# Revision

$$x^{(n)} + p_{n-1} x^{(n-1)} + p_{n-2} x^{(n-2)} + \dots + p_0 x = 0$$

$$x = e^{rt}$$

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$\downarrow$

$$r^n + p_{n-1} \cdot r^{n-1} + \dots + p_0 = 0$$

$\downarrow$ roots

- $r_1 \neq r_2 \neq r_3 \dots$ , $r \in \mathbb{R}$ or $r \in \mathbb{C}$

$x_1 = e^{r_1 t}$ , $x_2 = e^{r_2 t}$ & ....

$$x = c_1 \cdot x_1 + c_2 \cdot x_2 \dots$$

- $r_1 = r_2 = r$ , $r \in \mathbb{R}$

$x_1 = e^{rt}$ , $x_2 = t e^{rt}$

- $r_1 = r_2 = r_3 = r$

$x_1 = e^{rt}$ , $x_2 = e^{rt} \cdot t$ $\quad x_3 = e^{rt} \cdot t^2$ ...

$\dot{x}_1 = a_{11} \cdot x_1 + a_{12} x_2 + \dots + a_{1n} x_n$

$\dot{x}_2 = a_{21} x_1 + \dots\dots\dots$

$\vdots$

$\dot{x}_n = a_{n1} x_1 + \dots\dots\dots a_{nn} x_n$

$$\dot{X} = A \cdot X \quad , \quad A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

a soln $X(t)$ → satisfies it ($\dot{X} = A \cdot X$)

$$\dot{X} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \cdot X \qquad X^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\dot{X}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} \cdot (-1)$$

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{-t} = -e^{-t} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

test $X^{(2)} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} e^{-2t}$

if $X^{(1)}, X^{(2)} \ldots X^{(k)} \Rightarrow$

$\varphi = \sum_{i=1}^{k} c_i X^{(i)}$ is also a soln.

If n L.I. soln.

then any other soln

$\varphi = \sum_{i=n}^{n} c_i X^{(i)}$

$W = [X^{(1)} \quad X^{(2)} \quad \ldots \quad X^{(n)}]$

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$\dot{X} = AX$ I want to find my n L.I. soln

$X = e^{rt} \cdot e$
$\downarrow$
$n \times 1$

$(A - \lambda I) e = 0$ n+1 unknowns, n eqns

$\lambda$ = given $\rightarrow$ Linear $n \times n$

$|A - \lambda I| = 0 \Rightarrow a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_0 = 0$

$\downarrow$ roots

$\lambda_1$, $\lambda_2$, $\vdots$ $\Big\}$ $(A - \lambda I) e_i = 0 \Rightarrow$ $e_1 = \ldots$, $e_2 = \ldots$, $\ldots$

2nd order sys.

if $\lambda_1 \neq \lambda_2 \Rightarrow e_1, e_2$ L.I.

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- $\lambda_1 \neq \lambda_2 \neq \lambda_3 \neq \lambda_n \Rightarrow e_1, e_2, \dots, e_n$ L.I.

$$x = \sum_{i=1}^{n} e^{(i)} \cdot e^{\lambda_i t} c_i$$

$\lambda$ → real, complex

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- $\lambda_1 = \lambda_2 = \lambda$

$$x_1 = e^{(1)} e^{\lambda t}$$

$$x_2 = e^{(1)} \cdot t e^{\lambda t} + 0 e^{\lambda t} e^{(1)}$$

$\dot{x} = A \cdot x$

$e^{(1)} \cdot e^{\lambda t} + e^{(1)} \cdot t \cdot \lambda e^{\lambda t} = A \cdot e^{(1)} \cdot t e^{\lambda t}$

$\underbrace{e^{(1)} \lambda e^{\lambda t}} \cdot t + \underbrace{e^{(1)} e^{\lambda t}} = \underbrace{A e^{(1)} e^{\lambda t}} \cdot t + \underbrace{0 e^{(1)} e^{\lambda t}}$

$e^{(1)} \lambda \cancel{e^{\lambda t}} = A e^{(1)} \cancel{e^{\lambda t}}$

$(A - \lambda I) e^{(1)} = 0$

$e^{(1)} \cdot \cancel{e^{\lambda t}} = 0 \quad e^{(1)} \cancel{e^{\lambda t}} = 0$

How about $\quad x_2 = t e^{(1)} e^{\lambda t} + e^{\lambda t} e^{(2)}$

$$\dot{x}_2 = e^{(1)} e^{\lambda t} + e^{(1)} \cdot t \cdot \lambda e^{\lambda t} + \lambda e^{\lambda t} e^{(2)}$$

$$= \left(e^{(1)} + \lambda e^{(2)}\right) e^{\lambda t} + e^{(1)} \lambda e^{\lambda t} \cdot t$$

$$A \cdot x_2 = A \cdot t \cdot e^{(1)} e^{\lambda t} + A \cdot e^{\lambda t} e^{(2)}$$

$$e^{(1)} \lambda e^{\lambda t} = A e^{(1)} e^{\lambda t}$$

$$(A - \lambda I) e^{(1)} = 0$$

$$e^{(1)} + \lambda e^{(2)} = A e^{(2)}$$

$$e^{(1)} = (A - \lambda I) e^{(2)}$$ → Gen eigenvector

$$(A - \lambda I) \cdot (A - \lambda I) e^{(2)} = 0 \Rightarrow$$

$$(A - \lambda I)^2 e^{(2)} = 0$$

in general $(A - \lambda I)^k \cdot e^{(k)} = 0$

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$\dot{x} = A x \Rightarrow x = e^{At} \cdot x_0$

