Chapter #5

EEE8086-EEE8115

Robust and Adaptive Control Systems

Robust Control

1. Ideal Systems

Assume that we have a 2^{nd} order system:

$$\ddot{x} + A\dot{x} + Bx = u \tag{1}$$

and that we want to follow a specific signal x_d

then if we choose $u = \ddot{x}_d + A\dot{x}_d + Bx_d$:

$$\ddot{x} + A\dot{x} + Bx = \ddot{x}_d + A\dot{x}_d + Bx_d \Leftrightarrow$$
$$\left(\ddot{x} - \ddot{x}_d\right) + A\left(\dot{x} - \dot{x}_d\right) + B\left(x - x_d\right) = 0$$

We can define now as a tracking error:

$$\tilde{x} = x - x_d \tag{2}$$

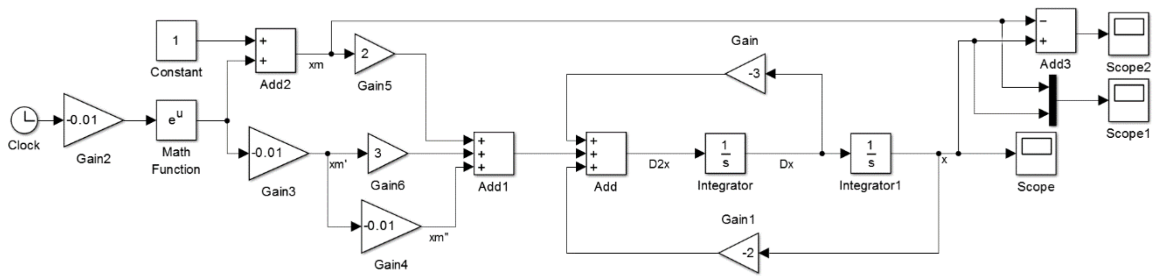
and hence the error dynamics are given by:

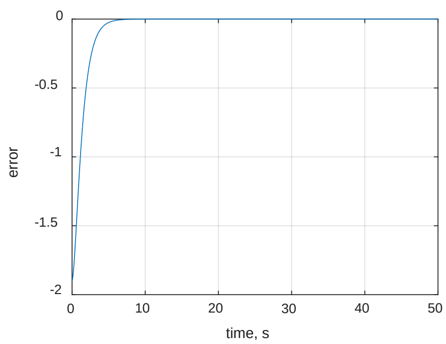
$$\ddot{\tilde{x}} + A\dot{\tilde{x}} + B\tilde{x} = 0 \tag{3}$$

So if the scalars A and B define a stable system the tracking error will converge to zero and hence the system will have the desired response.

Example 1.1: A system is given by the following 2^{nd} order ODE: $\ddot{x} + 3\dot{x} + 2x = u$ and we want it to track the desired trajectory $x_d = 1 + e^{-0.01t}$

Then we can set the control signal $u = \ddot{x}_d + 3\dot{x}_d + 2x_d$ and the response is:





If on the other hand the matrices A and B are not stable (or fast enough) we can use:

$$u = A\dot{x} + Bx - C\dot{\tilde{x}} - D\tilde{x} + \ddot{x}_d \tag{4}$$

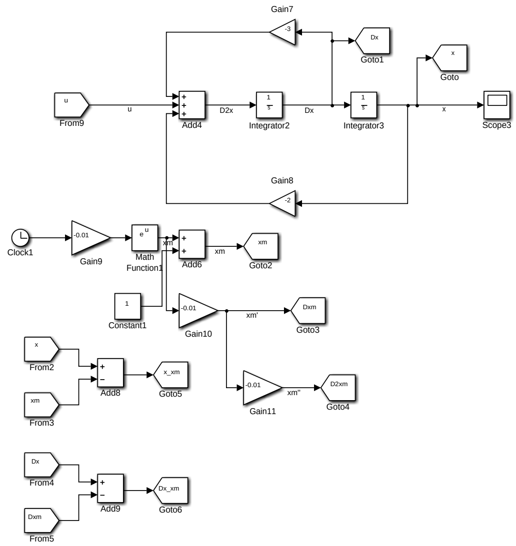
which will give me:

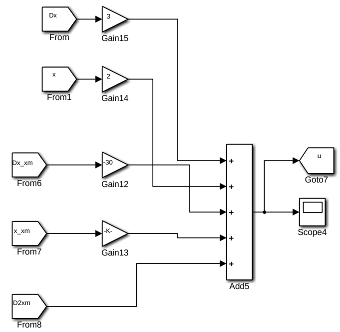
$$\ddot{x} + A\dot{x} + Bx = A\dot{x} + Bx - C\dot{\tilde{x}} - D\tilde{x} + \ddot{x}_d \Leftrightarrow$$
$$\ddot{x} - \ddot{x}_d + C\dot{\tilde{x}} + D\tilde{x} = 0 \Leftrightarrow$$
$$\ddot{\tilde{x}} + C\dot{\tilde{x}} + D\tilde{x} = 0$$

