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$\dot{x} = f(x,t) \rightarrow$ 1st order ODE

$\ddot{x} = f(\dot{x}, x, t) \rightarrow$ 2nd order ODE.

I.V.P = ODE + I.C.

Soln $x(t)$

e.g. $\dot{x} = x^2$ $\quad x(t) = \dfrac{1}{\frac{1}{x_0} - t}$

$$\dot{x} = \frac{-1}{\left(\frac{1}{x_0} - t\right)^2} \left(\frac{1}{x_0} - t\right)'$$

$$= \frac{1}{\left(\frac{1}{x_0} - t\right)^2} = x^2$$

$\dot{x} = -3x \quad x = \sqrt{\pi} \cdot e^{-3t}$

$$\left(\sqrt{\pi} \cdot e^{-3t}\right)' = -3 \cdot \left(\sqrt{\pi}\, e^{-3t}\right)$$

$$-3 \cdot \sqrt{\pi}\, e^{-3t} = -3\sqrt{\pi} \cdot e^{-3t} \quad \checkmark$$

eg $\ddot{x} + 3\dot{x} + 2x = 0$

$$x = e^{-t}$$

$$\cancel{(e^{-3t})'' + 3(e^{-3t})' + 2e^{-3t} = 0}$$

$$\cancel{-9e^{-3t} + 2e^{-3t} = 0}$$

$$(e^{-t})'' + 3(e^{-t})' + 2e^{-t} = 0$$

$$e^{-t} + (-3e^{-t}) + 2e^{-t} = 0$$

$$0 = 0$$

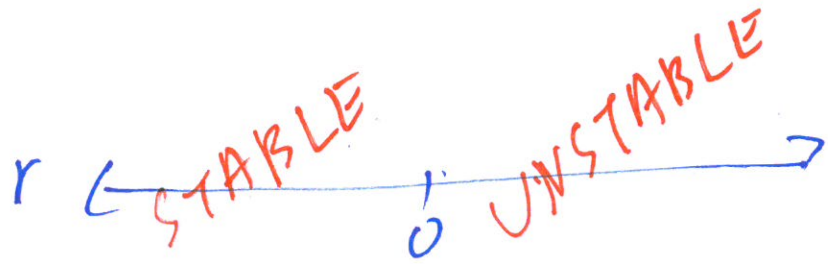
1st Linear ODE

$$\dot{x} + kx = u \Rightarrow x(t) = e^{-kt}\left(x_0 + \int_0^t e^{kt_1} u \, dt_1\right)$$

$\downarrow$ $u = 0$

$$x = e^{rt}$$

$$r + k = 0 \Rightarrow r = -k.$$



$\ddot{x} + A\dot{x} + Bx = 0$

$x = e^{rt}$

$r^2 + Ar + B = 0$

$r_{1,2} = \dfrac{-A \pm \sqrt{\Delta}}{2}$ $\qquad \Delta = A^2 - 4B$

- $\Delta > 0$, $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

  $x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$

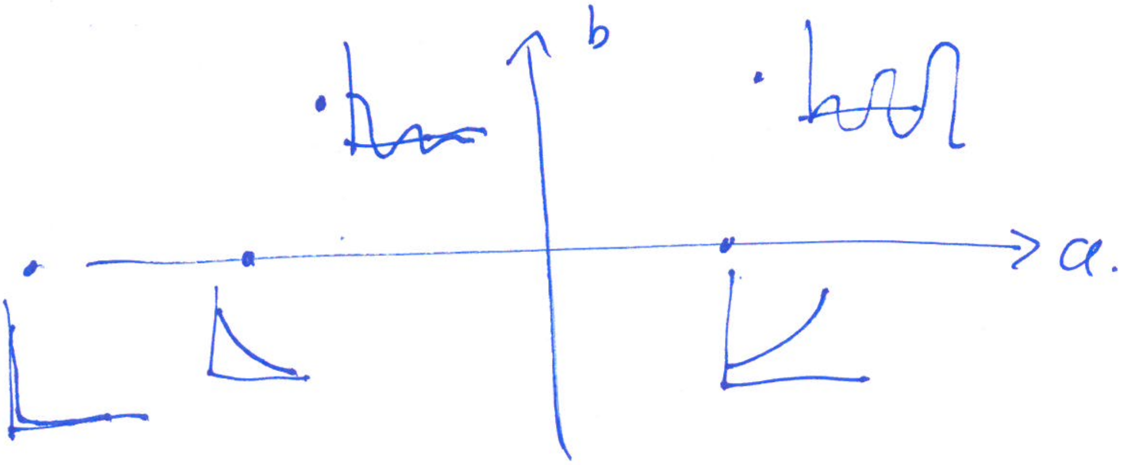
- $\Delta = 0$ $\quad r_1 = r_2 = r \in \mathbb{R}$

  $x_1 = e^{rt}$, $x_2 = t e^{rt}$

- $\Delta < 0$ $\quad r = a \pm bi$

  $x_1 = e^{rt}$, $x_2 = e^{\bar{r}t}$

$x = c_1 x_1 + c_2 x_2$



ODE 2 soln x_1, x_2

$$\Rightarrow \varphi = c_1 x_1 + c_2 x_2 = \text{soln.}$$

If x_1, x_2 are L.I. soln.

