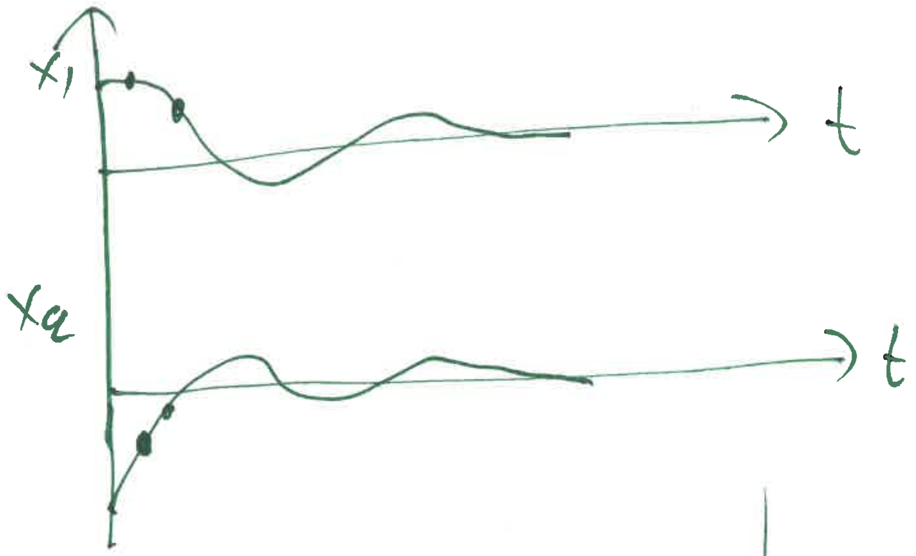
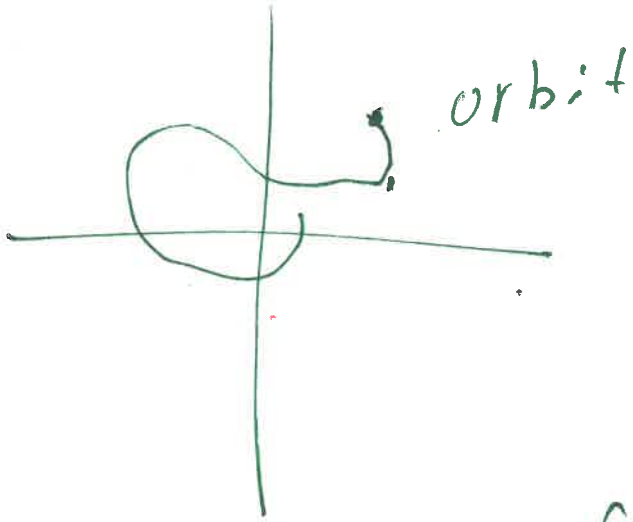
$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



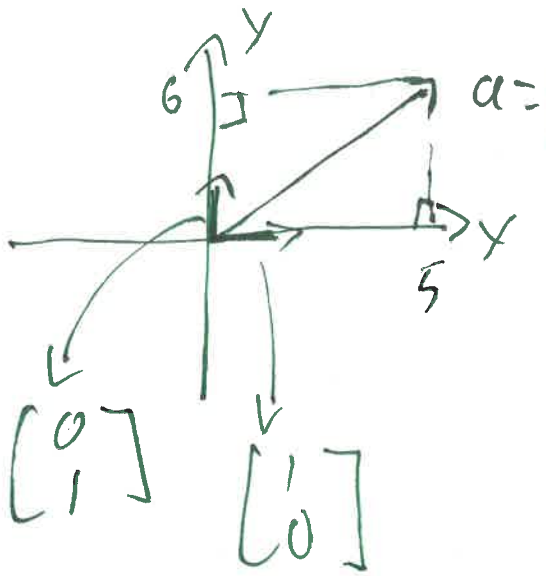


$$A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix} \quad \begin{matrix} \Rightarrow r_1 = -1, \; e_1 = [2 \quad 1] \\ \Rightarrow r_2 = -6, \; e_2 = [1 \quad -2] \end{matrix}$$

$$\left. X = c_1 e_1 e^{r_1 t} + c_2 e_2 \cdot e^{r_2 t} \quad X_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow$$