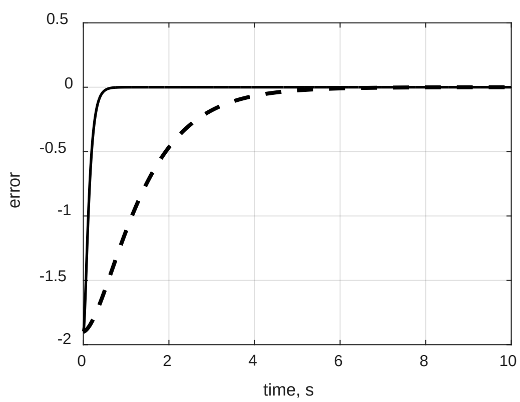
and hence if we properly choose C and D in order to have fast and stable dynamics for the error we can again ensure that the system will have the desired response.

Example 1.2: Assume that we want to make the speed of the error 10 times faster, i.e. to place the poles of the error dynamics at -10 and -20:

$$u = 3\dot{x} + 2x - 30\dot{\hat{x}} - 200\hat{x} + \ddot{x}_d$$







■

This method can be applied now to a nonlinear n^{th} order system as if

$$x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right) + g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)u$$

We can always choose:

$$u = \frac{-h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} - f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}{g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}$$

$$x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right) +$$

$$g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)\left(\frac{-h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} - f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}{g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)}\right)$$

$$x^{(n)} = -h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) + x_d{}^{(n)} \Leftrightarrow$$

$$x^{(n)} - x_d{}^{(n)} = -h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) \Leftrightarrow$$

$$\tilde{x}^{(n)} + h\left(\tilde{x},\dot{\tilde{x}},\ddot{\tilde{x}},\cdots,\tilde{x}^{(n-1)},t\right) = 0$$

Hence if we properly choose the function h we can make sure that regardless of the desired signal the system will behave satisfactory.

2. Sliding mode control

2.1 Ideal systems

Assume that we have a system:

$$x^{(n)} = f\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right) + g\left(x,\dot{x},\ddot{x},\cdots,x^{(n-1)},t\right)u \tag{5}$$

with a desired tracking trajectory $x_d(t)$, the error between the desired and real trajectory is defined as:

$$\tilde{x}(t) = x(t) - x_d(t) \tag{6}$$

We know that studying an n^{th} order nonlinear system is a cumbersome task, while linear systems are much easier to handle. So the first question that we have here is, can a linear system represent our system given in (5)? Let's denote the variable "*s*"[1] whose ODE describes our system. We impose 2 properties on *s* here:

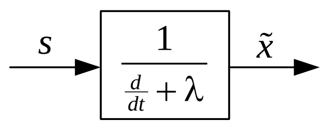
1. We need to differentiate *s* only once in order to have an expression of the control signal *u*.
2. When $s \to 0 \Rightarrow \tilde{x}(t) \to 0$

If we have a 2nd order system: $\ddot{x} = f(x,\dot{x},t) + g(x,\dot{x},t)u$ then the conditions are verified if we choose:

[1] Do not be confused, this *s* has nothing to do with the Laplace variable.

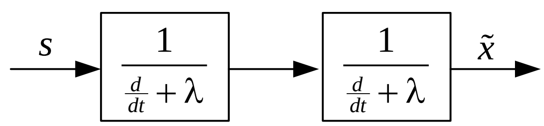
$$s = \dot{\tilde{x}} + \lambda\tilde{x} = \left(\frac{d}{dt} + \lambda\right)\tilde{x} \tag{7}$$

which can be seen as a stable linear filter:

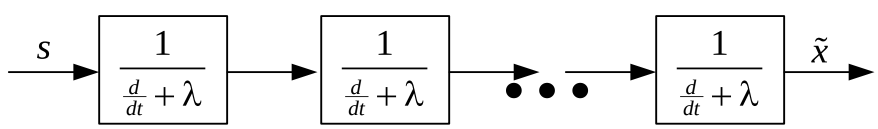


For 3rd order systems:

$$s = \left(\frac{d}{dt} + \lambda\right)^2 \tilde{x} = \left(\frac{d^2}{dt^2} + 2\lambda\frac{d}{dt} + \lambda^2\right)\tilde{x} = \frac{d^2\tilde{x}}{dt^2} + 2\lambda\frac{d\tilde{x}}{dt} + \lambda^2\tilde{x}$$



For the general case $s = \left(\frac{d}{dt} + \lambda\right)^{n-1} \tilde{x}$ (8)

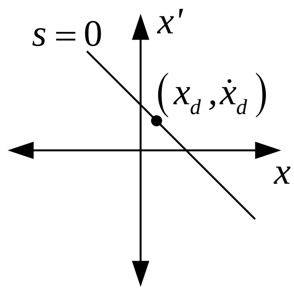


So by studying the error dynamics (given by (8)) we have replaced a nonlinear system with a linear one (and of a smaller order).

Our task now is to find the control law that will make the ODE given by (8) a stable one, i.e. that the error tends to zero (in finite time).

From this point we will focus on 2nd order systems, but the same analysis can be carried out in a general n^{th} order system.