$\nexists\, k : x_1 = k x_2$

$$W(x_1, x_2) = \begin{bmatrix} x_1 & x_2 \\ \dot{x}_1 & \dot{x}_2 \end{bmatrix}$$

$|W| \neq 0 \Rightarrow x_1, x_2$ are L.I.

$$\frac{d^{(n)} x}{dt} = f(x^{(n-1)}, x^{(n-2)}, \dots, \dot{x}, x, t)$$

$\downarrow$ Linear

$$x^{(n)} + P_{n-1} \cdot x^{(n-1)} + P_{n-2} x^{(n-2)} + \dots + P_0 x = 0$$

$x_1, x_2, \dots x_k$ = soln

$\varphi = c_1 x_1 + \dots + c_k x_k$ is also a soln

$$= \sum_{i=1}^{k} c_i x_i$$

$$x^{(4)} + 10x^{(3)} + 35\ddot{x} + 50\dot{x} + 24 \cdot x = 0$$ 5

$x_1 = 3e^{-t} \rightarrow$ see Matlab

$$W(x_1, x_2, \dots) = \begin{bmatrix} x_1 & x_2 & \dots & x_n \\ \dot{x}_1 & \dot{x}_2 & \dots & \dot{x}_n \\ \vdots & & & \\ x_1^{(n-1)} & x_2^{(n-1)} & \dots & x_n^{(n-1)} \end{bmatrix}$$

eg $\rightarrow$ Matlab + 1.17

$$x^{(n)} + p_{n-1} x^{(n-1)} + \dots + p_0 \cdot x = 0$$

$$x = e^{rt}$$

$\downarrow$

C.E. $\quad r^n + p_{n-1} \cdot r^{n-1} + \dots + p_0 = 0$

$\downarrow$ roots

- $r_1 \neq r_2 \Rightarrow e^{r_1 t}, e^{r_2 t} \dots$
- $r = a \pm bi \Rightarrow e^{(a+bi)t}, e^{(a-bi)t}$
- $r_1 = r_2 = r \Rightarrow e^{rt}, te^{rt}$
- $r = r_1 = r_2 = r_3 \Rightarrow e^{rt}, te^{rt}, t^2 e^{rt} \dots$

$e^{rt}, e^{\bar{r}t}, te^{rt}, te^{\bar{r}t}$

$r \in \mathbb{C}$

$$x^{(4)} + 10\dddot{x} + 35\ddot{x} + 50\dot{x} + 24 \cdot x = 0$$

$x = e^{rt}$

$$r^4 + 10r^3 + 35r^2 + 50r + 24 = 0$$

$\Big\downarrow$ roots([1 10 35 50 24])

$r_1 = -1, \quad r_2 = -2, \quad r_3 = -3, \quad r_4 = -4.$

$x_1 = e^{-t}, \quad x_2 = e^{-2t}, \quad x_3 = e^{-3t}, \quad x_4 = e^{-4t}.$

$$x = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_4.$$

$$\dddot{x} + 3\ddot{x} + 3\dot{x} + x = 0$$

$\downarrow$

$$r^3 + 3r^2 + 3 \cdot r + 1 = 0$$

$\Big\downarrow$ roots

$r_1 = r_2 = r_3 = -1$

$$x = c_1 e^{-t} + c_2 \cdot te^{-t} + c_3 t^2 \cdot e^{-t}$$

$$x^{(4)} + 4\cdot\dddot{x} + 6\cdot\ddot{x} + 4\dot{x} + x = 0$$

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$r_1 = r_2 = r_3 = r_4 = -1$

