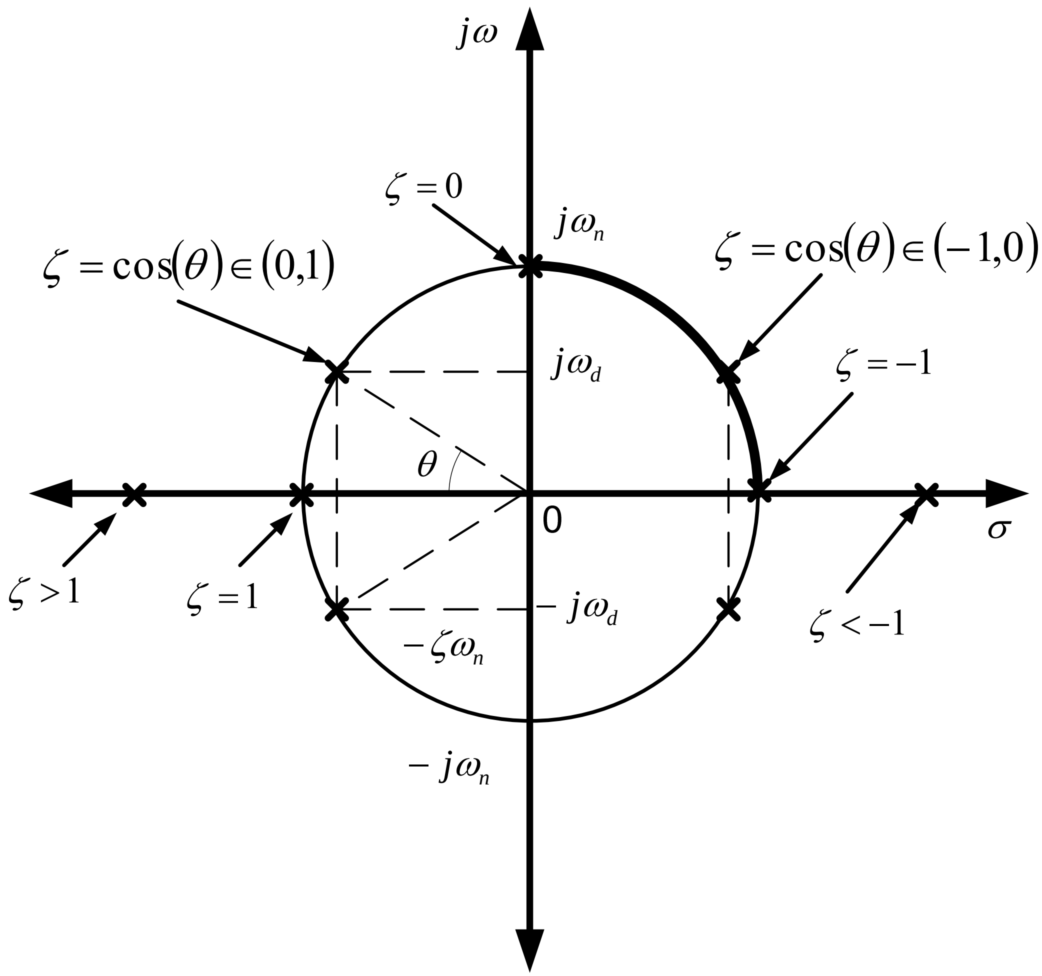
$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$