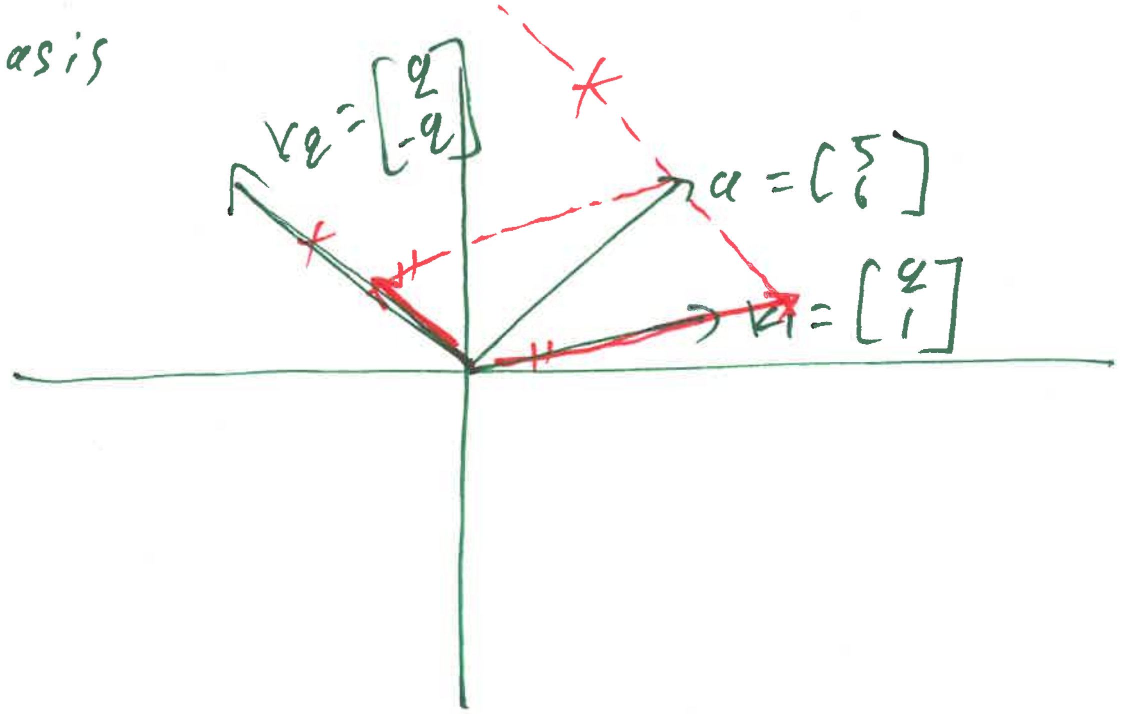
$c_1 = 0.6$
$c_2 = 0.2$

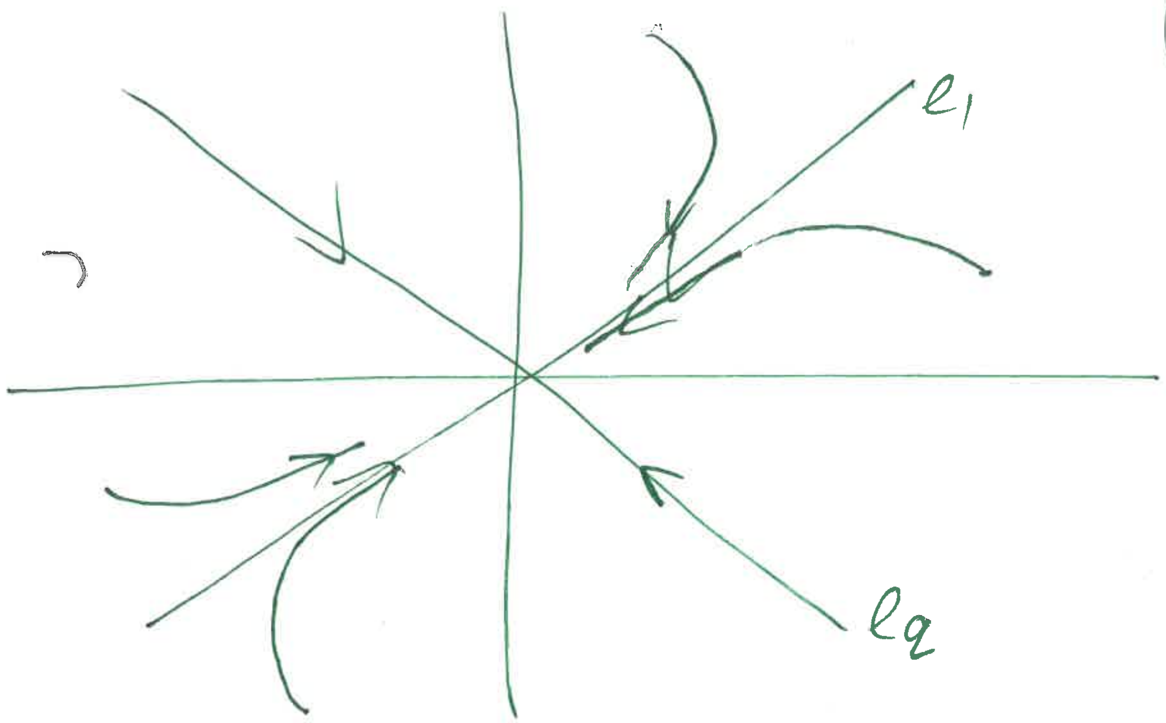
Revision



$$a = \begin{bmatrix} 5 \\ 6 \end{bmatrix} = 5 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