Geometrically the condition $s = \dot{\tilde{x}} + \lambda\tilde{x}$ with s is zero we have $\dot{\tilde{x}} + \lambda\tilde{x} = 0$ or $\left(\dot{x} - \dot{x}_d\right) + \lambda\left(x - x_d\right) = 0$. Which is the equation of a straight line (or a surface in n^{th} dimensional systems) in the state space:



Now let's try to solve the ODE of the error dynamics (by assuming that s=0): $\dot{\tilde{x}} + \lambda\tilde{x} = 0 \Rightarrow \tilde{x}(t) = \tilde{x}(t_0)e^{-\lambda t}$ which implies that $\dot{\tilde{x}} = 0, \tilde{x} = 0$. Hence, if the trajectory at some point hits the surface defined by s=0 at t=t_0 we have that $\tilde{x}(t) = 0, \forall t > t_0$. Hence the surface defined by s is invariant and this implies that we will have the desired response $\forall t > t_0$.

Note: At this point we have **NOT** solved our control problem. We have just changed a nonlinear problem with a linear one and we have seen its geometric interpretation in the state space. The task now is to find the control law u that will make s=0 in finite time.

Now assume that you have a 2nd order system:

$$\ddot{x} = f\left(x, \dot{x}, t\right) + g\left(x, \dot{x}, t\right)u$$

We want to converge to $s = 0$ by the appropriate choice of u. In order to guarantee that the above equation is stable we can look for a Lyapunov function like: $V(s)=\frac{1}{2}s^2$ with $V(0)=0$ and $V(s)>0, s\neq 0$. So now we have to find the appropriate u such as $\frac{dV(s)}{dt}=s\dot{s}<0$ and hence according to Lyapunov to have a stable system. One obvious way to make sure that is to set $\dot{s}=-s$:

$$s=\dot{\tilde{x}}+\lambda\tilde{x} \Leftrightarrow \dot{s}=\ddot{\tilde{x}}+\lambda\dot{\tilde{x}}=\ddot{x}-\ddot{x}_d+\lambda\dot{\tilde{x}}=f(x,\dot{x},t)+g(x,\dot{x},t)u-\ddot{x}_d+\lambda\dot{\tilde{x}}$$

Hence: $f(x,\dot{x},t)+g(x,\dot{x},t)u-\ddot{x}_d+\lambda\dot{\tilde{x}}=-\dot{\tilde{x}}-\lambda\tilde{x} \Leftrightarrow$

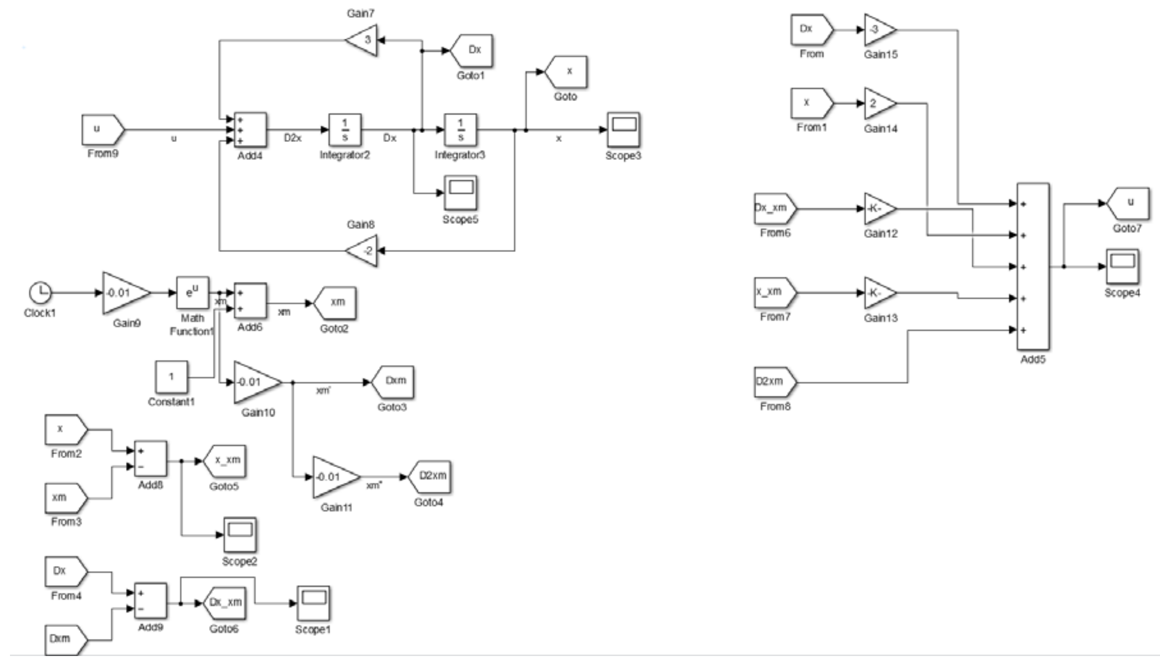
$$u=\frac{1}{g(x,\dot{x},t)}\left(-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}-f(x,\dot{x},t)\right)$$

