Chapter #1

EEE8115-EEE8086

Robust and Adaptive Control Systems

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Robust and Adaptive Control Systems

Dear Student,

Welcome to Robust and Adaptive Control Systems. This module continues from where EEE8031 (or EEE3001) stopped, so we will see more advanced controllers like sliding mode. It also will cover similar topics as in EEE8013 but in greater depth, for example we will also see how nonlinear systems behave. It is an interesting module that requires (as the EEE8013 did) constant work. In this module, very briefly we will work on the following topics:

1) Differential Equations, focusing on High order ODEs.
2) Normal or Canonical Form of state space models.
3) Geometry in the state space.
4) Nonlinear dynamics.
5) Robust and Adaptive control.

As with EEE8013, BB will not be used, but all the material, like handouts, lecture notes, Simulink files and others will be uploaded regularly at:

https://www.staff.ncl.ac.uk/damian.giaouris/teaching.html

Remember: "**There are no stupid questions, but only stupid answers**"!

Ordinary Differential Equations

1. Introduction

To understand the properties (dynamics) of a system, we can model (represent) it using differential equations (DEs). The response/behaviour of the system is found by solving the DEs. In our cases, the DE is an Ordinary DE (ODE), i.e. not a partial derivative. The main purpose of this Chapter is to learn how to solve first and second order ODEs in the time domain. This will serve as a building block to model and study more complicated systems. Our ultimate goal is to control the system when it does not show a "satisfactory" behaviour. Effectively, this will be done by modifying the ODE.

Note for EEE8013 students: There are footnotes throughout the notes, which **is assessed material**!

2. First Order ODEs

The general form of a first order ODE is:

$$\frac{dx(t)}{dt} = f\left(x(t),t\right) \quad (1)$$

where[1] $x,t \in \mathbb{R}$

Analytical solution: Explicit formula for *x*(*t*) (a solution which can be found using various methods) which satisfies $\frac{dx}{dt} = f(x,t)$

[1] The proper notation is *x*(*t*) and not *x* but we drop the brackets in order to simplify the presentation.

Example 1.1: Prove that $x = e^{-3t}$ and $x = -10e^{-3t}$ are solutions of $\frac{dx}{dt} = -3x$.

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(e^{-3t}\right)}{dt} = -3\left(e^{-3t}\right) \Leftrightarrow -3e^{-3t} = -3e^{-3t}$$

$$\frac{dx}{dt} = -3x \Leftrightarrow \frac{d\left(-10e^{-3t}\right)}{dt} = -3\left(-10e^{-3t}\right) \Leftrightarrow 30e^{-3t} = 30e^{-3t}$$ ■[2]

Obviously there are infinite solutions to an ODE and for that reason the found solution is called the **General Solution** of the ODE.

First order Initial Value Problem : $\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$

An initial value problem is an ODE with an initial condition, hence we do not find the general solution but the **Specific Solution** that passes through x_0 at $t=t_0$.

Analytical solution: Explicit formula for $x(t)$ which satisfies $\frac{dx}{dt} = f(x,t)$ and passes through x_0 when $t = t_0$.

Example 1.2: Prove that $x = e^{-3t}$ is a solution, while $x = -10e^{-3t}$ is not a solution of $\frac{dx}{dt} = -3x, x_0 = 1$

Both expressions ($x = e^{-3t}$ and $x = -10e^{-3t}$) satisfy the $\frac{dx}{dt} = -3x$ but at t=0

$$x(t) = e^{-3t} \Rightarrow x(0) = 1$$

$$x(t) = -10e^{-3t} \Rightarrow x(0) = -10 \neq 1$$ ■[3]

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[2]
```
clc, clear all, syms t, x1=exp(-3*t); dx=diff(x1,t); isequal(dx,-3*x1)
x2=-10*exp(-3*t); dx=diff(x2,t); isequal(dx,-3*x2)
```

[3]
```
clc, clear all, syms t, x1=exp(-3*t); x2=-10*exp(-3*t); x0=1;
x0_1=double(subs(x1,t,0)); x0_2=double(subs(x2,t,0)); isequal(x0,x0_1),
isequal(x0,x0_2),
```

For that reason some books use a different symbol for the specific solution: $\phi(t, t_0, x_0)$.

You must be clear about the difference between an ODE and the solution to an IVP! From now on we will just study IVP unless otherwise explicitly mentioned.

Linear First Order ODEs

A linear 1st order ODE is given by:

$$\begin{cases} a(t)x' + b(t)x = c(t), a(t) \neq 0 & Non \quad autonomous \\ ax' + bx = c, a \neq 0 & Autonomous \end{cases} \tag{2}$$

with $a, b, c \in \mathbb{R}$ and $a \neq 0$.

In engineering books the most common form of (2) is (since $a \neq 0$):

$$x' + k(t)x = u(t) \tag{3}$$

with $k, u \in \mathbb{R}$

Note: We say that *u* is the input to our system that is represented by (3)

The solution of (3) (using the integrating factor) is given by:

$$x(t) = e^{-kt}x(t_0) + e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)\,dt_1$$

The term $e^{-kt}x(t_0)$ is called transient response, while $e^{-kt}\int_{t_0}^{t} e^{kt_1}u(t_1)dt_1$ comes from the input signal u.

If we assume that u is constant:

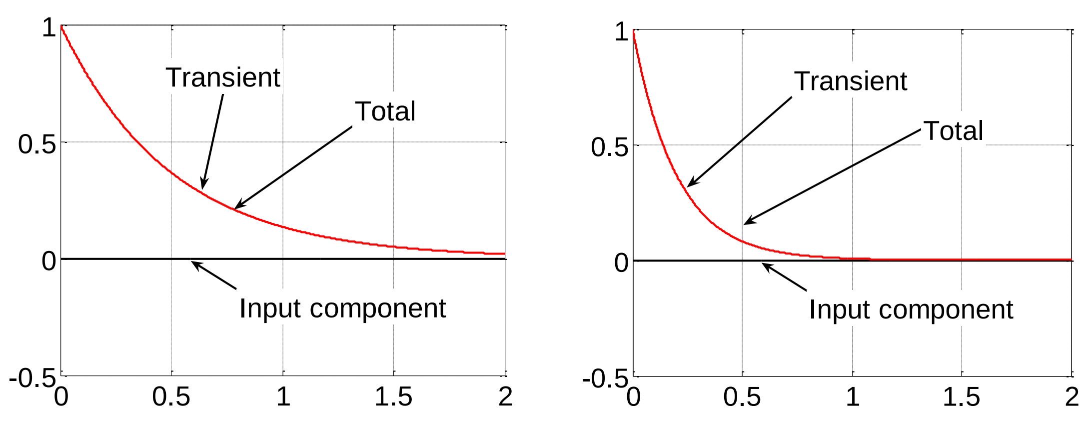
$$x(t) = e^{-kt}x(t_0) + e^{-kt}\int_{t_0}^{t} e^{kt_1}u\,dt_1 \Leftrightarrow x(t) = e^{-kt}x(t_0) + u\frac{1}{k}\left(1 - e^{-k(t-t_0)}\right)$$

Hence: $\lim_{t\to\infty} x(t) = \begin{cases} 0 + u\frac{1}{k}(1-0) = u/k, & k>0 \\ \pm\infty, & k<0 \end{cases}$

Thus we say that if k>0 the system is stable (and the solution converges exponentially at u/k) while if k<0 the system is unstable (and the solution diverges exponentially to $\pm\infty$,).