orthonormal basis



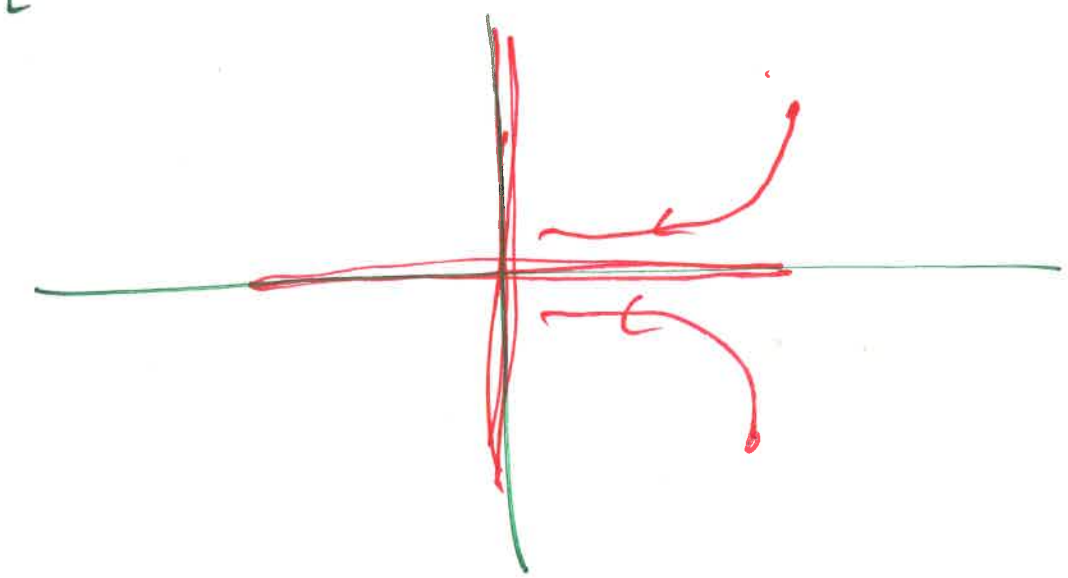


$$X = 0.6\cdot e^{-t}\begin{bmatrix}2\\1\end{bmatrix} - 0.2\, e^{-6t}\begin{bmatrix}1\\-2\end{bmatrix}$$

After 1s. $X \simeq 0.6\cdot e^{-t}\begin{bmatrix}2\\1\end{bmatrix}$

$\hat{A} = \begin{bmatrix}-1 & 0\\ 0 & -6\end{bmatrix}$ $\quad \lambda_1 = -1 \to e_1 = [1 \quad 0]$

$\lambda_2 = -6 \to e_2 = [0 \quad 1]$.



$$X = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1.2\, e^{-t} - 0.2\, e^{-6t}\\ 0.6\, e^{-t} + 0.4\, e^{-6t}\end{bmatrix}$$

- $t = 0 \Rightarrow x_1 = \dots \qquad x_2 = \dots$
- $t = 0.01 \Rightarrow$ . . . — — —

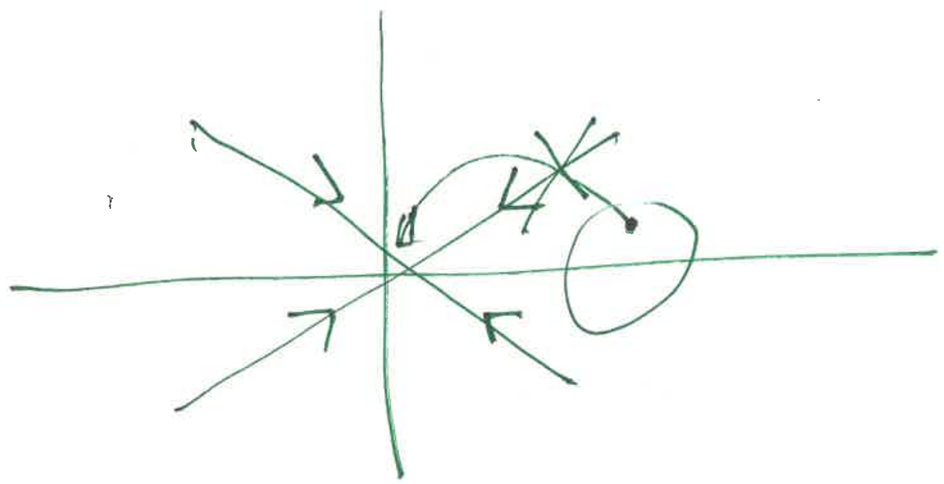
. — — — — .