$x_1 = e^{-t}$

$x_2 = t e^{-t}$

$x_3 = t^2 e^{-t}$

$x_4 = t^3 e^{-t}$

~~15th ODE~~ An ODE

$r_1 = -1$, $r_2 = -2$, $r_3 = -2$, $r_{4,5} = -3 \pm i$

$r_{6,7} = -4 \pm 2i$

$r_{8,9} = -4 \pm 2i$

$r_{10,11} = -5 \pm 3i$

$r_{12,13} = -5 \pm 3i$

$r_{14,15} = -5 \pm 3i$

$x_1 = e^{-t}$

$x_2 = e^{-2t}$

$x_3 = t e^{-2t}$

$x_4 = e^{(-3+i)\cdot t}$

$x_5 = e^{(-3-i)\cdot t}$

$x_6 = e^{(-4+2i)\cdot t}$

$x_7 = e^{(-4-2i)t}$

$x_8 = t\, e^{(-4+2i)t}$

$x_9 = t\, e^{(-4-2i)t}$

$x_{10} = e^{(-5+3i)t}$

$x_{11} = e^{(-5-3i)t}$

$x_{12} = t\, e^{(-5+3i)t}$

$x_{13} = t\, e^{(-5-3i)t}$

$x_{14} = t^2 e^{(-5+3i)t}$

$x_{15} = t^2 e^{(-5-3i)t}$

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n$$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n$$

$$\vdots$$

$$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n$$

$$\dot{X} = A \cdot X \,, \quad A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

~~n soln (x~~

a soln $X \in \mathbb{R}^{n \times 1}$

n solns $X^{(1)}, X^{(2)}, X^{(3)}$

$$X = \sum_{i=1}^{n} X^{(i)} \cdot c_i \quad \text{is also a soln}$$

if n solns are L.I. $\Rightarrow$

any soln $\varphi = \sum_{i=1}^{n} X^{(i)} \cdot c_i$

$$W = \begin{bmatrix} X^{(1)} & X^{(2)} & \dots & X^{(n)} \end{bmatrix}$$

$$\Phi(t) = [x^{(1)} \ x^{(2)} \ \dots \ x^{(n)}]$$

$$x(t) = \Phi(t) \cdot c \ , \quad c = [c_1, \ c_2, \dots \ c_n]$$

$$x(t) = \underbrace{\Phi(t) \cdot \Phi^{-1}(t_0)}_{S.T.M. \ \xrightarrow{LTI} \ e^{A(t-t_0)}} \cdot x_0$$

$$\dot{x} = A \cdot x$$

$x = e^{\lambda t} \cdot e \to$ vector $n \times 1$

$\lambda \cancel{e^{\lambda t}} \cdot e = A \cdot \cancel{e^{\lambda t}} \cdot e \Rightarrow$

$(A - \lambda I) \cdot e = 0$ $\quad$ $n+1$ unknowns

n eqs.

$\lambda =$ "Given" $\to$ $n \times n$

Linear

Homogeneous

Always $e=0$ is a soln

For non trivial solns $\Rightarrow$ $|A - \lambda I| = 0$

- $\lambda_1 \neq \lambda_2 \neq \lambda_3 \dots$ $\quad x = \sum_{i=1}^{n} e^{(i)} \cdot e^{\lambda_i t} c_i$

$\dot{x} = A\cdot x \qquad A \to \lambda_1 \to e^{(1)} \to e^{(1)} e^{\lambda_0 t}$

$\qquad\qquad\quad \hookrightarrow \lambda_1 \to e^{(1)} \to ?$

Assume $\quad x^{(2)} = t\cdot e^{(1)}\cdot e^{\lambda t}$

$$\left.\begin{array}{l} \dot{x} = e^{(1)} e^{\lambda t} + t e^{(1)} \lambda\cdot e^{\lambda t} \\ A\cdot x = A\cdot t e^{(1)} e^{\lambda t} \end{array}\right\} \Rightarrow$$

$$e^{(1)} e^{\lambda t} + t e^{(1)} \lambda e^{\lambda t} = A t e^{(1)} e^{\lambda t}$$

$$\underbrace{e^{(1)} e^{\lambda t}}_{0} + \underbrace{t e^{\lambda t}\cdot(\lambda e^{(1)} - A\cdot e^{(1)})}_{} = 0$$

$$e^{(1)} \cancel{e^{\lambda t}} = 0 \Rightarrow e^{(1)} = 0$$

$$\cancel{t e^{\lambda t}}\cdot(\lambda e^{(1)} - A e^{(1)}) = 0 \Rightarrow$$

$$\lambda e^{(1)} - A e^{(1)} = 0$$

$$(A - \lambda I) e^{(1)} = 0$$

If I try $\quad x^{(2)} = e^{\lambda t}\cdot(t e^{(1)} + e^{(2)})$

$$\dot{x} = \lambda e^{\lambda t}\cdot(t e^{(1)} + e^{(2)}) + e^{\lambda t}\cdot e^{(1)}$$

$$A\cdot x = A\cdot e^{\lambda t}(e^{(1)} t + e^{(2)}).$$

... Next week