Example 1.3: u=0 and k=2 & 5, x_0=1

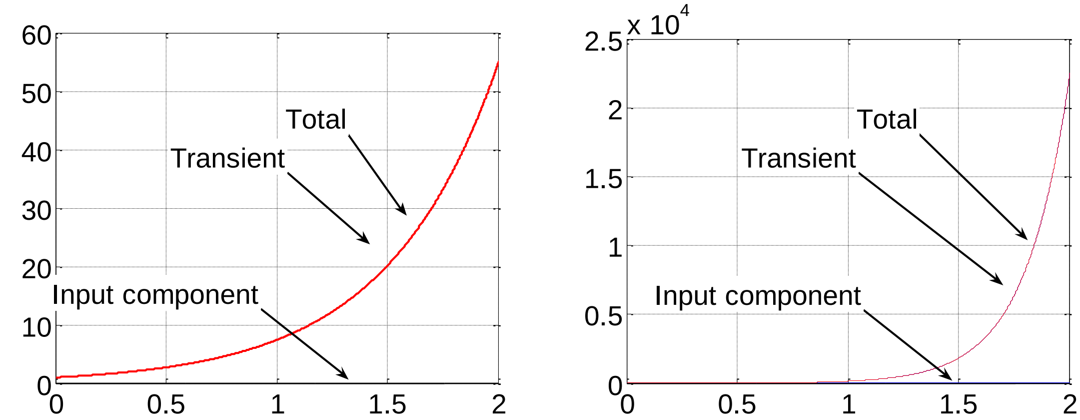
$$x(t) = e^{-2t}\cdot 1 + 0,\ \lim_{t\to\infty} x(t) = 0,\ \text{as } 2>0$$



■[4]

[4] clc, clear all, syms x(t), dx=diff(x); dsolve(dx+2*x, x(0)==1)

Example 1.4: u=0 and k=-2 & 5, x_0=1



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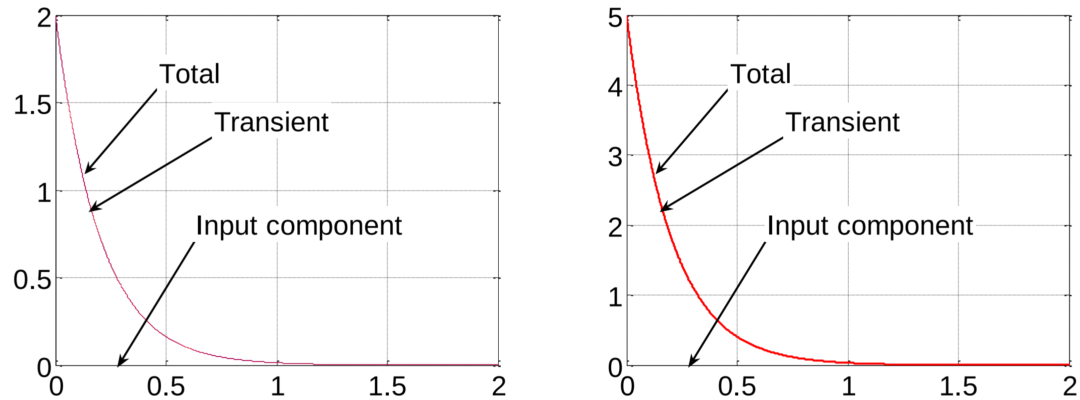
Example 1.5: u=0 and k=5, x_0=1 & 5



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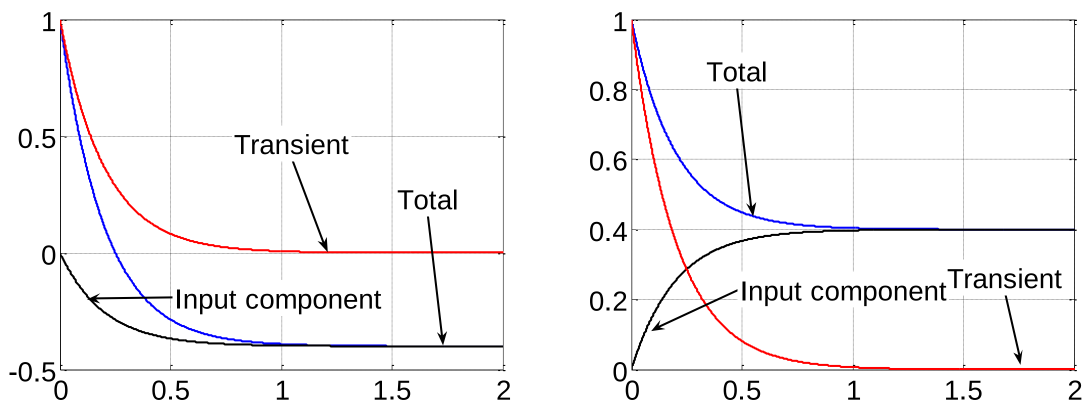
Example 1.6: u=-2 & 2 and k=5, x_0=1



■[5]