$$X = \underbrace{0.6\, e^{-t}}_{a(t)}\begin{bmatrix}2\\1\end{bmatrix} - \underbrace{0.2\, e^{-6t}}_{b(t)}\begin{bmatrix}1\\-2\end{bmatrix}$$

$X = a\cdot e_1 + b\cdot e_2 \qquad e_1, e_2$ eigenbasis

- if at $t = t_i \quad X(t_i) = e_1$ (or e_2)

$\Rightarrow \quad X(t) = e_1 \quad \forall\, t$
(or e_2)



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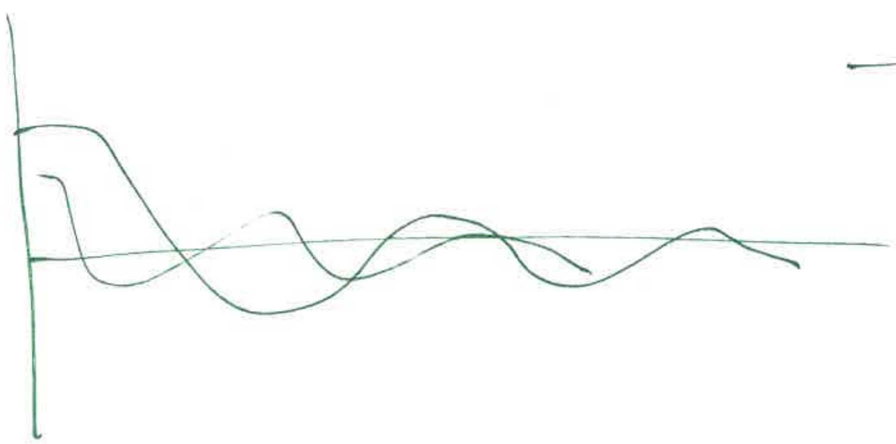
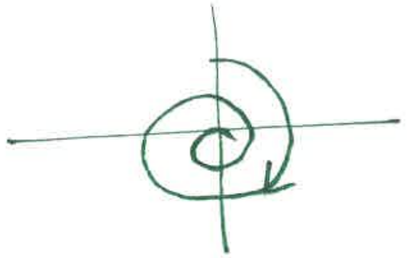
$A = \dots \quad \lambda = a \pm bi \to e = v \pm wi$

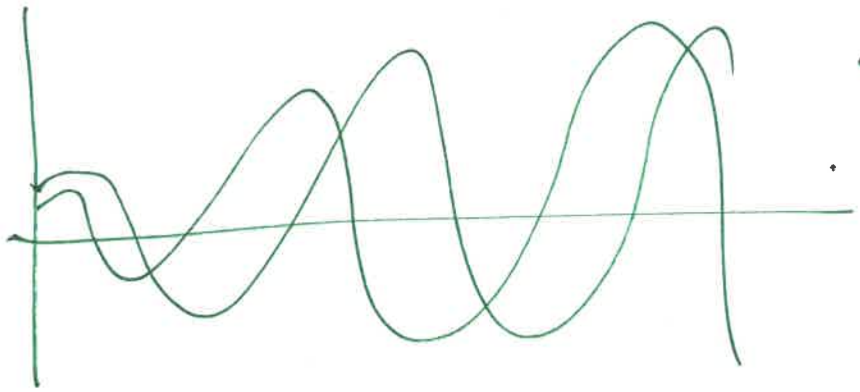
I cannot draw e in the state space

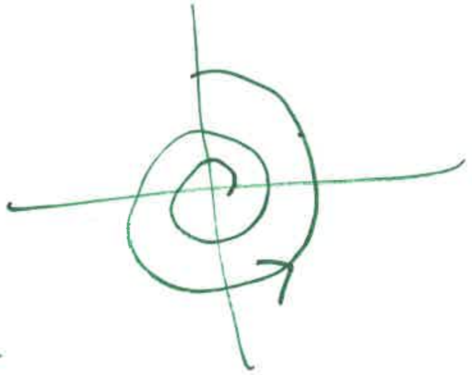
stable



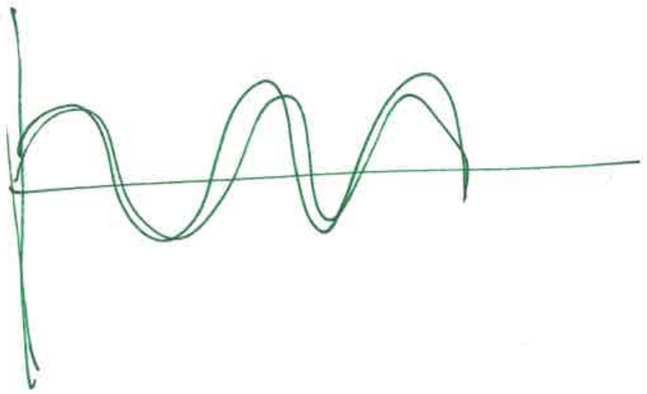


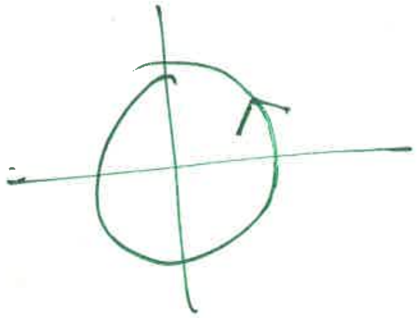
unst.