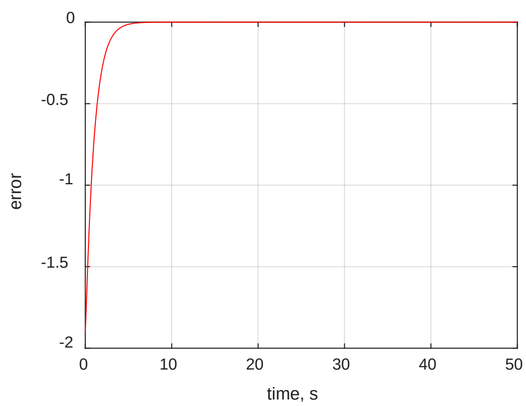
In this case we have:

$$\begin{aligned}\ddot{x}&=f(x,\dot{x},t)+g(x,\dot{x},t)u\\&=f(x,\dot{x},t)+g(x,\dot{x},t)\frac{1}{g(x,\dot{x},t)}\left(-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}-f(x,\dot{x},t)\right)\\&=-\dot{\tilde{x}}-\lambda\tilde{x}+\ddot{x}_d-\lambda\dot{\tilde{x}}\end{aligned}$$

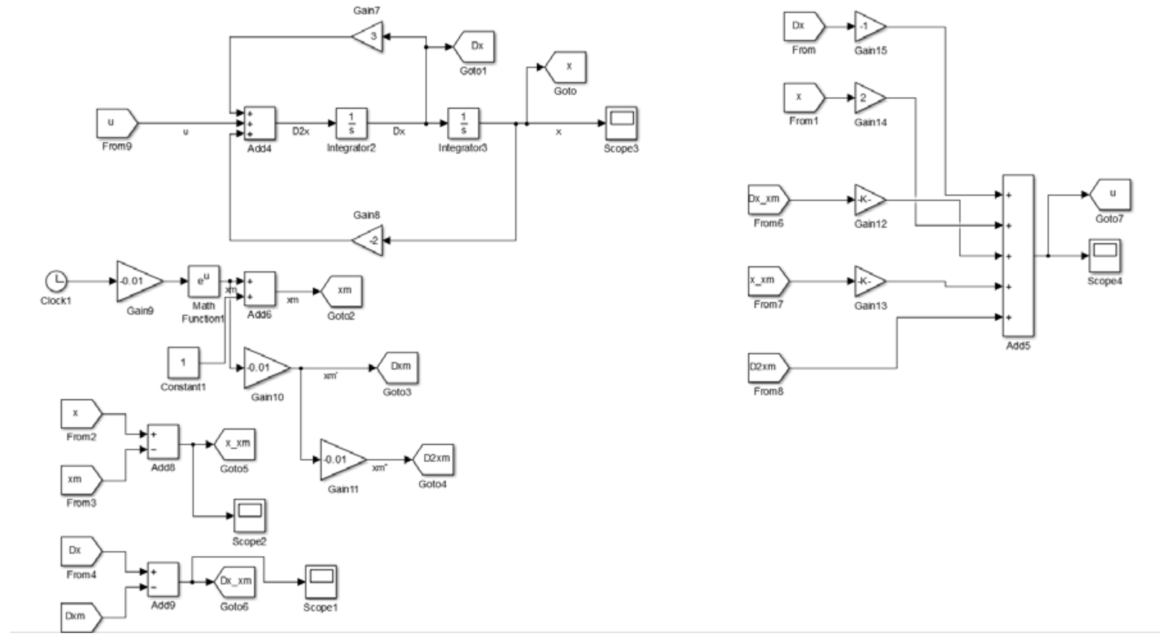
Hence, $\ddot{x}+\dot{\tilde{x}}+\lambda\tilde{x}-\ddot{x}_d+\lambda\dot{\tilde{x}}=0 \Leftrightarrow \ddot{\tilde{x}}+\dot{\tilde{x}}(1+\lambda)+\lambda\tilde{x}=0$

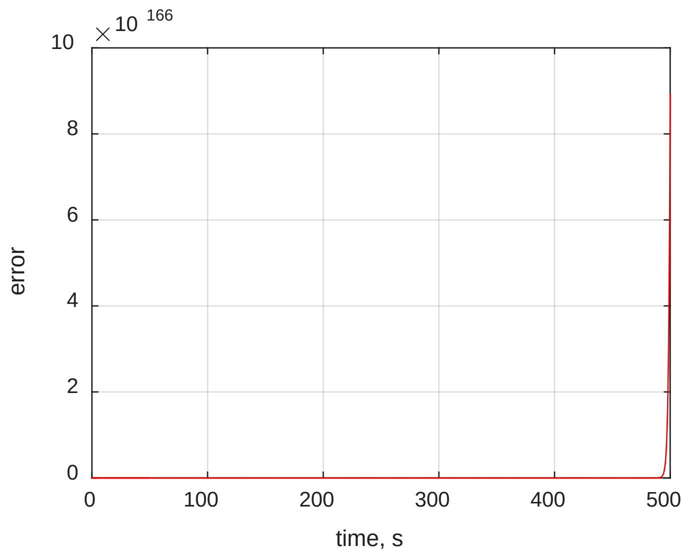
And hence we have a homogeneous ODE and with the appropriate choice of λ we can make sure that $\tilde{x}\to 0$