[5]
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clc, clear all, close all, syms t t1, x0=1; k=5; t0=0; u=2;
t2=0:0.01:2; x_x0=exp(-k*t)*x0; x_u=exp(-k*t)*int(exp(k*t1)*u,t0,t);
x_x0_t=double(subs(x_x0,t,t2)); x_u_t=double(subs(x_u,t,t2));
hold on, plot(t2,x_x0_t), plot(t2,x_u_t), plot(t2,x_u_t+x_x0_t)
```

Comments:

- In real systems we cannot have a state (say the speed of a mass-spring system) that becomes infinite, obviously the system will be destroyed when *x* gets to a high value.
- For the dynamics (settling time, stability…) of the system we should only focus on the homogenous ODE: $x' + k(t)x = 0$

3. Second Order ODEs

3.1 General Material

A second order ODE has as a general form:

$$\frac{d^2x(t)}{dt^2} = f\left(x'(t), x(t), t\right) \tag{4}$$

A linear 2nd order ODE is given by:

$$\begin{cases} x''(t) + A(t)x'(t) + B(t)x(t) = u(t), & Non\ autonomous \\ x''(t) + Ax'(t) + Bx(t) = u(t), & Autonomous \end{cases} \tag{5}$$

And again we focus on autonomous homogeneous systems:

$$x''(t) + A(t)x'(t) + B(t)x(t) = 0 \tag{6}$$

Again we define as an analytical solution of (6) an expression that satisfies it.

Example 1.7: Given $x''-2x'-3x=0$ prove that $x=e^{3t}$ and $x=e^{-t}$ are two solutions:

$$\left(e^{3t}\right)''-2\left(e^{3t}\right)'-3\left(e^{3t}\right)=0 \Leftrightarrow$$

$$9e^{3t}-6e^{3t}-3e^{3t}=0 \Leftrightarrow$$

$$0=0$$

$$\left(e^{-t}\right)''-2\left(e^{-t}\right)'-3\left(e^{-t}\right)=0 \Leftrightarrow$$

$$e^{-t}+e^{-t}-3e^{-t}=0 \Leftrightarrow$$

■[6]

$$0=0$$

Assume that you have 2 solutions for a 2nd order ODE x_1 and x_2 (we will see later how to get these two solutions), then:

$$\left.\begin{aligned} x_1''(t)+A(t)x_1'(t)+B(t)x_1(t)=0 \\ x_2''(t)+A(t)x_2'(t)+B(t)x_2(t)=0 \end{aligned}\right\}$$

obviously I can multiply these two equations with arbitrary constants:

$$\left.\begin{aligned} C_1x_1''(t)+C_1A(t)x_1'(t)+C_1B(t)x_1(t)=0 \\ C_2x_2''(t)+C_2A(t)x_2'(t)+C_2B(t)x_2(t)=0 \end{aligned}\right\}$$

and now I can add them and collect similar terms:

$$\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}''+A(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}'+B(t)\underbrace{\left(C_1x_1(t)+C_2x_2(t)\right)}_{\text{Common Term}}=0$$

which means that $C_1x_1(t)+C_2x_2(t)$ (i.e. the linear combination of x_1 and x_2) is also a solution of the ODE.

6
```
clc, clear all, close all, syms x(t) t
Dx=diff(x); D2x=diff(x,2); ODE=D2x-2*Dx-3*x;
subs(ODE, x, exp(-t)), subs(ODE, x, exp(3*t))
```

Example 1.8: Given $x''-2x'-3x=0$ prove that $x=e^{3t}+2e^{-t}$ is a solution:

$$\left(e^{3t}+2e^{-t}\right)''-2\left(e^{3t}+2e^{-t}\right)'-3\left(e^{3t}+2e^{-t}\right)=0 \Leftrightarrow$$

$$9e^{3t}+2e^{-t}-2\left(3e^{3t}-2e^{-t}\right)-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$

$$9e^{3t}+2e^{-t}-6e^{3t}+4e^{-t}-3e^{3t}-6e^{-t}=0 \Leftrightarrow$$

$$9e^{3t}-6e^{3t}-3e^{3t}+2e^{-t}+4e^{-t}-6e^{-t}=0 \Leftrightarrow$$

$$0=0$$

■[7]

Now, the question is, if we have x_1 and x_2, can ALL other solutions of the ODE, be expressed as a linear combination of x_1 and x_2? So assume a third solution $\varphi(t)$:

$$\varphi''(t)+A(t)\varphi'(t)+B(t)\varphi(t)=0$$

Now, the question can be written as, can we find constants C_1 and C_2 such as:

$$\begin{Bmatrix} \varphi(t)=C_1x_1(t)+C_2x_2(t) \\ \varphi'(t)=C_1x_1'(t)+C_2x_2'(t) \end{Bmatrix}$$

This equation can be seen as a 2by2 system with unknowns C_1 and C_2 as:

$$\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}=\begin{bmatrix} \varphi(t) \\ \varphi'(t) \end{bmatrix}$$

From linear algebra this system of equations has a unique solution if:

[7]
```
clc, clear all, close all, syms x(t) t; Dx=diff(x); D2x=diff(x,2);
ODE=D2x-2*Dx-3*x; subs(ODE, x, exp(3*t)+exp(-t))
```

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_2(t)x_1'(t) \neq 0$$

Note: The matrix $W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix}$ is called the Wronskian[8] of the ODE.

We also know from linear algebra that the determinant is not zero if:

$$\begin{bmatrix} x_1(t) \\ x_1'(t) \end{bmatrix} \neq C \begin{bmatrix} x_2(t) \\ x_2'(t) \end{bmatrix}$$

So if the two solutions x_1 and x_2 are linear independent (LI) then ANY other solution can be described by the linear combination of x_1 and x_2. So now we have to look for two LI solutions for the 2nd order ODE.

Example 1.9: Prove that two solutions of $x'' - 2x' - 3x = 0$, $x_1 = e^{3t}$ and $x_2 = e^{-t}$ are linear independent.

$$W(x_1(t), x_2(t)) = \begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{bmatrix} = \begin{bmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{bmatrix} \Rightarrow |W| = \begin{vmatrix} e^{3t} & e^{-t} \\ 3e^{3t} & -e^{-t} \end{vmatrix} \Rightarrow$$

$$|W| = e^{3t}\left(-e^{-t}\right) - 3e^{3t}e^{-t} = -e^{2t} - 3e^{2t} = -4e^{2t}$$ ■[9]

[8] From the Polish mathematician Józef Maria Hoëne-Wroński

[9]
```
clc, clear all, close all, syms t, x1=exp(3*t); x2=exp(-t);
Dx1=diff(x1); Dx2=diff(x2);  W=[x1, x2; Dx1, Dx2], det(W)
```

Example 1.10: Prove that two solutions of $x''-2x'-3x=0$, $x_1=e^{3t}$ and $x_2=2e^{3t}$ are NOT linear independent.

$$W\left(x_1(t),x_2(t)\right)=\begin{bmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t)\end{bmatrix}=\begin{bmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t}\end{bmatrix}\Rightarrow$$

$$|W|=\begin{vmatrix} e^{3t} & 2e^{3t} \\ 3e^{3t} & 6e^{3t}\end{vmatrix}=6e^{6t}-6e^{6t}=0$$ ■

Example 1.11: For the ODE $x''-2x'-3x=0$ prove that the solution $x=-e^{3t}+2e^{t}$ cannot be written as any combination of $x_1=e^{3t}$ and $x_2=2e^{3t}$.

$$x=C_1x_1+C_2x_2 \Leftrightarrow -e^{3t}+2e^{t}=C_1e^{3t}+C_2e^{3t}=\left(C_1+C_2\right)e^{3t}$$

From this expression we have that $C_1+C_2=-1$ (and hence we have the term $-e^{3t}$) but there is no term e^{t} for $2e^{t}$. ■

