1

$\dot{x} = f(x,t) \rightarrow$ 1st order ODE

$\ddot{x} = f(x,\dot{x},t) \rightarrow$ 2nd order ODE

ODE + I.C. = I.V.P.

a soln $x(t)$

$\dot{x} = x^2$, $x = \dfrac{1}{\frac{1}{x_0} - t}$

$\dot{x} = -\dfrac{1}{\left(\frac{1}{x_0} - t\right)^2}$ $(1/x_0 - t)' \rightarrow -1$

$\dot{x} = \dfrac{1}{\left(\frac{1}{x_0} - t\right)^2} = x^2$

For linear sys. if $x_1 =$ soln.

then $k \cdot x_1 \rightarrow$ also a soln.

$\dot{x} = -3x$

$x_1 = e^{-3t}$

$\dot{x}_1 = -3e^{-3t} = -3 \cdot x_1$

$x_2 = 10\, e^{-3t}$

$\dot{x}_2 = -30 \cdot e^{-3t} = -3 \cdot x_2$

H.W. prove that

$$\dot{x} = x^2 \quad , \quad x_1 = \frac{1}{1/x_0 - t} \rightarrow \text{soln.}$$

$$x_2 = \frac{3}{1/x_0 - t} \quad \cancel{\rightarrow} \text{ soln.}$$

$\dot{x} = -3x$ $\quad x_1 = e^{-3t}$, $x_2 = 10e^{-3t}$

$x_0 = 10$ $\quad x_1$ is NOT a soln.

$$\dot{x} + kx = u \longrightarrow x = e^{-kt} \cdot x_0 + e^{-kt} \cdot \int_0^t e^{kt_1} \cdot u \, dt_1$$

$u = \text{const.}$ $\quad x = e^{-kt} \cdot x_0 + \frac{u}{k}(1 - e^{-kt})$

- $k > 0$ $\quad x_{ss} = \frac{u}{k}$ $\quad$ stable
- $k < 0$ $\quad x \rightarrow \pm\infty$ $\quad$ unstable