But if there is an imperfection in the system then the convergence will not happen:


Gain7
Dx
Goto1
x
Goto
u
From9
u
Add4
D2x
Integrator2
Dx
Integrator3
x
Scope3
Gain8
Clock1
-0.01
Gain9
Math Function1
xm
Add6
xm
Goto2
1
Constant1
-0.01
Gain10
xm'
Dxm
Goto3
x
From2
xm
From3
Add8
x_xm
Goto5
Scope2
-0.01
Gain11
xm''
D2xm
Goto4
Dx
From4
Dxm
Add9
Dx_xm
Goto6
Scope1
Dx
From
-1
Gain15
x
From1
2
Gain14
Dx_xm
From6
-K-
Gain12
x_xm
From7
-K-
Gain13
D2xm
From8
Add5
u
Goto7
Scope4


× 10 166
10
8
6
4
2
0
error
0
100
200
300
400
500
time, s

2.2 Nonsmooth control law

In order to make the controller more robust to parameter changes we impose a different condition on the Lypunov function:

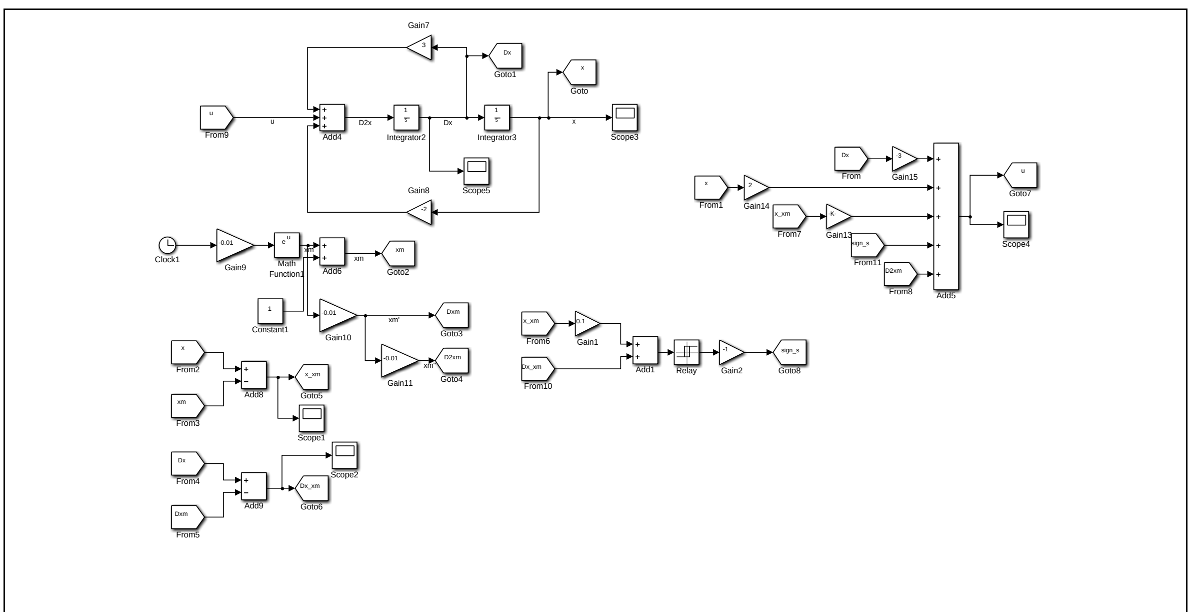
$$\dot{s} = -k \cdot \text{sign}(s)$$

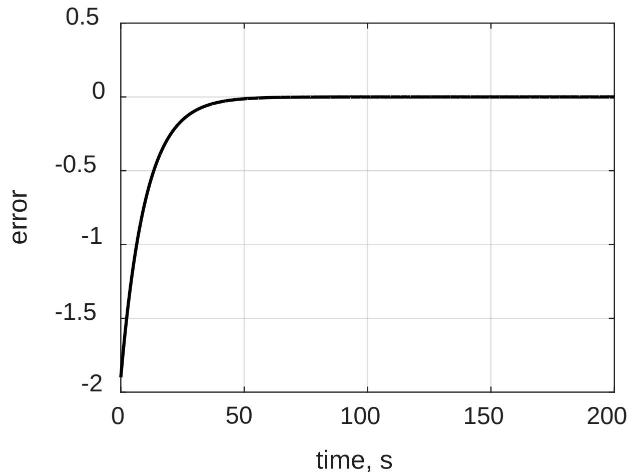
In this case $\dfrac{dV(s)}{dt} = s\dot{s} = -s \cdot k \cdot \text{sign}(s) = -k \cdot |s|$

We know that:

$$\dot{s} = f(x,\dot{x},t) + g(x,\dot{x},t)u - \ddot{x}_d + \lambda\dot{\tilde{x}} = -k \cdot sign(s) \Leftrightarrow$$

$$u = \frac{1}{g(x,\dot{x},t)}\left(-k \cdot sign(s) - f(x,\dot{x},t) + \ddot{x}_d - \lambda\dot{\tilde{x}}\right)$$





2.3 Uncertainty

Assume that we have a system:

$$\ddot{x} = f\left(x,\dot{x},t\right) + g\left(x,\dot{x},t\right)u \tag{9}$$

with *f* and *g* being two **NOT "COMPLETELY"** known functions.