But how can we find two LI solutions? For homogeneous 1st order ODEs with *u*=0 the solution was: $x(t)=e^{-kt}C$ so we will try a similar approach for 2nd order ODEs:

$x''+Ax'+Bx=0$, assume[10] $x=e^{rt}$ => $x'=re^{rt}$ & $x''=r^2e^{rt}$ =>

$$x''+Ax'+Bx=0 \Leftrightarrow r^2e^{rt}+Are^{rt}+Be^{rt}=0 \Leftrightarrow$$

$$r^2+Ar+B=0 \tag{7}$$

This is called the Characteristic Equation (CE) and we have to check its roots:

$r=\dfrac{-A\pm\sqrt{A^2-4B}}{2}$, these are the Characteristic values or Eigenvalues.

[10] Notice that we do NOT know what is the value of *r*.

3.2 Roots are real and unequal

If $A^2 > 4B$ the system is called **Overdamped** and the two roots are r_1 and r_2 with $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{R}$. Then $x_1 = e^{r_1 t}$ and $x_2 = e^{r_2 t}$ are two linear independent solutions as:

$$\begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} \neq 0$$

hence the general solution is $x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$ (8)

If r_1 and $r_2 < 0$ then $x \to 0$ and the system is stable.

If r_1 or $r_2 > 0$ then $x \to \pm\infty$ and the system is unstable.

Example 1.12: The CE of $x'' + 11x' + 30x = 0$ is $r^2 + 11r + 30 = 0$ which means that the two roots are: $r_{1,2} = \frac{-11 \pm \sqrt{11^2 - 4 \cdot 1 \cdot 30}}{2} = \frac{-11 \pm 1}{2} \Rightarrow \begin{cases} r_1 = -5 \\ r_2 = -6 \end{cases}$ [11]

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{-5t} \\ x_2 = e^{r_2 t} = e^{-6t} \end{cases}$

This means that the general solution is $x = C_1 e^{-5t} + C_2 e^{-6t}$ and hence the ODE is stable[12]. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-5t} & e^{-6t} \\ -5e^{-5t} & -6e^{-6t} \end{vmatrix} = -6e^{-5t}e^{-6t} + 5e^{-6t}e^{-5t} = -e^{-11t} \neq 0$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

[11] `roots([1 11 30])`

[12] `clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2); ODE=D2x+11*Dx+30*x; dsolve(ODE)`

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ -5C_1 - 6C_2 &= 0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} C_1 &= 6 \\ C_2 &= -5 \end{aligned}\right\} \Rightarrow x = 6e^{-5t} - 5e^{-6t}$$ ■[13]

3.4 Roots are Complex (and hence not equal)

If $A^2 < 4B$ then the system is called **Underdamped** and the two roots are $r_1 = a + bj$ and $r_2 = \overline{r_1} = a - bj$ with $r_1 \neq r_2$, $r_1, r_2 \in \mathbb{C}$. Then $x_1 = e^{r_1 t} = e^{(a+bj)t}$ and $x_2 = e^{r_2 t} = e^{(a-bj)t}$ are two linear independent solutions as

$$\begin{vmatrix} e^{(a+bj)t} & e^{(a-bj)t} \\ (a+bj)e^{(a+bj)t} & (a-bj)e^{(a-bj)t} \end{vmatrix} = e^{(a+bj)t}(a-bj)e^{(a-bj)t} - e^{(a-bj)t}(a+bj)e^{(a+bj)t} =$$

$$(a-bj)e^{2at} - (a+bj)e^{2at} = e^{2at}(a - bj - a - bj) = -2e^{2at}bj \neq 0$$

Hence the general solution is

$$x = C_1 x_1 + C_2 x_1 = C_1 e^{rt} + C_2 e^{\overline{r}t} \tag{9}$$

but remember that C_1 and C_2 are complex now variables such as $x \in \mathbb{R}$.

Example 1.13: The CE of $x'' + 2x' + 5x = 0$ is $r^2 + 2r + 5 = 0$ which means that the two roots are: $r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{r_1 t} = e^{(-1+2j)t} \\ x_2 = e^{r_2 t} = e^{(-1-2j)t} \end{cases}$

[13] `clc, clear all, close all, syms x(t) Dx=diff(x,1); D2x=diff(x,2); ODE=D2x+11*Dx+30*x; dsolve(ODE, x(0)==1, Dx(0)==0)`

This means that the general solution is $x = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{(-1+2j)t} & e^{(-1-2j)t} \\ (-1+2j)e^{(-1+2j)t} & (-1-2j)e^{(-1-2j)t} \end{vmatrix} =$$

$$(-1-2j)e^{(-1+2j)t}e^{(-1-2j)t} - (-1+2j)e^{(-1+2j)t}e^{(-1-2j)t} =$$

$$(-1-2j)e^{-2t} - (-1+2j)e^{-2t} = (-1-2j+1-2j)e^{-2t} =$$

$$-4je^{-2t} \neq 0$$

If the initial condition is $x(0)=1, x'(0)=0$ then:

$$\left.\begin{array}{l} C_1 + C_2 = 1 \\ (-1+2j)C_1 + (-1-2j)C_2 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} C_1 = \frac{1}{2} + \frac{1}{4}j \\ C_2 = \frac{1}{2} - \frac{1}{4}j \end{array}\right\} \Rightarrow$$

$$x = \left(\frac{1}{2} + \frac{1}{4}j\right)e^{(-1+2j)t} + \left(\frac{1}{2} - \frac{1}{4}j\right)e^{(-1-2j)t}$$ ■

An alternative approach is not to use x_1 & x_2 but a linear combination of them:

$$y_1 = e^{rt} + e^{\bar{r}t},\ y_2 = e^{rt} - e^{\bar{r}t}$$

Note that $\begin{vmatrix} e^{rt} + e^{\bar{r}t} & e^{rt} - e^{\bar{r}t} \\ re^{rt} + \bar{r}e^{\bar{r}t} & re^{rt} - \bar{r}e^{\bar{r}t} \end{vmatrix} \neq 0$

Using Euler's formula: $e^{(a+bj)t} = e^{at}(\cos bt + j\sin bt)$ and hence:

$$y_1 = e^{(a+bj)t} + e^{(a-bj)t} = e^{at}(\cos bt + j\sin bt + \cos bt - j\sin bt) = 2e^{at}\cos bt$$

$$y_2 = e^{(a+bj)t} - e^{(a-bj)t} = e^{at}(\cos bt + j\sin bt - \cos bt + j\sin bt) = j2e^{at}\sin bt$$

As y_1 and y_2 are solutions so do $y_1 \times \frac{1}{2}, y_2 \times \frac{1}{2j}$. So the general solution when we have complex roots is:

$$x(t) = e^{at}\left(C_1 \cos bt + C_2 \sin bt\right), C_1, C_2 \in \mathbb{R} \tag{10}$$

Example 1.14: The CE of $x'' + 2x' + 5x = 0$ is $r^2 + 2r + 5 = 0$ which means that the two roots are: $r_{1,2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j \Rightarrow \begin{cases} r_1 = -1 + 2j \\ r_2 = -1 - 2j \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t}\cos(2t) \\ x_2 = e^{-t}\sin(2t) \end{cases}$

This means that the general solution is $x = e^{-t}\left(C_1 \cos 2t + C_2 \sin 2t\right)$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^{-t}\cos(2t) & e^{-t}\sin(2t) \\ -e^{-t}\cos(2t) - 2e^{-t}\sin(2t) & -e^{-t}\sin(2t) + 2e^{-t}\cos(2t) \end{vmatrix} \neq 2e^{-2t}$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

$$\left.\begin{matrix} C_1 = 1 \\ -C_1 + 2C_2 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} C_1 = 1 \\ C_2 = 0.5 \end{matrix}\right\} \Rightarrow$$

$$x = e^{-t}\left(\cos 2t + 0.5 \sin 2t\right)$$ ■

3.3 Roots are real and equal