$u = 0$ $\quad \dot{x} + kx = 0$ $\quad$ Assume $x = e^{rt}$

$r + k = 0 \Rightarrow r = -k.$

C.E $\quad$ eigenvalue

$$\ddot{x} + A\dot{x} + Bx = u$$

$$\downarrow u = 0$$

$\ddot{x} + A\dot{x} + Bx = 0$ Assume $x = e^{rt}$

$$r^2 + Ar + B = 0$$

$$\Delta = A^2 - 4B$$

$$r_{1,2} = \frac{-A \pm \sqrt{\Delta}}{2}$$

- $\Delta > 0 \to r_1, r_2 \in \mathbb{R},\ r_1 \neq r_2$

  $x_1 = \cancel{c_1} e^{r_1 t} \quad x_2 = \cancel{c_2} e^{r_2 t}$

  $$x = c_1 x_1 + c_2 x_2$$

- $\Delta = 0 \to r_1 = r_2 = r \in \mathbb{R}$

  $x_1 = e^{rt} \quad x_2 = e^{rt} \cdot t$

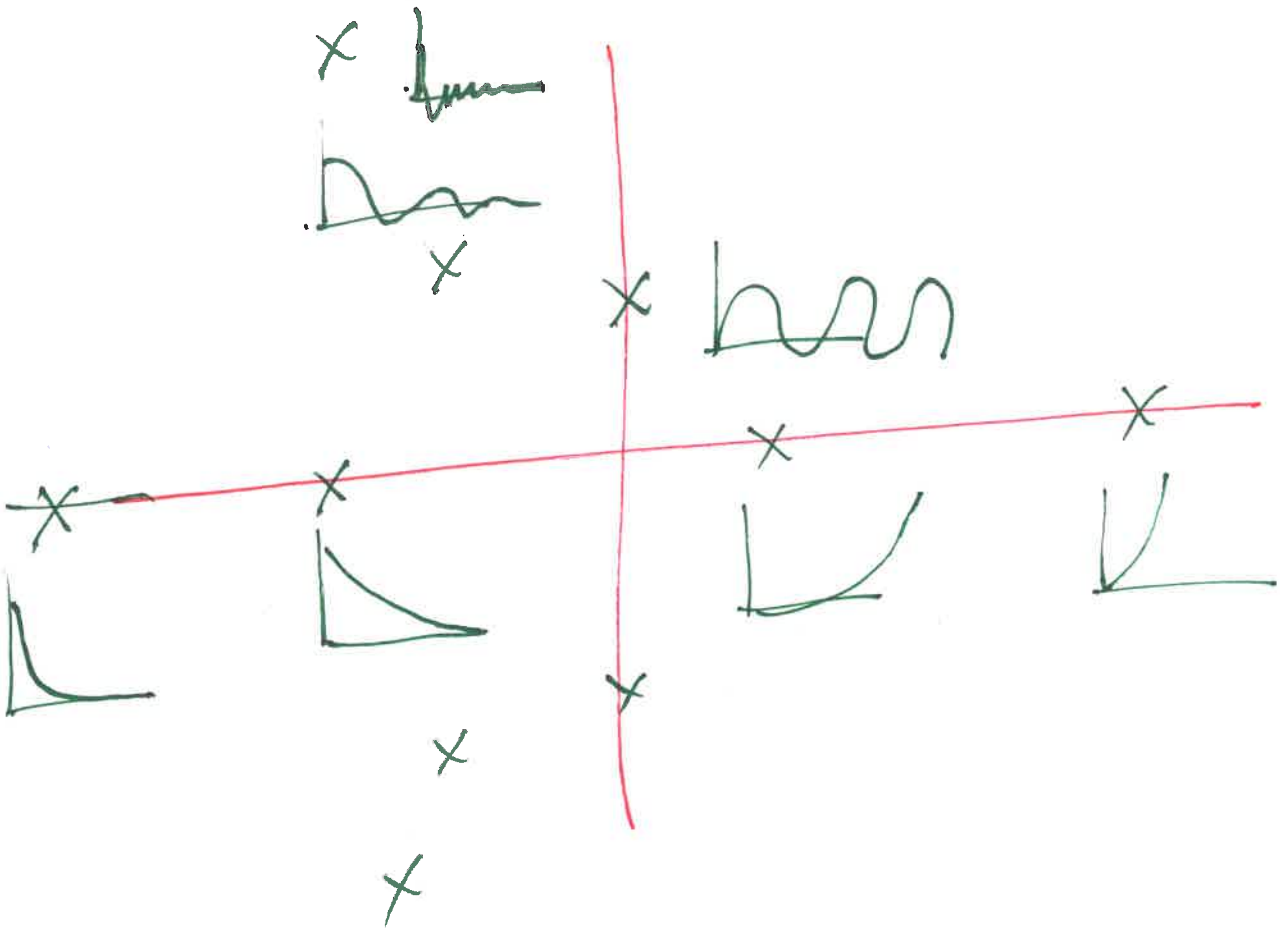
- $\Delta < 0$ $\quad r_1 = a \pm bi,\ r_2 = \bar{r}_1,\ a, b \in \mathbb{R}$

  $x_1 = e^{r_1 t} \quad x_2 = e^{r_2 t}$

  $x_1 = \mathrm{Re}(e^{r_1 t}),\ x_2 = \mathrm{Im}(\quad)$

  $x_1 = c_1 \cdot e^{at} \cdot \cos bt$

  $x_2 = c_2 e^{at} \cdot \sin bt$



2nd order ODEs x_1, x_2 are L.I

$\nexists k : x_1 = k \cdot x_2$

$$W(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}$$

$|W| \neq 0 \rightarrow$ L.I.

n^{th} order ODE.

$$\frac{d^{(n)}x(t)}{dt} = f(x^{(n-1)}, x^{(n-2)}, \ldots, \ddot{x}, \dot{x}, t)$$

$$x^{(k)} \neq x^{k}$$

k^{th} time derivative

Linear

e.g. $x^{(5)} = \dot{x} + t$

$$x^{(n)} + P_{n-1}x^{(n-1)} + P_{n-2}x^{(n-2)} + \ldots + P_0 x = 0$$

- $n=2 \rightarrow \ddot{x} + P_1\dot{x} + P_0 x = 0$
- $n=3 \rightarrow \dddot{x} + P_2\ddot{x} + P_1\dot{x} + P_0 \cdot x = 0$

If I have n L.I. soln $x_1, x_2 \ldots x_n$

Any other soln $x = \sum_{i=1}^{n} c_i x_i$

ex. 1.16

$$x^{(n)} + P_{n-1} x^{(n-1)} + P_{n-2} x^{(n-2)} + \dots + P_0 x = 0 \quad (6)$$

if x_1, x_2 are soln. $\rightarrow$ $x = c_1 x_2 + c_2 x_2$ also a soln.

if $x_1, x_2, \dots x_k$ ($\cancel{k < n}$) are soln $\rightarrow$

$$x = \sum_{i=1}^{k} c_i x_i \text{ also a sol.}$$

$$W(x_1, x_2, \dots x_n) = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ \dot{x}_1 & \dot{x}_2 & \dots & \dot{x}_n \\ \dots & \dots & \dots & \dots \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{bmatrix}$$

if $|W| \neq 0 \rightarrow x_1, x_2, \dots x_n$ are L.I. soln,

<u>ALL</u> $x = \sum_{i=1}^{n} x_i c_i$

$x^{(4)} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24x = 0$ 7

H.W.: $x_1 = e^{-t}$ $x_2 = e^{-2t}$ $x_3 = e^{-3t}$

one sol.

$W = ?$

$\downarrow 4\times 3$

$x = 3 \cdot x_1 + 5x_2 + 3x_3$

Add $x_4 = 2 \cdot x_2 \rightarrow$ soln.