Our model based on an estimation of *f* and *g* is written as:

$$\ddot{x} = \hat{f}\left(x,\dot{x},t\right) + \hat{g}\left(x,\dot{x},t\right)u \tag{10}$$

And we only know that the difference between the real and the estimated functions is bounded:

$$\left|\hat{f} - f\right| \leq F, \left|\hat{g} - g\right| \leq G \tag{11}$$

To simplify the analysis we assume that $g\left(x,\dot{x},t\right) = \hat{g}\left(x,\dot{x},t\right) = 1$

As before we can find when $\dot{s} = 0$ but now

$$u = -\hat{f}\left(x,\dot{x},t\right) + \ddot{x}_d - \lambda\dot{\tilde{x}} - k \cdot sign\left(s\right) \tag{12}$$

Hence,

$$\dot{s} = f\left(x,\dot{x},t\right) + u - \ddot{x}_d + \lambda\dot{\tilde{x}} = f\left(x,\dot{x},t\right) + \left(-\hat{f}\left(x,\dot{x},t\right) + \ddot{x}_d - \lambda\dot{\tilde{x}} - k \cdot sign\left(s\right)\right) - \ddot{x}_d + \lambda\dot{\tilde{x}} =$$
$$= f\left(x,\dot{x},t\right) - \hat{f}\left(x,\dot{x},t\right) - k \cdot sign\left(s\right)$$

Or $\dfrac{dV\left(s\right)}{dt} = s\left(f\left(x,\dot{x},t\right) - \hat{f}\left(x,\dot{x},t\right) - k \cdot sign\left(s\right)\right)$

So if we choose $k > F$ we can be sure that $\dfrac{dV\left(s\right)}{dt} \le 0$

2.4 Finite Convergence

We have seen that $\dfrac{dV\left(s\right)}{dt} = s\dot{s} \le 0$. We can impose a stricter condition to ensure that *s* becomes zero in finite time: $\dfrac{dV\left(s\right)}{dt} = s\dot{s} = -\eta\left|s\right| < 0, \eta > 0$

This is happening as if at some instant $t=t_0$ we have $s(t_0)>0$:

$$s\dot{s} = -\eta\left|s\right| \Leftrightarrow s\dot{s} = -\eta s \Leftrightarrow s = -\eta t + s\left(t_0\right)$$

which means that *s* will decrease until it becomes zero at $\eta t = s\left(t_0\right) \Leftrightarrow t = \dfrac{s\left(t_0\right)}{\eta}$

Similarly if $s(t_0)<0$:

$$s\dot{s} = -\eta|s| \Leftrightarrow s\dot{s} = \eta s \Leftrightarrow s = \eta t + s(t_0) \Leftrightarrow t = -\frac{s(t_0)}{\eta} > 0, s(t_0) < 0$$

Hence, we have

$$\frac{dV(s)}{dt} = s\left(f(x,\dot{x},t) - \hat{f}(x,\dot{x},t) - k \cdot sign(s)\right)$$

And

$$\frac{dV(s)}{dt} \le -\eta|s| \le 0$$

Which is true if $k = F + \eta$

3. Model Reference Adaptive Control

Assume the following system:

$$\ddot{x} = f\left(x,\dot{x},t\right) + u \tag{13}$$

with

$$f\left(x,\dot{x},t\right) = f_1\left(x,\dot{x},t\right)p_1 + f_2\left(x,\dot{x},t\right)p_2 + \cdots \tag{14}$$

for example:

- $f\left(x,\dot{x},t\right) = \dot{x}p_1 + xp_2$ (linear 2nd order ODE)
- $f\left(x,\dot{x},t\right) = \dot{x}^2 p_1 + x|x||\dot{x}|p_2 + cos\left(\sqrt{x}\right)$ (nonlinear but time invariant)
- $f\left(x,\dot{x},t\right) = cos\left(t\right)\dot{x}\,cos\left(x\right)p_1$ (nonlinear and time varying)

Now, we assume that $f_1\left(x,\dot{x},t\right), f_2\left(x,\dot{x},t\right)\cdots$ are known functions of the state vector, while the $p_1, p_2 \cdots$ are unknown **constants**.