If $A^2 = 4B$ then the system is called **Critically damped** and the two roots are $r = r_1 = r_2$ with $r \in \mathbb{R}$. One solution is $x_1 = e^{rt}$ but how about x_2? We can use $x_2 = te^{rt}$ and the general solution:

$$x = C_1 x_1 + C_2 x_2 = C_1 e^{r_1 t} + C_2 t e^{r_1 t} \tag{11}$$

The Wronskian is:

$$\begin{vmatrix} e^{r_1 t} & te^{r_1 t} \\ r_1 e^{r_1 t} & r_1 te^{r_1 t} + e^{r_1 t} \end{vmatrix} = e^{r_1 t}\left(r_1 te^{r_1 t} + e^{r_1 t}\right) - r_1 e^{r_1 t} te^{r_1 t} = r_1 te^{2r_1 t} + e^{2r_1 t} - r_1 te^{2r_1 t} = e^{2r_1 t} \neq 0$$

Example 1.15: The CE of $x'' + 2x' + x = 0$ is $r^2 + 2r + 1 = 0$ which means that the two roots are: $r_{1,2} = \dfrac{-2 \pm \sqrt{0}}{2} \Rightarrow \begin{cases} r_1 = -1 \\ r_2 = -1 \end{cases}$

and hence the 2 LI solutions are $\begin{cases} x_1 = e^{-t} \\ x_2 = te^{-t} \end{cases}$

This means that the general solution is $x = C_1 e^{-t} + C_2 te^{-t}$ and hence the ODE is stable. The Wronskian is

$$\begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = e^{2t} \neq 0$$

If the initial condition is $x(0) = 1, x'(0) = 0$ then:

$$\left.\begin{matrix} C_1 = 1 \\ -C_1 + C_2 = 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} C_1 = 1 \\ C_2 = 1 \end{matrix}\right\}$$

$$x = e^{-t} + te^{-t}$$

■

Not assessed material

To see why $x_2 = te^{rt}$ is the 2nd solution go to the ODE and place $x = e^{rt}$:

$$\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt}\left(r^2 + Ar + B\right)$$

Since r_1 is a double root of the CE: $r^2 + Ar + B = a\left(r - r_1\right)^2$ for some constant

a. So: $\left(e^{rt}\right)'' + A\left(e^{rt}\right)' + Bx = e^{rt} a\left(r - r_1\right)^2$

Taking the time derivative wrt r:

$$\frac{d\left(\left(e^{rt}\right)''\right)}{dr}+A\frac{d\left(\left(e^{rt}\right)'\right)}{dr}+B\frac{d\left(e^{rt}\right)}{dr}=\frac{d\left(e^{rt}a\left(r-r_1\right)^2\right)}{dr}$$

And as we can change the sequence of the differentiation:

$$\left(\frac{d\left(e^{rt}\right)}{dr}\right)''+A\left(\frac{d\left(e^{rt}\right)}{dr}\right)'+B\frac{d\left(e^{rt}\right)}{dr}=\frac{d\left(e^{rt}a\left(r-r_1\right)^2\right)}{dr}$$

By using simple calculus:

$$\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=\frac{d\left(e^{rt}\right)}{dr}a\left(r-r_1\right)^2+e^{rt}\frac{d\left(a\left(r-r_1\right)^2\right)}{dr}\Leftrightarrow$$

$$\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=e^{rt}ta\left(r-r_1\right)^2+e^{rt}2a\left(r-r_1\right)$$

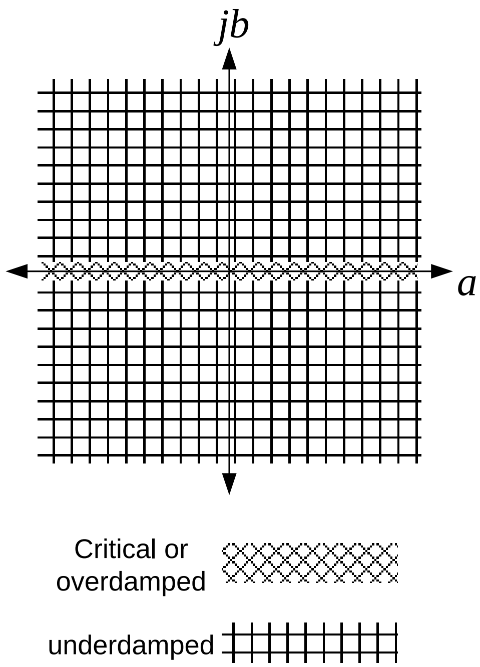
By placing now where $r=r_1$: $\left(e^{rt}t\right)''+A\left(e^{rt}t\right)'+Be^{rt}t=0$

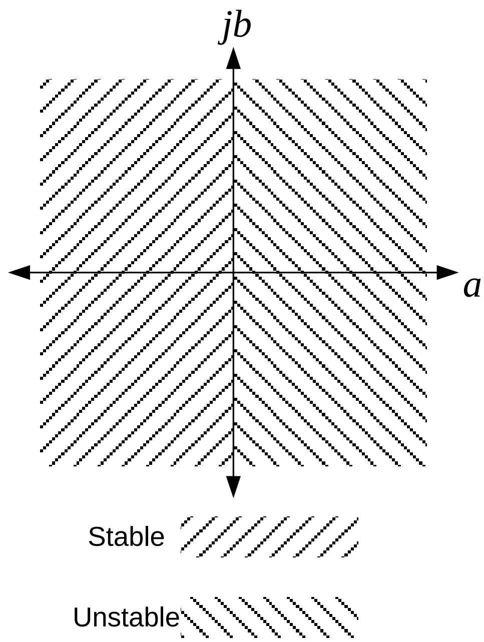
Which means that $e^{rt}t$ must be a solution of my ODE and:

$$\begin{vmatrix} e^{r_1t} & te^{r_1t} \\ r_1e^{r_1t} & tr_1e^{r_1t}+e^{r_1t} \end{vmatrix}=e^{r_1t}\cdot\left(tr_1e^{r_1t}+e^{r_1t}\right)-te^{r_1t}\cdot r_1e^{r_1t}=tr_1e^{2r_1t}+e^{2r_1t}-tr_1e^{2r_1t}=e^{2r_1t}\neq 0$$

And hence $x_2(t)=e^{rt}t$ is my second solution.

Root Space





Name	Oscillations?	Components of solution
Overdamped	No	Two exponentials: $e^{k_1 t}, e^{k_2 t}, \; k_1, k_2 < 0$
Critically damped	No	Two exponentials: $e^{kt}, te^{kt}, \; k < 0$
Underdamped	Yes	One exponential and one cosine $e^{kt}, \cos(\omega t), \; k < 0$
Undamped	Yes	one cosine $\cos(\omega t)$

4. n^{th} Order ODEs

4.1 General Material

In general, the theory that we have developed until now can be applied to an n^{th} order system. So a general n^{th} order ODE is given by:

$$\frac{d^n x(t)}{dt^n} = f\left(x^{(n-1)}(t), x^{(n-2)}(t), \ldots, x'(t), x(t), t\right) \quad (12)$$

The homogeneous linear n^{th} order ODE is given by:

$$x^{(n)}(t) + p_{n-1}x^{(n-1)}(t) + \cdots + p_0 x(t) = 0 \quad (13)$$

If we have *n* solutions $x_1(t), x_2(t), x_3(t), \ldots x_n(t)$ then their linear combination $x(t) = C_1 x_1(t) + C_1 x_2(t) + C_3 x_3(t) + \ldots + C_n x_n(t)$ is also a solution.

Example 1.16: Given the ODE $\ddddot{x} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0$ prove that $x_1(t) = e^{-t}, x_2(t) = 3e^{-2t}$ are solutions:

$$\left.\begin{array}{l} x_1(t) = e^{-t} \\ x_2(t) = 3e^{-2t} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \dot{x}_1(t) = -e^{-t} \\ \dot{x}_2(t) = -6e^{-2t} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \ddot{x}_1(t) = e^{-t} \\ \ddot{x}_2(t) = 12e^{-2t} \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} \dddot{x}_1(t) = -e^{-t} \\ \dddot{x}_2(t) = -24e^{-2t} \end{array}\right\} \Rightarrow \left.\begin{array}{l} \ddddot{x}_1(t) = e^{-t} \\ \ddddot{x}_2(t) = 48e^{-2t} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} e^{-t} - 10e^{-t} + 35e^{-t} - 50e^{-t} + 24e^{-t} = 0 \\ 48e^{-2t} + 10\left(-24e^{-2t}\right) + 35\left(12e^{-2t}\right) + 50\left(-6e^{-2t}\right) + 24\left(3e^{-2t}\right) = 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} 1 - 10 + 35 - 50 + 24 = 0 \\ 48 - 240 + 420 - 300 + 72 = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 = 0 \\ 0 = 0 \end{array}\right\}$$

■[14]

[14] syms t; x1=exp(-t); diff(x1,4)+10*diff(x1,3)+35*diff(x1,2)+50*diff(x1,1)+24*x1

The Wronskian is given by:

$$W\left(x_1(t),x_2(t),\cdots x_n(t)\right)=\begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ x_1'(t) & x_2'(t) & \cdots & x_n'(t) \\ \vdots & \vdots & & \vdots \\ x_1^{(n-1)}(t) & x_2^{(n-1)}(t) & \cdots & x_n^{(n-1)}(t) \end{bmatrix} \tag{14}$$

and if its determinant is nonzero then the n solutions are linear independent and they describe any other solution φ by taking their linear combination:

$$\varphi(t)=C_1x_1(t)+C_1x_2(t)+C_3x_3(t)+\ldots+C_nx_n(t) \tag{15}$$

Example 1.17: Given the ODE $\ddddot{x}+10\dddot{x}+35\ddot{x}+50\dot{x}+24x=0$ prove that $x_1(t)=e^{-t}, x_2(t)=e^{-2t}, x_3(t)=e^{-3t}, x_4(t)=e^{-4t}$ are LI solutions:

$$\left.\begin{aligned} x_1(t)&=e^{-t}\\ x_2(t)&=e^{-2t}\\ x_3(t)&=e^{-3t}\\ x_4(t)&=e^{-4t}\end{aligned}\right\}\Rightarrow \left.\begin{aligned} \dot{x}_1(t)&=-e^{-t}\\ \dot{x}_2(t)&=-2e^{-2t}\\ \dot{x}_3(t)&=-3e^{-3t}\\ \dot{x}_4(t)&=-4e^{-4t}\end{aligned}\right\}\Rightarrow \left.\begin{aligned} \ddot{x}_1(t)&=e^{-t}\\ \ddot{x}_2(t)&=4e^{-2t}\\ \ddot{x}_3(t)&=9e^{-3t}\\ \ddot{x}_4(t)&=16e^{-4t}\end{aligned}\right\}\Rightarrow \left.\begin{aligned} \dddot{x}_1(t)&=-e^{-t}\\ \dddot{x}_2(t)&=-8e^{-2t}\\ \dddot{x}_3(t)&=-27e^{-3t}\\ \dddot{x}_4(t)&=-64e^{-4t}\end{aligned}\right\}\Rightarrow$$

$$W\left(x_1(t),x_2(t),x_3(t),x_4(t)\right)=\begin{bmatrix} e^{-t} & e^{-2t} & e^{-3t} & e^{-4t} \\ -e^{-t} & -2e^{-2t} & -3e^{-3t} & -4e^{-4t} \\ e^{-t} & 4e^{-2t} & 9e^{-3t} & 16e^{-4t} \\ -e^{-t} & -8e^{-2t} & -27e^{-3t} & -64e^{-4t} \end{bmatrix}\Rightarrow$$

$$|W|=\begin{vmatrix} e^{-t} & e^{-2t} & e^{-3t} & e^{-4t} \\ -e^{-t} & -2e^{-2t} & -3e^{-3t} & -4e^{-4t} \\ e^{-t} & 4e^{-2t} & 9e^{-3t} & 16e^{-4t} \\ -e^{-t} & -8e^{-2t} & -27e^{-3t} & -64e^{-4t} \end{vmatrix}=12e^{-10t}\neq 0$$

■[15]

[15] syms t; x1=exp(-t);x2=exp(-2*t); x3=exp(-3*t); x4=exp(-4*t); W1=[x1 x2 x3 x4]; W2=diff(W1); W3=diff(W2); W4=diff(W3); W=[W1;W2;W3;W4]; det(W)

The CE is given by an n^{th} order polynomial:

$$r^n + p_{n-1}r^{n-1}(t) + \cdots + p_0 r = 0 \qquad (16)$$

And if all the eigenvalues are negative the ODE is stable. If only one is positive (or with positive real part, it is unstable).

Example 1.18: Given the ODE $\ddddot{x} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0$ find the CE:

$$\ddddot{x} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0 \overset{x=e^{rt}}{\Rightarrow}$$
$$r^4e^{rt} + 10r^3e^{rt} + 35r^2e^{rt} + 50re^{rt} + 24e^{rt} = 0 \Rightarrow$$
$$r^4 + 10r^3 + 35r^2 + 50r + 24 = 0 \Rightarrow$$
$$\{r_1 = -1, r_2 = -2, r_3 = -3, r_4 = -4\}$$ ■[16]

Of course an obvious problem here is that, in general it is very difficult to solve an n^{th} order polynomial expression. Having said that, the analysis remains the same as before regarding stable/unstable, real/complex and distinct/repeated eigenvalues:

- If r is a distinct (complex or real) root, then $x(t) = e^{rt}$ is a solution.
- If r is a repeated root (complex or real) of multiplicity k, then $x_1(t) = e^{rt}, x_2(t) = te^{rt}, x_3(t) = t^2e^{rt} \cdots x_{k-1}(t) = t^{k-1}e^{rt}$ are solutions.

Example 1.19: The ODE $\dddot{x} + 3\ddot{x} + 3\dot{x} + x = 0$ has the following eigenvalues: $r_1 = r_2 = r_3 = -1$, the 3 LI solutions and hence the general solution are: $x_1 = e^{-t}, x_2 = te^{-t}, x_3 = t^2e^{-t} \Rightarrow x(t) = C_1e^{-t} + C_2te^{-t} + C_3t^2e^{-t}$ ■[17]

[16] roots([1 10 35 50 24])
[17] syms x(t); Dx=diff(x); D2x=diff(x,2); D3x=diff(x,3); dsolve(D3x+D2x*3+Dx*3+x)

Example 1.20: The ODE $\ddddot{x}+4\dddot{x}+6\ddot{x}+4\dot{x}+x=0$ has the following eigenvalues: $r_1=r_2=r_3=r_4=-1$, the 4 LI solutions and hence the general solution are: $x_1=e^{-t}, x_2=te^{-t}, x_3=t^2e^{-t}, x_4=t^3e^{-t}$

$\Rightarrow x(t)=C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4t^3e^{-t}$ ■[18]

Example 1.21: The ODE $\ddddot{x}+5\dddot{x}+9\ddot{x}+7\dot{x}+2x=0$ has the following eigenvalues: $r_1=r_2=r_3=-1, r_4=-2$, the 4 LI solutions and hence the general solution are: $x_1=e^{-t}, x_2=te^{-t}, x_3=t^2e^{-t}, x_4=e^{-2t}$

$\Rightarrow x(t)=C_1e^{-t}+C_2te^{-t}+C_3t^2e^{-t}+C_4e^{-2t}$ ■

Example 1.22: The ODE $\ddddot{x}+9\dddot{x}+30\ddot{x}+42\dot{x}+20x=0$ has the following eigenvalues: $r_1=-1, r_2=-2, r_3=-3+i, r_4=-3-i$, the 4 LI solutions and hence the general solution are: $x_1=e^{-t}, x_2=e^{-2t}, x_3=e^{(-3+i)t}, x_4=e^{(-3+i)t}$

$\Rightarrow x(t)=C_1e^{-t}+C_2e^{-2t}+C_3e^{(-3+i)t}+C_4e^{(-3-i)t}$ ■

Example 1.23: The ODE $\ddddot{x}+12\dddot{x}+59\ddot{x}+138\dot{x}+130x=0$ has the following eigenvalues: $r_1=-3+2i, r_2=-3-2i, r_3=-3+i, r_4=-3-i$, the 4 LI solutions and hence the general solution are: $x_1=e^{(-3+2i)t}, x_2=e^{(-3-2i)t}, x_3=e^{(-3+i)t}, x_4=e^{(-3+i)t}$

$\Rightarrow x(t)=C_1e^{(-3+2i)t}+C_2e^{(-3+2i)t}+C_3e^{(-3+i)t}+C_4e^{(-3-i)t}$ ■

Example 1.24: The ODE $\ddddot{x}+12\dddot{x}+56\ddot{x}+120\dot{x}+10x=0$ has the following eigenvalues: $r_1=-3+i, r_2=-3-i, r_3=-3+i, r_4=-3-i$, the 4 LI solutions and hence the general solution are: $x_1=e^{(-3+i)t}, x_2=e^{(-3-i)t}, x_3=te^{(-3+i)t}, x_4=te^{(-3+i)t}$

$\Rightarrow x(t)=C_1e^{(-3+i)t}+C_2e^{(-3+i)t}+C_3te^{(-3+i)t}+C_4te^{(-3-i)t}$ ■

Example 1.25: A 15th order ODE has the following eigenvalues: $r_1=-1, r_2=-2, r_3=-2, r_{4,5}=-3\pm i, r_{6,7}=-4\pm 2i, r_{8,9}=-4\pm 2i,$

[18] syms x(t); Dx=diff(x); D2x=diff(x,2); D3x=diff(x,3); D4x=diff(x,4); solve(D4x+4*D3x+D2x*6+4*Dx+x)

$r_{10,11} = -5 \pm 3i,\ r_{12,13} = -5 \pm 3i,\ r_{14,15} = -5 \pm 3i$, the 14 LI solutions are:
$x_1 = e^{-t}, x_2 = e^{-2t}, x_3 = te^{-2t}, x_4 = e^{(-3+i)t}, x_5 = e^{(-3-i)t}$,
$x_6 = e^{(-4+2i)t}, x_7 = e^{(-4-2i)t}, x_8 = te^{(-4+2i)t}, x_9 = te^{(-4-2i)t}$,
$x_{10} = e^{(-5+3i)t}, x_{11} = e^{(-5-3i)t}, x_{12} = te^{(-5+3i)t}, x_{13} = te^{(-5-3i)t}$,
$x_{14} = t^2 e^{(-5+3i)t}, x_{15} = t^2 e^{(-5-3i)t}$ ■

As it can be seen by the last examples, the complexity is increasing as the order increases to levels that it is not possible to use that approach. The next section will address this issue.

5. Systems of first order ODEs

5.1 General Material

An n^{th} order homogeneous system can be written as:

$$\begin{aligned} x_1'(t) &= a_{1,1}x_1(t) + a_{1,2}x_2(t) + \cdots + a_{1,n}x_n(t) \\ x_2'(t) &= a_{2,1}x_1(t) + a_{2,2}x_2(t) + \cdots + a_{2,n}x_n(t) \\ &\vdots \\ x_n'(t) &= a_{n,1}x_1(t) + a_{n,2}x_2(t) + \cdots + a_{n,n}x_n(t) \end{aligned} \tag{17}$$

which can be written in a matrix form as