$\lambda = \pm bi$





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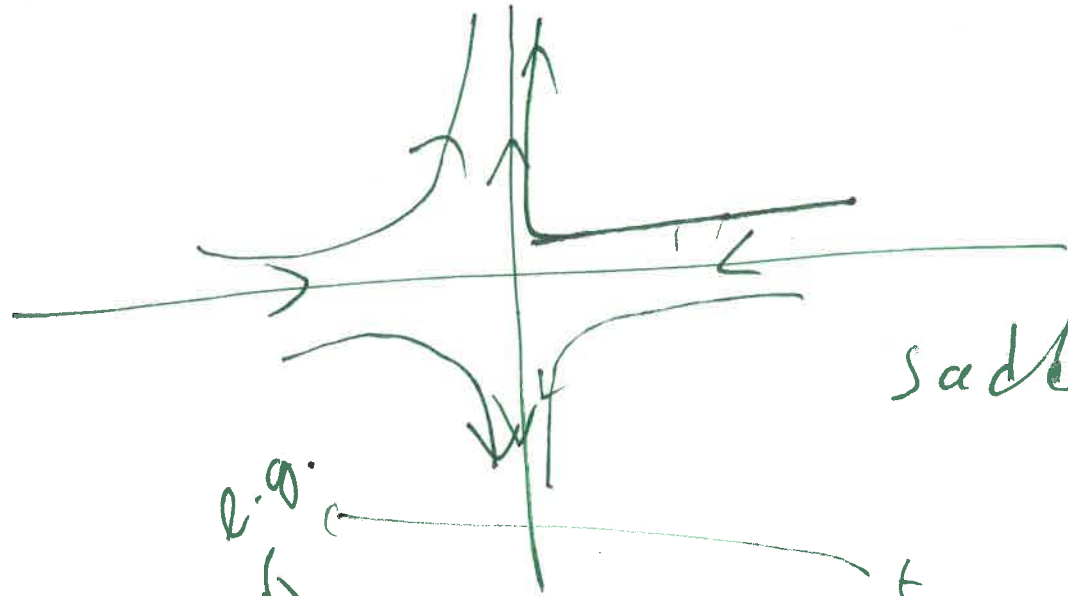


saddle

e.g.

$x = c_1 \cdot e_1 \cdot e^{-t} + c_2 \cdot e_2 \, e^{t}$

$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \lambda_1 = \lambda_2 = -1 = \lambda$

1 L.I. $\quad e = [1 \;\; 0]$

$x = \cancel{\quad} \; c_1 (et + b) e^{\lambda t} + c_2 e e^{\lambda t}$

$x_0 = k \cdot e$

$c_1 \cdot (e \cdot 0 + b) \cdot + c_2 e = k \cdot e$

$c_1 = 0 \Rightarrow x = \cancel{c_2} e e^{\lambda t}$

$k e e^{\lambda t}$

$e \to$ invariat

$x_0 = k \cdot b \qquad c_1 (e \cdot 0 + b) + c_2 e = k \cdot b$

if $c_2 = 0 \Rightarrow x = c_1 (et + b)$

$\dot{x} = Ax$

$x = e^{At} \cdot x_0$

if $x_0 = 0$

$\Big\} \Rightarrow x = 0 \quad \forall t$

$x_{ss} = 0$

So x_0 is an invariant point

$\dot{x} = Ax + B \cdot u$

A = stable $\quad x \to x_{ss}$

$\dot{x}_{ss} = 0$

$0 = A \cdot x_{ss} + Bu \Rightarrow x_{ss} = -A^{-1} \cdot B \cdot u$

it can be proved that if $x_0 = x_{ss}$

$x = x_{ss} \quad \forall t$

x_{ss} → Eq. point — Equilibrium

→ Fixed point

→ St. point — stationary

stable

unstable

Ch. 4

$\dot{x} = f(x, u)$, $\quad f \in \mathbb{R}^{n \times 1}$

$u \in \mathbb{R}^{p \times 1}$

L.O.D.E : if x_1 is a soln

$\Rightarrow k \cdot x_1$ also a soln.