Now as in the previous section let's choose a Lyapunov function such as:

$$V\left(s\right) = \frac{1}{2}s^2 \tag{15}$$

where $s = \dot{\tilde{x}} + \lambda\tilde{x}$

and hence: $\dot{s} = \ddot{\tilde{x}} + \lambda\dot{\tilde{x}} = \ddot{x} - \ddot{x}_d + \lambda\dot{\tilde{x}} = f\left(x,\dot{x},t\right) + u - \ddot{x}_d + \lambda\dot{\tilde{x}}$

from (14) we have that

$$\dot{s} = u + \left(f_1(x,\dot{x},t)\, p_1 + f_2(x,\dot{x},t)\, p_2 + \cdots \right) - \ddot{x}_d + \lambda \dot{\tilde{x}}$$

Now this can be written as:

$$\begin{aligned}\dot{s} &= u + \left(f_1(x,\dot{x},t)\, p_1 + f_2(x,\dot{x},t)\, p_2 + \cdots - \ddot{x}_d + \lambda \dot{\tilde{x}} \right) \\ &= u + F \cdot p\end{aligned}$$

So: $\dot{V}(s) = s\dot{s} = s(u + F \cdot p)$

with $F = \left[f_1(x,\dot{x},t) \quad f_2(x,\dot{x},t) \quad \cdots \quad \ddot{x}_d \quad \dot{\tilde{x}} \right]$ & $p = \left[p_1 \quad p_2 \quad \cdots \quad -1 \quad \lambda \right]^T$

Hence, if we knew the vector p we can choose: $u = -F \cdot p - ks, k > 0$

which would have given:

$$\dot{V}(s) = -ks^2 < 0$$

Unfortunately we do not the vector p but we can have an estimate $\hat{p}(t)$ and hence we can define the parameter error $\tilde{p}(t) = p - \hat{p}(t)$

So actually our control signal is: $u = -F \cdot \hat{p} - ks$

So $\dot{V}(s) = s\dot{s} = s(u + F \cdot p) = s(-F \cdot \hat{p} - ks + F \cdot p) = -ks^2 + sF \cdot \tilde{p}$ (16)

Previously when we had only the tracking error we used as $V(s) = \frac{1}{2}s^2$, now we also have the parameter error and hence we can use:

$$V(s) = \frac{1}{2}s^2 + \frac{1}{2}\tilde{p}_1^2 + \frac{1}{2}\tilde{p}_2^2 + \cdots$$

As not all parameter errors are equally important:

$$V(s)=\frac{1}{2}s^2+h_1\frac{1}{2}\tilde{p}_1^2+h_2\frac{1}{2}\tilde{p}_2^2+\cdots$$

And in a matrix form[2]:

$$V(s)=\frac{1}{2}s^2+\frac{1}{2}\tilde{p}^T H\tilde{p}$$

Now, $\frac{d\left(\frac{1}{2}\tilde{p}^T H\tilde{p}\right)}{dt}=\frac{1}{2}\dot{\tilde{p}}^T H\tilde{p}+\frac{1}{2}\tilde{p}^T H\dot{\tilde{p}}=\dot{\tilde{p}}^T H\tilde{p}\ \left(\text{ or } \tilde{p}^T H\dot{\tilde{p}}\right)$

Also, $\tilde{p}(t)=p-\hat{p}(t)\Rightarrow \dot{\tilde{p}}(t)=\cancel{\dot{p}}-\dot{\hat{p}}(t)$

So $\frac{d\left(\frac{1}{2}\tilde{p}^T H\tilde{p}\right)}{dt}=-\dot{\hat{p}}^T H\tilde{p}$

Hence the time derivative of the chosen Lyapunov function is:

$$\dot{V}(s)=-ks^2+sF\cdot\tilde{p}-\dot{\hat{p}}^T H\tilde{p}$$

Hence, if we set $sF\cdot\tilde{p}=\dot{\hat{p}}^T H\tilde{p}\ \Leftrightarrow sF=\dot{\hat{p}}^T H\Leftrightarrow \dot{\hat{p}}^T=sFH^{-1}$

So the adaptation law is: $\dot{\hat{p}}^T=sFH^{-1}$ (17)

And the control law is: $u=-F\cdot\hat{p}-ks$ (18)

---

[2] In general the matrix H does not have to be diagonal, but only symmetric.

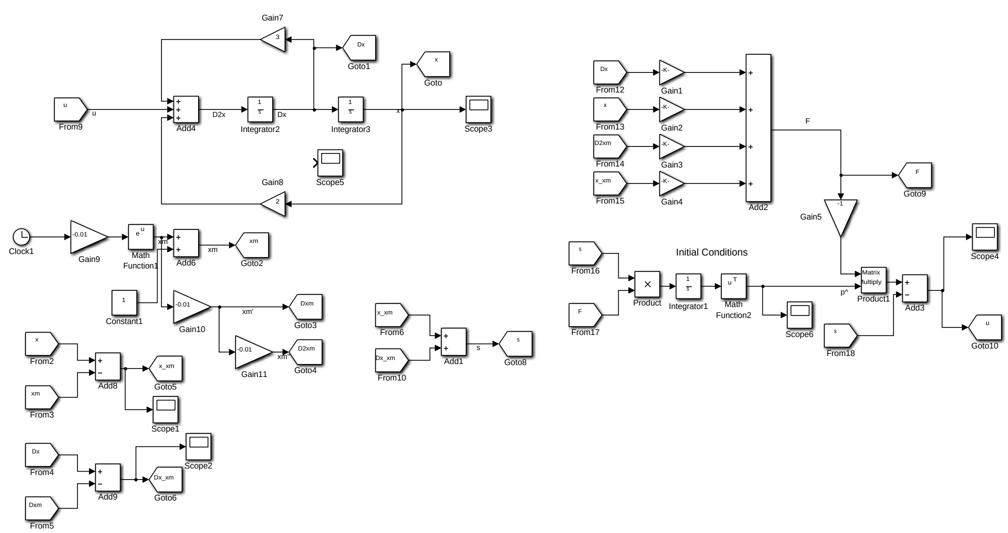
Another way to see this is to place (xx) into (xx):