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \tag{18}$$

or $x'(t) = Ax(t)$ (19)

with $x \in \mathbb{R}^{n\times 1}, A \in \mathbb{R}^{n\times n}$

In order to simplify the analysis and as we do not have n^{th} order derivatives the n solutions are denoted as: $x^{(1)}(t), x^{(2)}(t), \cdots x^{(n)}(t)$ (with $x^{(1)}, x^{(2)}, \cdots x^{(n)} \in \mathbb{R}^{n\times 1}$) and as before their linear combination $x(t) = C_1 x^{(1)}(t) + C_2\, x^{(2)}(t) + \cdots + C_n x^{(n)}(t)$ is also a solution.

The Wronskian is given by:

$$W\left(x^{(1)}(t), x^{(2)}(t), \cdots x^{(n)}(t)\right) = \left[x^{(1)}(t), x^{(2)}(t), \cdots x^{(n)}(t)\right] \quad (20)$$

And if its determinant is nonzero then any other solution $\varphi(t)$ can be written as their linear combination:

$$\varphi(t) = C_1 x^{(1)}(t) + C_2\, x^{(2)}(t) + \cdots + C_n x^{(n)}(t) \quad (21)$$

Example 1.26: For the system $x' = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} x$, prove that $x^{(1)} = e^{-3t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, x^{(2)} = e^{-2t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, x^{(3)} = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are 3 LI solutions.

$$x^{(1)} = e^{-3t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \dot{x}^{(1)} = -3e^{-3t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} e^{-3t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2+1 \\ 0 \\ -1-2 \end{bmatrix}$$

$$x^{(2)} = e^{-2t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \Rightarrow \dot{x}^{(2)} = -2e^{-2t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} e^{-2t}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$$

$$x^{(3)} = e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \dot{x}^{(3)} = -e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} e^{-t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2+1 \\ 0 \\ 1-2 \end{bmatrix}$$

$$W = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow |W| = -1\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1 - 0 + 1 = 2 \neq 0 \blacksquare$$

5.2 Solution matrices

We can define now another matrix which is called the solution matrix as

$$\Phi(t) = \left[x^{(1)}(t), x^{(2)}(t), \cdots x^{(n)}(t) \right] \tag{22}$$

and hence the general solution is given by:

$$x(t) = \Phi(t)C \tag{23}$$

where C is a matrix with constants, $C = [C_1 \quad C_2 \quad \cdots \quad C_n]$

At $t=t_0$ we have that:

$x(t_0) = \Phi(t_0)C \Leftrightarrow C = \Phi^{-1}(t_0)x(t_0)$ and hence the general solution is given by:

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x(t_0) \tag{24}$$

Now the matrix $\Phi_{STM}(t,t_0) = \Phi(t)\Phi^{-1}(t_0)$ (25)

is called the state transition matrix or the normalised (at $t = t_0$) solution matrix as: $\Phi_{STM}(t_0,t_0) = \Phi(t_0)\Phi^{-1}(t_0) = I$

But how can we find the matrix $\Phi(t)$? As before let's assume that one solution is $x(t) = e \cdot e^{rt}$ with $e \in \mathbb{R}^{n\times 1}$ which we do Anot know. Then we have:

$$x'(t) = Ax(t) \Leftrightarrow r \cdot e \cdot e^{rt} = A \cdot e \cdot e^{rt}$$

As the exponential term is not zero we have that:

$$r \cdot e = A \cdot e \Leftrightarrow A \cdot e - r \cdot e = 0 \Leftrightarrow$$

$$(A - r \cdot I) \cdot e = 0 \qquad (26)$$

with *I* being the identity matrix and of course $I \in \mathbb{R}^{n\times n}$.

Note: You may remember from linear algebra that (26) is effectively the expression that will give us the eigenvalues and eigenvectors.

If we see (26) as a system of linear equations with unknowns being the elements of *e*, then in order to have a nontrivial solution (which is the zero solution) we must impose:

$$\left| A - r \cdot I \right| = 0 \qquad (27)$$

And now for each eigenvalue *r* we try to find an eigenvector *e* and as before we have 3 different cases:

5.3 Roots are real and not equal

In this case it can be proved that for each eigenvalue r_1, r_2, … r_n we can find a linear independent eigenvector $e^{(1)}$, $e^{(2)}$, … $e^{(n)}$ and the general solution is given by:

$$x(t) = \sum_{i=1}^{n} C_i e^{(i)} e^{r_i t} \tag{28}$$

5.4 Roots are Complex and not equal

If we have a complex eigenvalue $r = a + bj$ then we will have a complex eigenvector e associated it. We will also have the complex conjugate eigenvalue $\bar{r} = a - bj$ and eigenvector $\bar{e}$. Hence from these complex eigenvalues we have the two linear independent solutions:

$$x^{(1)}(t) = e e^{rt},\ x^{(2)}(t) = \bar{e} e^{\bar{r}t} \tag{29}$$

5.5 Roots are real and equal

If we have a root with multicity 2 (for example) then we can have 2 subcases if we have or not linear independent eigenvectors. If we have[19] then the two solutions are:

$$x^{(1)}(t) = e^{(1)} e^{rt},\ x^{(2)}(t) = e^{(2)} e^{rt}$$

If we do not have linear independent eigenvectors then we have to try other solutions (as we have previously tried te^{rt}). So how about if we try $te^{(1)}e^{rt}$:

[19] This will be the case when the state matrix is a multiple of the identity matrix.

$$\dot{x}(t) = Ax(t) \Leftrightarrow e^{(1)}e^{rt} + te^{(1)}re^{rt} = Ate^{(1)}e^{rt} \Leftrightarrow e^{(1)}e^{rt} + \left(e^{(1)}r - Ae^{(1)}\right)te^{rt} = 0$$

Which means that $e^{(1)} = 0$ and $e^{(1)}r - Ae^{(1)} = 0$, hence only the trivial (zero) exists if we try $te^{(1)}e^{rt}$. As $e^{(1)}e^{rt} + \left(e^{(1)}r - Ae^{(1)}\right)te^{rt}$ contains terms with e^{rt} and te^{rt} a new solution that we try is $te^{(1)}e^{rt} + e^{(2)}e^{rt}$, with $e^{(2)}$ being an unknown vector:

$$\dot{x}(t) = Ax(t) \Leftrightarrow \left(te^{(1)}e^{rt} + e^{(2)}e^{rt}\right)' = A\left(te^{(1)}e^{rt} + e^{(2)}e^{rt}\right) \Leftrightarrow$$
$$e^{(1)}e^{rt} + te^{(1)}re^{rt} + e^{(2)}re^{rt} = Ate^{(1)}e^{rt} + Ae^{(2)}e^{rt} \Leftrightarrow$$