~~N.L. ODE~~

$\dot{x} = -x^2$

$x = \dfrac{1}{t+c}$, $\quad -x^2 = -\dfrac{1}{(t+c)^2}$

$\dot{x} = -\dfrac{1}{(t+c)^2}$

is $x = 3 \dfrac{1}{t+c}$ a soln => ?

$-x^2 = -9 \cdot \dfrac{1}{(t+c)^2}$

$\dot{x} = -3 \cdot \dfrac{1}{(t+c)^2}$

$$\left.\begin{array}{l}\dot{x} = x^2 - a \\ \dot{x} = 0\end{array}\right\} \Rightarrow \quad a \in \mathbb{R}$$

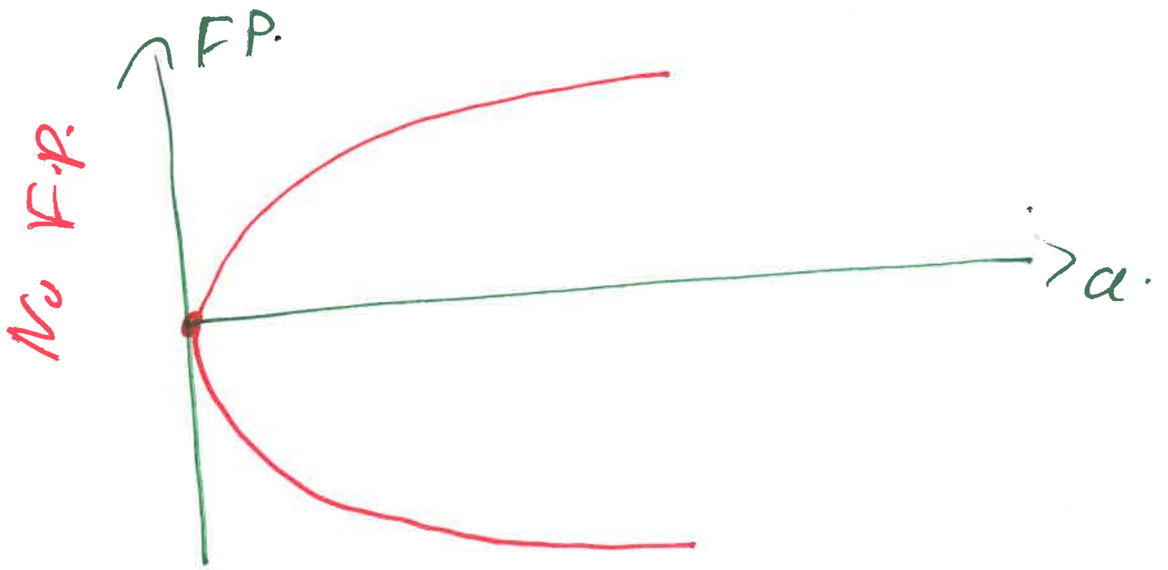
$$x^2 - a = 0 \Rightarrow x = \pm\sqrt{a}$$

$a < 0$ N.F.P.

$a = 0 \Rightarrow x = 0$ 1 F.P.

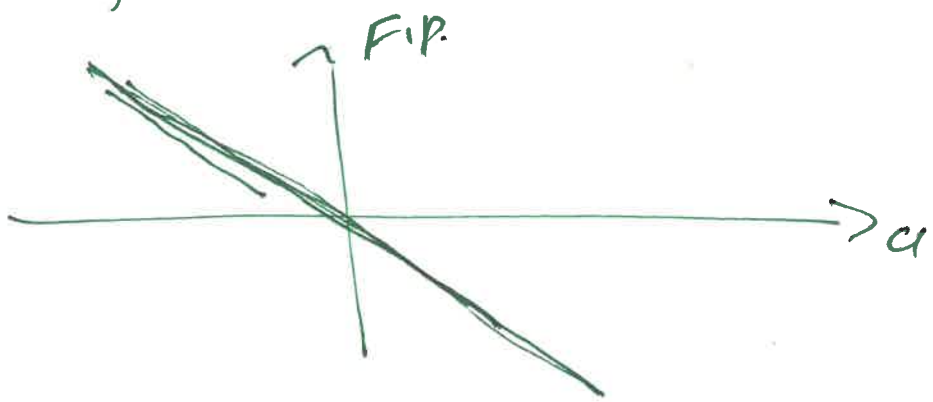
$a > 0 \Rightarrow x = \pm\sqrt{a}$ 2 F.P.