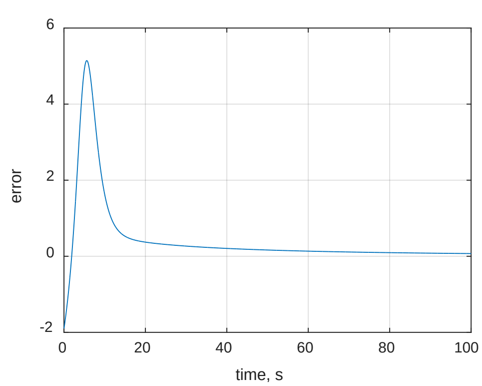
$$\hat{p} = \left(\int sFH^{-1}dt\right)^T \Rightarrow u = \underbrace{-F \cdot \left(\int sFH^{-1}dt\right)^T}_{Integral\ term} - \underbrace{ks}_{PD\ term} \tag{19}$$

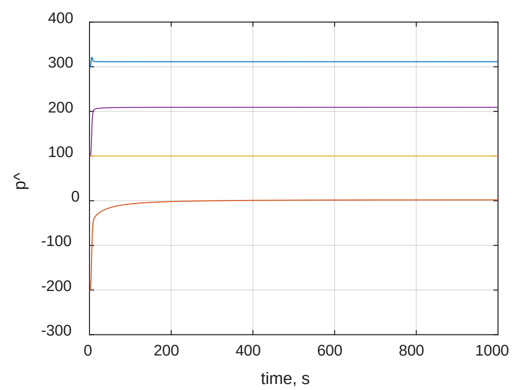
i.e. an adaptive PID controller.

Example 1:

$$\ddot{x} = 3\dot{x} + 2x + u$$

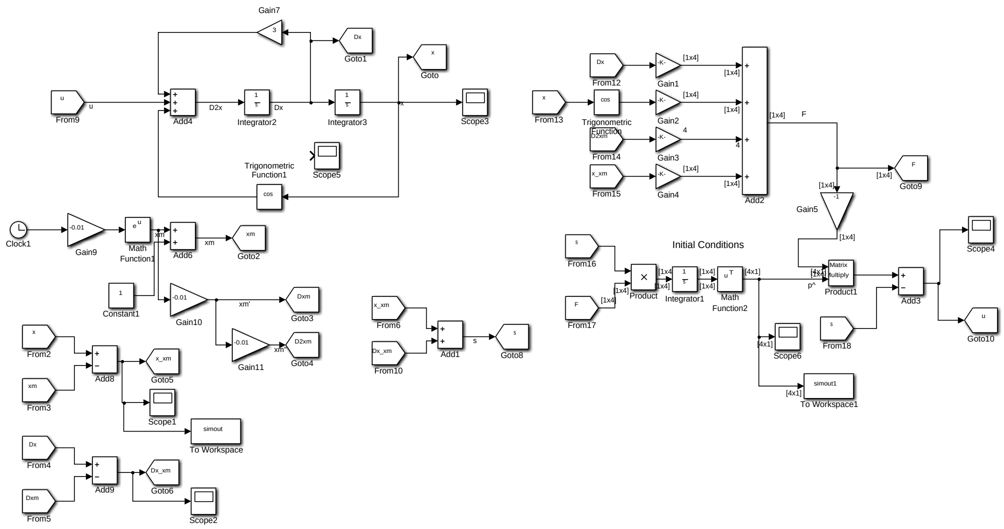


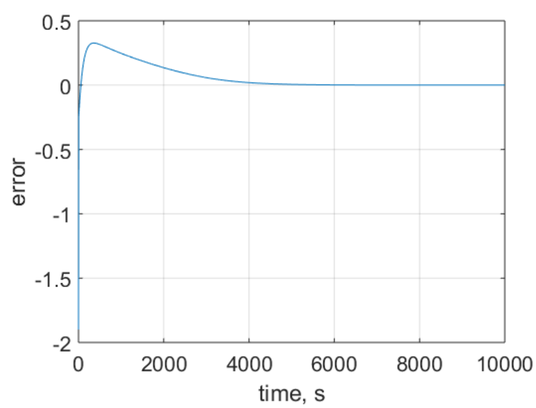




Example 2:

$$\ddot{x} = 3\dot{x} + \cos(x) + u$$





Matlab Based Exercises for EEE8086

Reproduce all Simulink files of chapter 5.