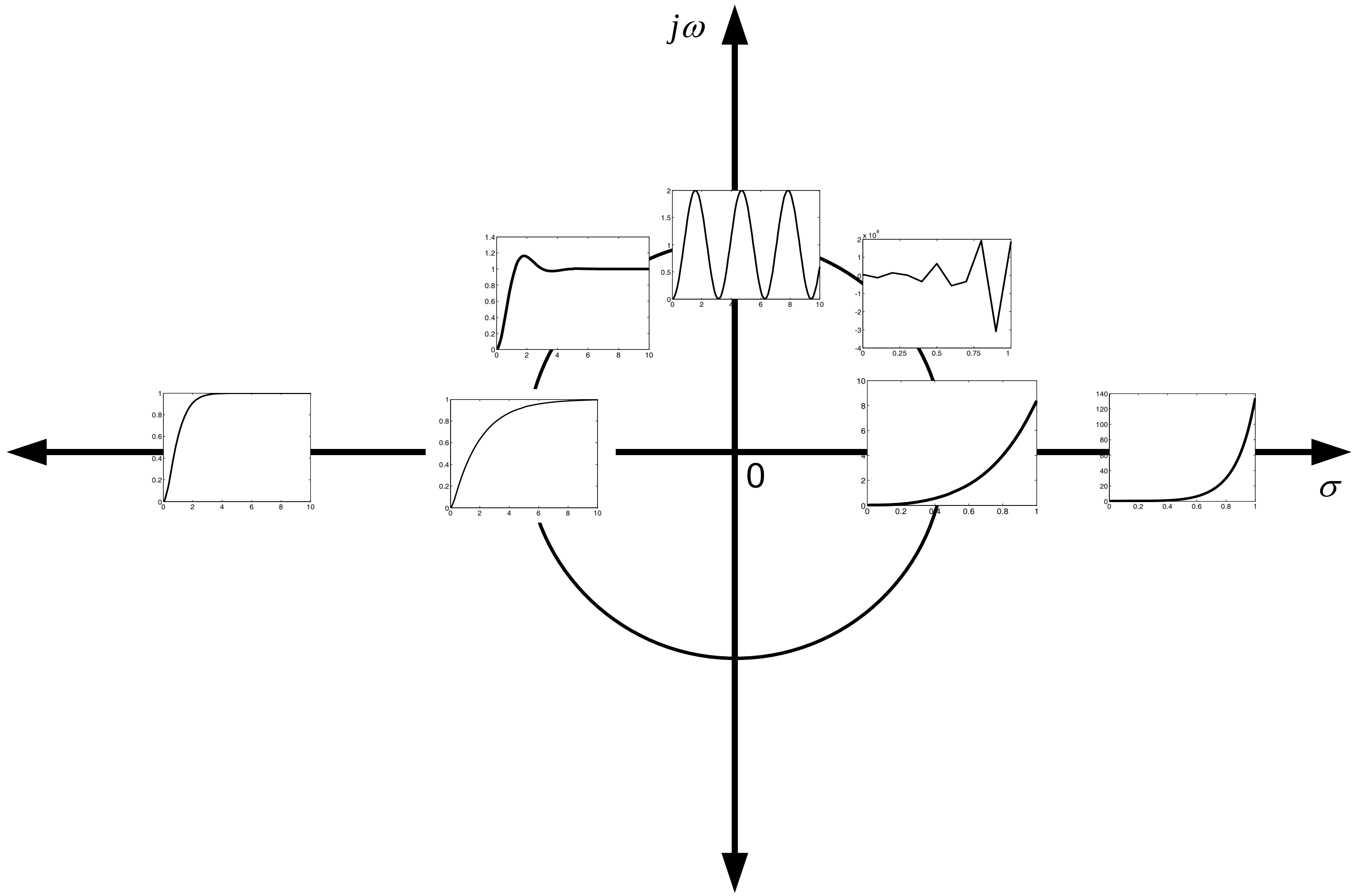
Since $\zeta < 0$ then angle θ defined in the s-plane ($\cos\theta = \frac{\omega_n\zeta}{\omega_n} \Leftrightarrow \theta = \cos^{-1}(\zeta)$) has to be greater than 90°:



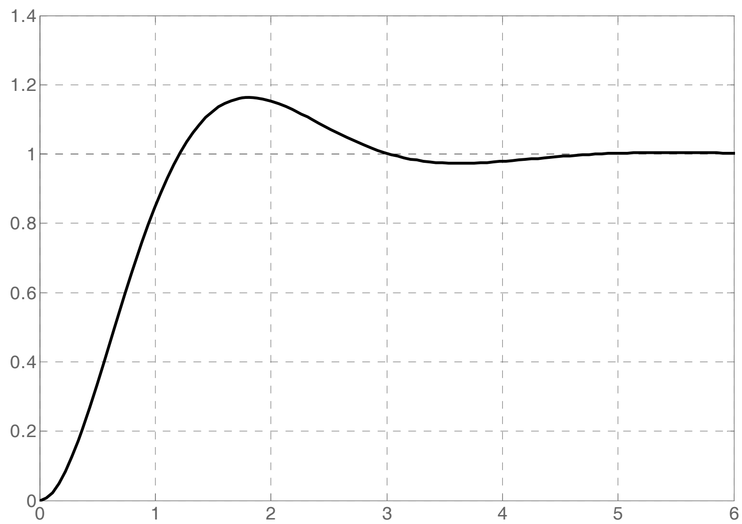
By combining the previous s-planes we have:



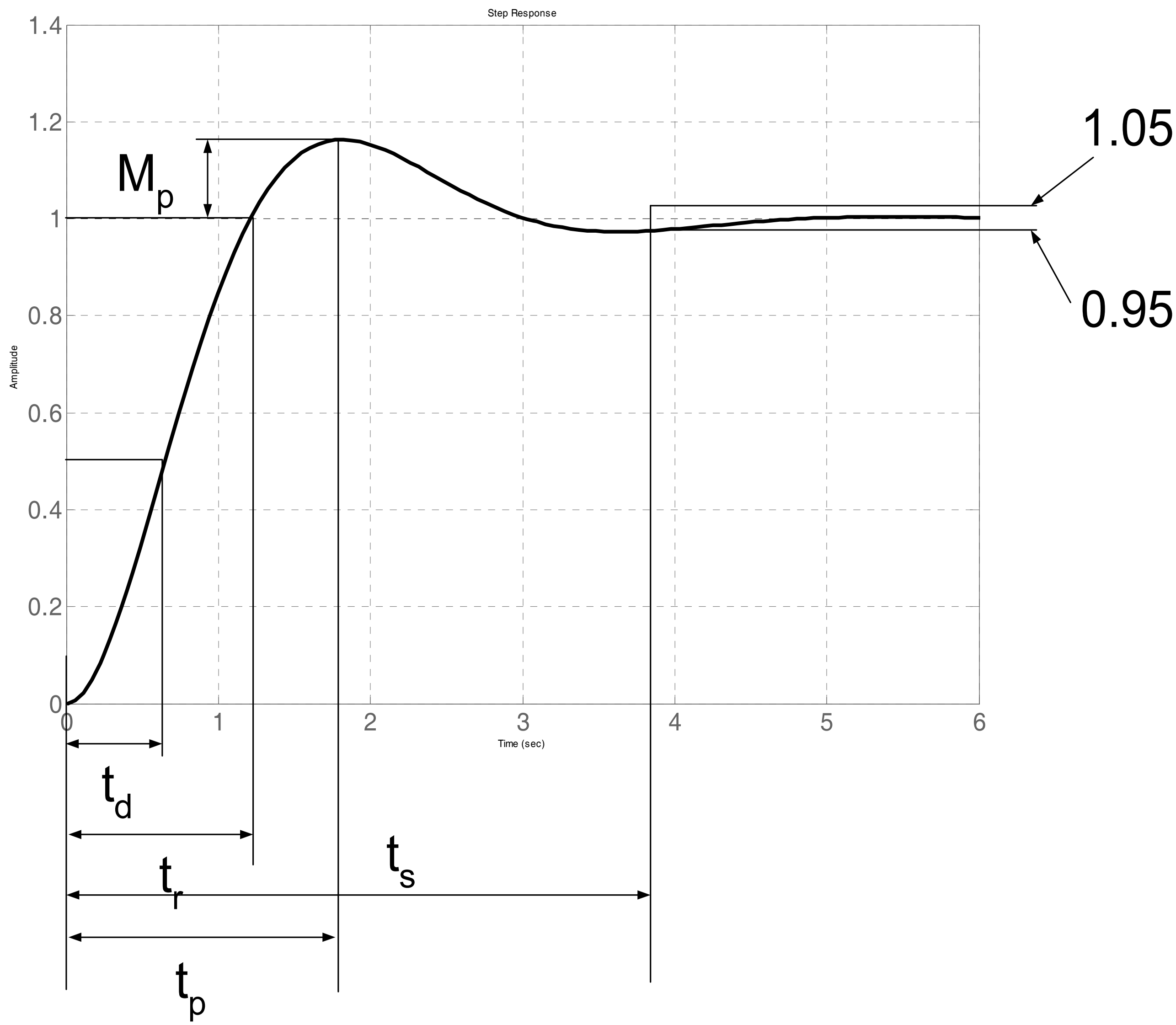
$j\omega$
$\zeta = 0$
$j\omega_n$
$\zeta = \cos(\theta) \in (0,1)$
$\zeta = \cos(\theta) \in (-1,0)$
$j\omega_d$
$\zeta = -1$
θ
0
σ
$\zeta > 1$
$\zeta = 1$
$-j\omega_d$
$\zeta < -1$
$-\zeta\omega_n$
$-j\omega_n$