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots$$

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \dots$$

scalar

$$\dot{x} = A \cdot x$$

$$P = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \qquad |P - \lambda I| = 0$$

$$\begin{vmatrix} a-\lambda & 0 \\ 0 & b-\lambda \end{vmatrix} = 0 \Rightarrow (a-\lambda)(b-\lambda) = 0$$

$a \neq b$

$\Rightarrow \lambda_{1,2} = a$ and $b$

• $\lambda = a$ $\quad (P - aI) \cdot e_a = 0$

$$\begin{bmatrix} a-a & 0 \\ 0 & b-a \end{bmatrix} \cdot \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} 0 \cdot V_1 + 0 V_2 = 0 \\ 0 \cdot V_1 + (b-a) \cdot V_2 = 0 \end{matrix}\right\} \begin{matrix} V_1 = \text{Any} = 1 \\ V_2 = 0 \end{matrix}$$

$$e_a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

• $\lambda = b \Rightarrow \ldots \quad e_b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$e^{Pt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix} + \frac{\begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix}^2}{2!}$$

$$\begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix}^2 = \begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix} \cdot \begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix} = \begin{bmatrix} (at)^2 & 0 \\ 0 & (bt)^2 \end{bmatrix}$$

$$\begin{bmatrix} at & 0 \\ 0 & bt \end{bmatrix}^K = \begin{bmatrix} (at)^K & 0 \\ 0 & (bt)^K \end{bmatrix}$$

ele 1,1: $1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots = e^{at}$

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$\Rightarrow$

$$e^{Pt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$$

Gen: 
$$P = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & \dots & & \lambda_n \end{bmatrix}$$

eigenvalues $\lambda_1 \quad \lambda_2 \quad \dots \lambda_n$

eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \dots$

$$e^{Pt} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \dots \\ 0 & e^{\lambda_2 t} & \dots \\ \vdots & & \\ \vdots & & \end{bmatrix}$$

$$P = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

$$\begin{vmatrix} \lambda - a & 1 \\ 0 & \lambda - a \end{vmatrix} = 0 \Rightarrow (\lambda - a)^2 = 0 \Rightarrow$$

$$\lambda_1 = \lambda_2 = a.$$

- $\lambda = a.$ $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$0 \cdot V_1 + V_2 = 0 \Rightarrow V_1 = \text{any} = 1$$

$$V_2 = 0 \quad \text{Always}$$

$$e = [1 \quad 0]^T$$

$$e^{(1)} = (P - \lambda I) e^{(2)} \Rightarrow \ldots \quad b_1 = \text{any} = 0$$

$$b_2 = 1$$

$$e^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$e^{Pt} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} at & t \\ 0 & at \end{bmatrix} + \frac{\begin{bmatrix} at & t \\ 0 & at \end{bmatrix}^2}{2!} + \ldots$$

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}^2 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix}$$

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$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}^3 = \begin{bmatrix} a^2 & 2a \\ 0 & a^2 \end{bmatrix} \cdot \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

$$= \begin{bmatrix} a^3 & 3a^2 \\ 0 & a^3 \end{bmatrix}$$

$$\begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}^n = \begin{bmatrix} a^n & na^{n-1} \\ 0 & a^n \end{bmatrix}$$

el 1: $1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{3!} + \dots = e^{at}$

2.: $0 + t + \frac{2at^2}{2!} + \frac{3a^2t^3}{3!} + \frac{4a^3t^4}{4!} + \dots$

$= t\left(1 + \frac{2at}{2!} + \frac{3a^2t^2}{3!} + \dots\right)$

$\frac{2}{2!} \to 1$, $\frac{3}{3!} \to 2!$, $\to 3!$

$= te^{at}$

$$e^{Pt} = \begin{bmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{bmatrix}$$

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$P = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$  $\lambda_1 = \lambda_2 = a$

eigv: $\begin{bmatrix} a-a & 0 \\ 0 & a-a \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$0 \cdot v_1 + 0 \cdot v_2 = 0$

2 L.I. eigev.

$e^{Pt} = \begin{bmatrix} e^{at} & 0 \\ 0 & e^{at} \end{bmatrix}$

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$P = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \Rightarrow P = \begin{bmatrix} Re(\lambda) & Im(\lambda) \\ -Im(\lambda) & Re(\lambda) \end{bmatrix}$

$|P - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ -b & a-\lambda \end{vmatrix} = 0 \Rightarrow$