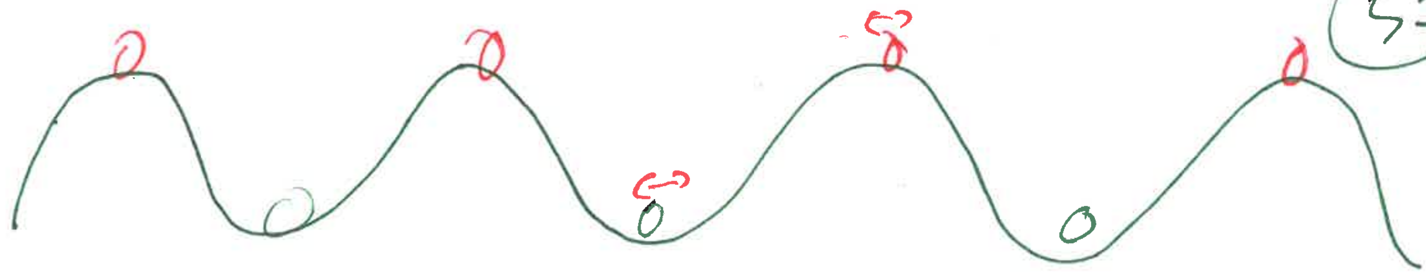
Bifurcation.



Bifurcation Diagram

$$\dot{x} = 5x + a, \quad \dot{x} = 0 \Rightarrow 5x + a = 0, \quad x = -a/5$$





How to find F.P.s = ?

$$\dot{x} = \cancel{f(x,a)} = f(x), \quad \dot{x} = 0$$

or $f(x) = 0$

$$\left.\begin{array}{l}\dot{x} = x^2 \\ \dot{x} = 0\end{array}\right\} \Rightarrow x^2 = 0 \Rightarrow x = 0$$

$$\dot{x} = Ax \Rightarrow x = 0$$

$$\dot{x} = Ax + Bu = 0 \Rightarrow x = -A^{-1} \cdot B \cdot u$$

$$\left.\begin{array}{l}\dot{x} = x^2 - 1 \\ \dot{x} = 0\end{array}\right\} \Rightarrow x^2 - 1 = 0$$

$$x^2 = 1 \Rightarrow x = +1, \quad x = -1$$

$$\left.\begin{array}{l}\dot{x} = x^2 + 1 \\ \dot{x} = 0\end{array}\right\} \Rightarrow x^2 + 1 = 0$$

There are no F.P.s

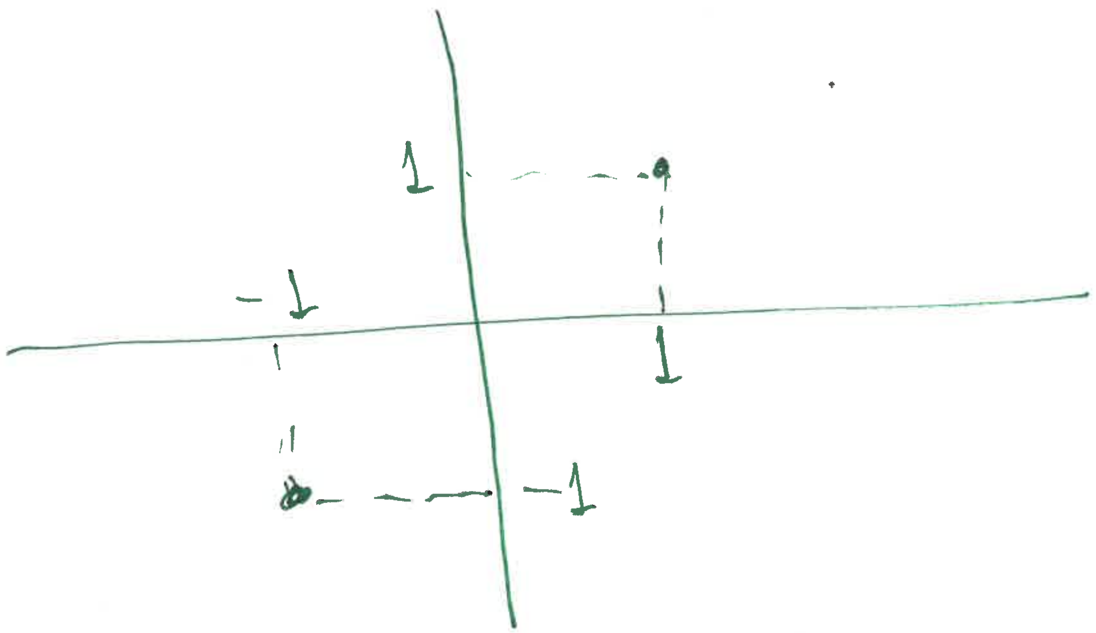
$$\left.\begin{aligned}\dot{x}_1 &= x_1 - x_2 \\ \dot{x}_2 &= x_1^2 + x_2^2 - 2\end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} x_1 - x_2 &= 0 \\ x_1^2 + x_2^2 - 2 &= 0\end{aligned}\right\} \Rightarrow x_1 = x_2$$

$$x_1^2 + x_1^2 - 2 = 0 \iff x_1^2 = 1$$

$$x_1 = \pm 1$$

$X_{EP1} = (1, 1)$, $X_{EP2} = (-1, -1)$



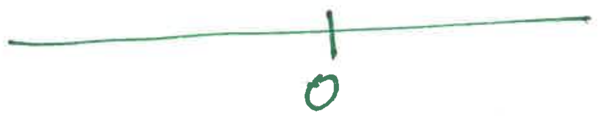
$\dot{x} = x^2 - a$.