$$e^{rt}\left(e^{(1)} + e^{(2)}r - Ae^{(2)}\right) + te^{rt}\left(e^{(1)}r - Ae^{(1)}\right) = 0 \Rightarrow \begin{cases} e^{(1)} + e^{(2)}r - Ae^{(2)} = 0 \\ e^{(1)}r - Ae^{(1)} = 0 \end{cases}$$

From these two equations we have that:

$$re^{(1)} - Ae^{(1)} = 0 \Leftrightarrow (rI - A)e^{(1)} = 0 \Leftrightarrow (A - rI)e^{(1)} = 0$$

and $e^{(1)} + e^{(2)}r - Ae^{(2)} = 0 \Leftrightarrow Ae^{(2)} - e^{(2)}r = e^{(1)} \Leftrightarrow (A - rI)e^{(2)} = e^{(1)}$

which implies that $e^{(1)}$ is an eigenvector of A (as expected) and by setting the 2nd expression into the first:

$$(A - rI)e^{(1)} = 0 \Leftrightarrow (A - rI)(A - rI)e^{(2)} = 0 \Leftrightarrow (A - rI)^2 e^{(2)} = 0$$

Hence from linear algebra we know that $e^{(2)}$ is a generalised eigenvector[20].

[20] Note that *e* is the first generalised eigenvector of *A*.

More specifically, a nonzero vector $e^{(k)}$ is a called a "rank k generalised eigenvector" associated with the eigenvalue λ when:

$$(A - rI)^k e^{(k)} = 0 \quad \text{and} \quad (A - rI)^{k-1} e^{(k)} \neq 0$$

5.6 General Case

To summarise the aforementioned analysis for a system $x'(t) = Ax(t)$ we have the following cases:

- For distinct eigenvalue λ (complex or real), we have an eigenvector e that will give a term $e \cdot e^{rt}$ to the solution.
- For a repeated eigenvalue λ (complex or real), of multiplicity k, we have the generalised eigenvectors $(A - rI)^k e^{(k)} = 0$ (or $(A - rI)e^{(k)} = e^{(k-1)}, (A - rI)e^{(k-1)} = e^{(k-2)}, \ldots, (A - rI)e^{(2)} = e^{(1)}$) which will create the solutions[21]: $\left(\dfrac{e^{(1)} t^{k-1}}{(k-1)!} + \cdots \dfrac{e^{(k-2)} t^2}{2!} + \cdots e^{(k-1)} t + e^{(k)} \right) e^{\lambda t}$

Example 1.27: An ODE has eigenvalues -1 and -2, find its general solution by assuming that the 2 eigenvectors are $e^{(1)}$ and $e^{(2)}$:

$x(t) = C_1 e^{-t} e^{(1)} + C_2 e^{-2t} e^{(2)}$ ■

Example 1.28: An ODE has eigenvalues -1+i and -1-i, find its general solution by assuming that the 2 eigenvectors are $e^{(1)}$ and $e^{(2)}$:

$x(t) = C_1 e^{(-1+i)t} e^{(1)} + C_2 e^{(-1-i)t} e^{(2)}$ ■

Example 1.29: An ODE has eigenvalues -1 and -1, find its general solution by assuming that the only LI eigenvector is $e^{(1)}$:

$x(t) = C_1 \left(e^{(1)} t + e^{(2)} \right) e^{-t} + C_2 e^{(1)} e^{-t}$ ■

[21] In this module k=2, for a deeper analysis see the book from EDWARDS & PENNEY

5.7 Exponential Matrix

For a scalar ODE: $\dot{x} = ax$ the solution was $x(t) = e^{at} x(0)$ (no special cases) so can we do the same with $\dot{x} = Ax$, i.e. $x(t) = e^{At} x(0)$?

If only we knew how to calculate e^{At} then no special cases are needed.

It can be proved that for LTI systems $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$

Obviously this series is not very easy to calculate and in the next chapter we will use "Similarity Transformations" to overcome this problem.

6. Exercises

1. Determine the stability of the following ODEs:
   i. $x' - 3x = 5$
   ii. $x' + 3x = 5$
   iii. $x' - 3x = -5$
   iv. $x' + 3x = -5$
   v. $-x' - 3x = 5$
   vi. $-x' + 3x = 5$
2. For the stable systems of Q1, determine if you will converge to zero or to a nonzero value.
3. Find the nonzero values of Q2.
4. Prove that two solutions to $x'' + 7x' + 12x = 0$ are: $x = e^{-3t}$ are $x = e^{-4t}$. Also prove that the following are also solutions: $x = 3e^{-3t} + 4e^{-4t}$ and $x = -1e^{-3t} + 7e^{-4t}$
5. For the ODE $x'' + 7x' + 12x = 0$ and with solutions $x = e^{-3t}$ are $x = e^{-4t}$ find the determinant of the Wronskian and prove that it is not zero.
6. Repeat Q5 for $x = e^{-3t}$ are $x = 3e^{-3t}$

7. Determine the stability of the following ODEs:
   i. $x''+3x'+2x=0$
   ii. $x''-x'-2x=0$
   iii. $x''+x'-2x=0$
   iv. $x''-3x'+2x=0$
   v. $x''+4x'+4x=0$
   vi. $x''-4x'+4x=0$
   vii. $x''-4x'+8x=0$
   viii. $x''+4x'+8x=0$
8. For the stable systems of Q7, determine if you will converge exponentially or with oscillations.
9. Find the general solutions of Q7.
10. Find the specific solutions of Q7, for $x(0)=1,\ x'(0)=0$.
11. For the following ODE $\ddot{\ddot{x}}+2\dddot{x}-13\ddot{x}-14\dot{x}+24x=0$
   i. Prove that the following are its solutions
   $x_1(t)=e^t,\ x_2(t)=-3e^{-2t},\ x_3(t)=\pi e^{3t},\ x_4(t)=\sqrt{3}e^{-4t},\ x_5(t)=3e^t$
   $x(t)=C_1x_1(t)+C_2x_2(t)+C_3x_4(t)+C_4x_4(t)$
   ii. Prove that x_1, x_2, x_3, x_4 are LI solutions, while x_1, x_2, x_3, x_5 are not.
   iii. Find its characteristic equation.
   iv. Given that its roots are 1, -2, 3, -4, find the general solution.
12. Given that an ODE has the following eigenvalues, find its general solution:
   i. $r_1=r_2=r_3=r_4=-2$
   ii. $r_1=r_2=r_3=-2,\ r_4=-3$
   iii. $r_1=r_2=-2,\ r_3=r_4=-3$
   iv. $r_1=1,\ r_2=2,\ r_3=4+i,\ r_4=4-i$
   v. $r_1=5+3i,\ r_2=5-3i,\ r_3=5+5i,\ r_4=5-5i$
   vi. $r_1=-6+2i,\ r_2=-6-2i,\ r_3=-6+2i,\ r_4=-6-2i$
13. Solve numerically the following systems
   i. $x'=\begin{bmatrix}1 & 2\\ 3 & 2\end{bmatrix},\ x(0)=\begin{bmatrix}0\\ -4\end{bmatrix}$.
   ii. $x'=\begin{bmatrix}7 & 1\\ -4 & 3\end{bmatrix},\ x(0)=\begin{bmatrix}2\\ -5\end{bmatrix}$.

iii. $x' = \begin{bmatrix} 3 & -9 \\ 4 & -3 \end{bmatrix}, x(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}.$

14. Find the analytical solutions of Q13.
15. Find the Wronskian matrices of the solutions of Q13.

7. Matlab Based Exercises for EEE8086

An ODE is given by $x''+3x'+2x=0,\ x(0)=1,\ x'(0)=0$:

1. Find its analytical solution.
2. Find its numerical solution.
3. Simulate the analytical solution.
4. Transform the ODE into a system of first order ODEs.
5. Find the analytical solution of Q4.
6. Find the numerical solution of Q4.
7. Simulate the analytical solution of Q4.
8. Use the exponential matrix to simulate the analytical solution of Q4.

Try to repeat the above for Q13 from section 6.