$(a-\lambda)^2 + b^2 = 0$

$a^2 - 2a\lambda + b^2 + \lambda^2 = 0$

$\lambda^2 - 2a\lambda + (a^2 + b^2) = 0$

$\Delta = 4a^2 - 4 \cdot a^2 - 4b^2 = -4b^2 = (2bi)^2$

$\lambda_{1,2} = \dfrac{-(-2a) \pm 2bi}{2} = a \pm bi$

$e_1 = [1 \quad j]^T$

$e_2 = [1 \quad -j]^T$

$$e^{Pt} = e^{Re(\lambda)\cdot t} \cdot \begin{bmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{bmatrix}$$

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$$P = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{bmatrix} \rightarrow \lambda_1 = \lambda_2 = \lambda_3 = a.$$

$0 \cdot v_1 + v_2 + 0 \cdot v_3 = 0$ I must have

$0 v_1 + 0 v_2 + v_3 = 0$ $v_2 = v_3 = 0$

$e = [1 \quad 0 \quad 0]^T$ $v_1 = \text{any} = 1$

$$e^{Pt} = \begin{bmatrix} e^{at} & te^{at} & \frac{t^2 e^{at}}{2} \\ 0 & e^{at} & te^{at} \\ 0 & 0 & e^{at} \end{bmatrix}$$

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$$P = \begin{bmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{bmatrix}$$

$\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = a$

1 L.I. eig. $= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

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$$P = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \qquad \lambda_1 = \lambda_2 = \lambda_3 = a.$$

$$0 \cdot v_1 + v_2 + 0 \cdot v_3 = 0$$

$v_2 = 0$

- $v_1 = 1 \quad v_3 = 0$
- $v_1 = 0 \quad v_3 = 1$
- $v_1 = 1 \quad v_3 = 1$

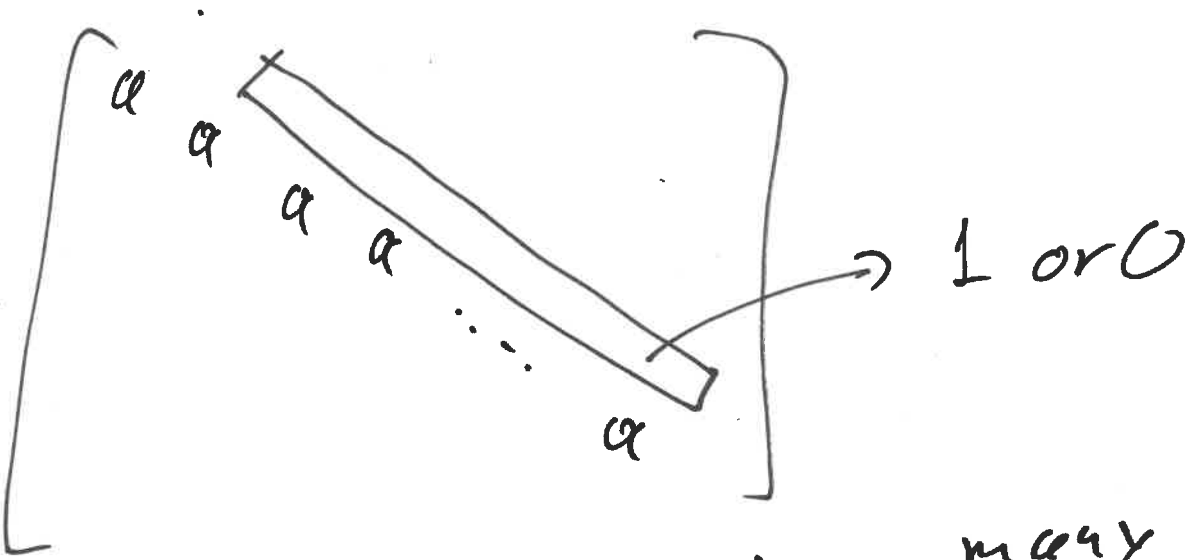
$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad e_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so $e_3$ is <u>not</u> L.I. of $e_1$, $e_2$

$$P = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \rightarrow \begin{array}{l} \lambda_1 = \lambda_2 = \lambda_3 = a \\ e_1 = [1 \quad 0 \quad 0]^T \\ e_2 = [0 \quad 1 \quad 0]^T \\ e_3 = [0 \quad 0 \quad 1]^T \end{array}$$

so




# of zeros → how many +L
L.I. eigvectors

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$$P_1 = \begin{bmatrix} a & b & 0 & 0 \\ -b & a & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix}$$

$$P_1 = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$$

↑

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

↓

$$P_2 = \begin{bmatrix} B & I \\ 0 & B \end{bmatrix}$$

$$P_2 = \begin{bmatrix} a & b & 1 & 0 \\ -b & a & 0 & 1 \\ 0 & 0 & a & b \\ 0 & 0 & -b & a \end{bmatrix}$$

$\lambda_1 = a+bi$

$\lambda_2 = a-bi$

$\lambda_3 = a+bi$

$\lambda_4 = a-bi$

$$P = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

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P1=[P zeros(2);
    zeros(2) P] ↵
[x,v]=eigs(P1)
det(x)
P2=[P ~~ones~~ eye(2);
    zeros(2) P] ↵
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