$x = 3x_1 + 5x_2 + 3x_3 + x_4$

$W = ?$

$\downarrow 4\times 4.$

$x^{(n)} + p_{n-1} x^{(n-1)} + \dots\dots + p_0 x = 0$

Assume $x = e^{rt}$

$r^n \cancel{e^{rt}} + p_{n-1} r^{n-1} \cancel{e^{rt}} + \dots + p_0 \cancel{e^{rt}} = 0$

$r^n + p_{n-1} r^{n-1} + p_{n-2} r^{n-2} + \dots + p_0 = 0$

$\downarrow$ roots

roots([1 p_{n-1} p_{n-2} ..... p_0])

n roots

$k < 4$ (8)

- $r_1, r_2, \ldots r_k$, $r_1 \neq r_2$, $r_1 \neq r_3$, $\vdots$, $r_1 \neq r_k$ $\in \mathbb{R}$

$e^{r_1 t}$ $e^{r_2 t}$ $e^{r_k t}$

- $r_{k+1} = a + bi$ $r_{k+2} = a - bi$

$e^{(r_{k+1}) \cdot t}$ $e^{r_{k+2} \cdot t}$

- $r_{k+3} = r_{k+4} = r \rightarrow e^{rt}, te^{rt}$

$\in \mathbb{R}$

- $r_{k+5} = r_{k+6} = r_{k+7} = r_A \in \mathbb{R}.$

$e^{r_A t}$, $e^{r_A t} \cdot t$ $e^{r_A t} \, t^2$

- $r_{k+8} = r_{k+9} = r_{k+10} = r_{k+11} = r_b$

$e^{r_b t}$, $t e^{r_b t}$, $t^2 e^{r_b t}$, $t^3 e^{r_b t}$

- $r_{k+12} = c + di$, $r_{k+13} = c - di$

$r_{k+14} = r_{k+12}$, $r_{k+15} = r_{k+13}$

$e^{r_{k+12} \cdot t}$, $e^{r_{k+13} t}$, $e^{(r_{k+12}) t} \cdot t$, $e^{r_{k+13} \cdot t} \cdot t$

$r_1 = -1, \quad r_2 = -2, \quad r_3 = -3$

$x = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}$

$r_1 = -1, \quad r_2 = -1, \quad r_3 = -3$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$e^{-t} \qquad t \cdot e^{-t} \qquad e^{-3t}$

$r_1 = -1 \quad r_2 = -1, \quad r_3 = -1$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$e^{-t} \qquad t\, e^{-t} \qquad t^2 e^{-t}$

$r_1 = r_2 = r_3 = r_4 = -1$

$e^{-t}, \quad t e^{-t}, \quad t^2 e^{-t}, \quad t^3 e^{-t}$

$r_1 = -1 \quad r_2 = -1 - i, \quad r_3 = -1 + i$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$e^{-t} \qquad e^{(-1-i)t}, \quad e^{(-1+i)t}$

$r_1 = -1 \quad r_2 = -1 - i \quad r_3 = -1 - 2i$

Not a real sys.

(10)

$r_1 = -1+i \quad r_2 = -1-i, \quad r_3 = -1$

$r_4 = -5.$

$e^{-t}, \quad e^{-5t}, \quad e^{(-1+i)t}, \quad e^{(-1-i)t}$

$r_1 = -1+i \quad r_2 = -1-i \quad r_3 = -1, \quad r_4 = -1$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$e^{(-1+i)t} \quad e^{(-1-i)t} \quad e^{-t} \quad te^{-t}$

$r_1 = -1+i \quad r_2 = -1-i \quad r_3 = -1+i$

$r_4 = -1-i$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$e^{(-1+i)t} \quad e^{(-1-i)t} \quad te^{(-1+i)t}$

$te^{(-1-i)t}$

$r_1 = -1$, $r_2 = -2$, $r_3 = -2$

$r_{4,5} = -3 \pm i$, $r_{6,7} = -4 \pm 2i$, $r_{8,9} = -4 \pm 2i$

$r_{10,11} = -5 \pm 3i$, $r_{12,13} = -5 \pm 3i$

$r_{14,15} = -5 \pm 3i$

$x_1 = e^{-t}$, $x_2 = e^{-2t}$, $x_3 = e^{-2t} \cdot t$

$x_4 = e^{(-3+i)t}$, $x_5 = e^{(-3-i)t}$

$x_6 = e^{(-4+2i)t}$, $x_7 = e^{(-4-2i)t}$

$x_8 = e^{(-4+2i)t} \cdot t$, $x_9 = e^{(-4-2i)t} \cdot t$

$x_{10} = e^{(-5+3i)t}$, $x_{11} = e^{(-5-3i)t}$

$x_{12} = e^{(-5+3i)t}\, t$, $x_{13} = e^{(-5-3i)t} \cdot t$

$x_{14} = e^{(-5+3i)t}\, t^2$, $x_{15} = e^{(-5-3i)t}\, t^2$