$a = 9$

$a = 6$

$a = 1$

$a = 0$



$a < 0$

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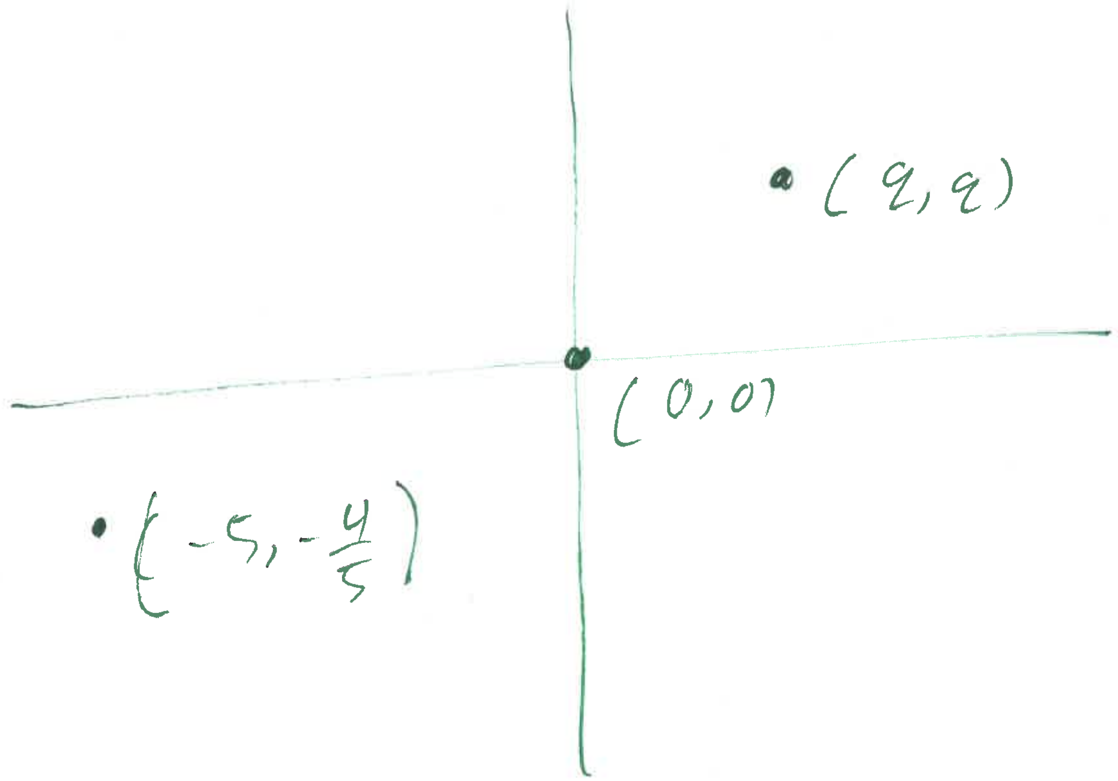

01

$$\left.\begin{aligned}\dot{x}_1 &= x_1^2 x_2 + 3\cdot x_1\cdot x_2 - 10x_2\\ \dot{x}_2 &= x_1^2 x_2 - 4\cdot x_1\end{aligned}\right\} \Rightarrow$$

$$x_1^2\cdot x_2 + 3x_1x_2 - 10x_2 = 0$$

$$x_1^2\cdot x_2 - 4\cdot x_1 = 0$$

$$x_2(x_1^2 + 3x_1 - 10) = 0$$

$x_2 = 0$ or $x_1^2 + 3x_1 - 10 = 0$

$$x = \frac{-3 \pm \sqrt{49}}{2} \to 2,\ -5$$

or $x_2 = 0$ or $x_1 = 2$ or $x_1 = 5$

if $x_2 = 0$: $x_1^2\, 0 - 4\cdot x_1 = 0 \Rightarrow x_1 = 0$

My 1st F.P. $(x_1, x_2) = (0,0)$

if $x_1 = 2$: $\cancel{x}\ 2^2\cdot x_2 - 4\cdot 2 = 0$

$4x_2 - 8 = 0 \Rightarrow x_2 = 2.$

My 2d F.P. $(x_1, x_2) = (2,2)$

if $x_1 = -5$ $\quad 25\cdot x_2 + 20 = 0 \Rightarrow$

$$x_2 = \frac{-20}{25} = -\frac{4}{5}$$

My 3d F.P. $(x_1, x_2) = (-5, -4/5)$