$j\omega$
0
σ

A general response is:



- Time that the system needs to reach half of its final value:
- Rise time (10%-90% or 5%-95% or 0%-100%) $t_r = \frac{\pi - \theta}{\omega_d}$
- Peak time: $t_p = \frac{\pi}{\omega_d}$
- Maximum overshoot: $Mp = e^{-\left(\zeta / \sqrt{1-\zeta^{"}}\right)\pi}$
- Settling time: $t_{s\,5\%} = \frac{3}{\zeta\omega_n}$ and $t_{s\,2\%} = \frac{4}{\zeta\omega_n}$


Step Response
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Amplitude
0
1
2
3
4
5
6
Time (sec)
M_p
1.05
0.95
t_d
t_r
t_s
t_p

Extra poles and zeros

General form of a TF: $\frac{C(s)}{R(s)} = \frac{b_0 s^m + ... + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_{n-1}s + a_n}$

For a step input:

$$C(s) = \frac{1}{s} + \sum_{j=1}^{q} \frac{a_j}{s + p_j} + \sum_{k=1}^{r} \frac{b_k(s + \zeta_k \omega_k) + c_k \omega_k \sqrt{1 - \zeta_k}}{s^2 + 2\zeta_k \omega_k s + \omega_k^2}$$

$(q + 2r = n)$, i.e. combination of first and second order systems.

Example: $\frac{1}{s^3 + as^2 + bs + c} = \frac{1}{(s + f)(s^2 + ds + e)}$ =>

$$s^3 + as^2 + bs + c = (s + f)(s^2 + ds + e) \Leftrightarrow$$

$$\Leftrightarrow s^3 + as^2 + bs + c = s^3 + (d + f)s^2 + (e + fd)s + fe$$

$$\Leftrightarrow \begin{Bmatrix} 1 = 1 \\ a = d + f \\ b = e + fd \\ c = fe \end{Bmatrix}$$

- The response of a higher order system is the sum of exponential and damped sinusoidal curves.
- Assuming that all poles are at the left hand side then the final value of the output is “1” since all exponential terms will converge to 0.

- Let's assume that some poles have real parts that are far away from the imaginary axis=>

$$c(t)=1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t+\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right) => e^{-\zeta\omega_n t}\rightarrow 0$$

  - Overall performance is characterised by the isolated (far away from zeros) poles that are close to the imaginary axis.
  - If we have only one pole (or a pair for complex roots) that is closed to the real axis then we say that this pole (or pair of poles) is (are) the DOMINANT pole(s) for the system.
  - A simple rule is that the dominant poles must be at least five to ten times closer to the imaginary axis than the other ones.

$$C(s)=\frac{1}{s}+\sum_{j=1}^{q}\frac{a_j}{s+p_j}+\sum_{k=1}^{r}\frac{b_k\left(s+\zeta_k\omega_k\right)+c_k\omega_k\sqrt{1-\zeta_k}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

$$c(t)=1+\sum_{j=1}^{q}a_j e^{-p_j t}+\sum_{k=1}^{r}b_k e^{-\zeta_k\omega_k t}\cos\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right)+$$
$$+\sum_{k=1}^{r}c_k e^{-\zeta_k\omega_k t}\sin\left(\omega_k\sqrt{1-\zeta_k{}^2}\,t\right)$$

The values of b (numerator coefficients) determine the amplitude of the oscillations of the system.