Chapter 4

EEE8115

Robust and Adaptive Control Systems

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Nonlinear Dynamics

1. Linear Combination

Until now we have seen only linear (autonomous) systems which unfortunately can only fully describe only a small class of real systems. In general a nonlinear system is given by:

\[ \dot{x}(t) = f(x(t), u(t)) \]  \hspace{1cm} (1)

with \( x \in \mathbb{R}^{n \times 1}, u \in \mathbb{R}^{q \times 1}, f \in \mathbb{R}^{n \times 1} \)

The main property of a linear system (as we have seen) is that if you have 2 solutions \( x_1 \) and \( x_2 \) then also their linear combination is a solution.

For example, if we have: \( \dot{x}(t) = -3x \) and \( x_1 = e^{-3t} \) and \( x_2 = 7e^{-3t} \), then \( x_3 = 10x_1 - 2x_2 = 10e^{-3t} - 14e^{-3t} \) is also a solution as:

\[
\begin{align*}
\frac{d}{dt}(10e^{-3t} - 14e^{-3t}) &= -30e^{-3t} + 42e^{-3t} = 12e^{-3t} \\
-3(10e^{-3t} - 14e^{-3t}) &= -30e^{-3t} + 42e^{-3t} = 12e^{-3t}
\end{align*}
\]

However this is not the case in nonlinear systems: \( \dot{x}(t) = -x^2 \) and a solution:

\[ x = \frac{1}{t + C} \] as \( \dot{x} = -\frac{1}{(t + C)^2} \) and \( -x^2 = -\left(\frac{1}{t + C}\right)^2 = -\frac{1}{(t + C)^2} \) but

\[ x_A = 3\frac{1}{t + C} \] is not a solution:
\[ \dot{x} = -\frac{3}{(t + C)^2} \quad \text{and} \quad -x^2 = -\left( \frac{3}{t + C} \right)^2 = -\frac{9}{(t + C)^2} \neq -\frac{3}{(t + C)^2} = \dot{x} \]

2. Equilibrium points

Before we see the more differences between linear and nonlinear systems it is important to understand the concept of the equilibrium or rest or singular point which is defined as the point where

\[ \dot{x}(t) = 0 \]  \hspace{1cm} (2)

For first order linear systems we have that

\[ \dot{x}(t) = 0 \iff A(t)x_{EP}(t) + B(t)u(t) = 0 \iff x_{EP}(t) = -A^{-1}(t)B(t)u(t) \]  \hspace{1cm} (3)

For autonomous systems: \( x_{EP}(t) = 0 \) and for that reason in Chapter 3 all the state spaces were drawn with respect to the origin:
One of the biggest differences between linear and nonlinear systems is that a nonlinear system may have more than one EPs.

As an example see the following nonlinear system:

\[ x_1' = x_1 - x_2 \]
\[ x_2' = x_1^2 + x_2^2 - 2 \]

The EPs are:

\[
\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = 0 \iff \begin{bmatrix} x_1 - x_2 \\ x_1^2 + x_2^2 - 2 \end{bmatrix} = 0 \iff \begin{cases} x_1 = x_2 \\ x_1^2 + x_1^2 = 2 \end{cases} \iff \begin{cases} x_1 = x_2 \\ 2x_1^2 = 2 \end{cases}
\]

\[ \Rightarrow \begin{cases} (x_1, x_2) = (1,1) \\ (x_1, x_2) = (-1,-1) \end{cases} \]

So we have 2 Eps at (1,1) and at (-1,-1).

Another example is:

\[ x_1' = x_1^2 x_2 + 3x_1 x_2 - 10x_2 \]
\[ x_2' = x_1^2 x_2 - 4x_1 \]

which means:

\[
\begin{cases} x_1^2 x_2 + 3x_1 x_2 - 10x_2 = 0 \\ x_1^2 x_2 - 4x_1 = 0 \end{cases}
\]

From the 2\(^{nd}\) expression we have that \( x_1 = 0 \) or \( x_2 = 4 \)
From the 1st expression we have for \( x_1 = 0 \), \( x_2 = 0 \) i.e. \((x_1, x_2) = (0, 0)\) and for \( x_1x_2 = 4 \) we have that either \( x_2 = 0 \) which implies that \( x_1 = 0 \) (as before) or \( x_1^2 + 3x_1 - 10 = 0 \) which is a 2nd order polynomial and can easily be solved as:

\[
\Delta = 49 \Rightarrow x_{1,2} = -\frac{3 \pm 7}{2} \Rightarrow \begin{cases} x_1 = -5 \Rightarrow (x_1, x_2) = (-5, 4) \\ x_1 = 2 \Rightarrow (x_1, x_2) = (2, 4) \end{cases}
\]

So the system has the following EPs:

\((x_1, x_2) = (0, 0), (x_1, x_2) = (-5, 4), (x_1, x_2) = (2, 4)\)

But how about the stability of these EPs? Previously we had the state matrix and we could find its eigenvalues.

### 3. Linearisation

In order to determine if each of the EPs of a nonlinear system is stable or not we have to take a “local” picture of each EP. This local picture is called a linearization. I.e. we will describe each EP in a neighbourhood around it. The main tool for that is the Taylor series. Remember that the Taylor series around a point \( x_0 \) is defined as:

\[
f(x) = f(x_0) + \frac{\partial f(x)}{\partial x} \bigg|_{x=x_0} (x - x_0) + \frac{\partial^2 f(x)}{\partial x^2} \bigg|_{x=x_0} \frac{(x - x_0)^2}{2!} + \frac{\partial^3 f(x)}{\partial x^3} \bigg|_{x=x_0} \frac{(x - x_0)^3}{3!} + \ldots
\]

For example the TS of \( \sin(x) \) around \( x=0 \) is:

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} + O(x^{11})
\]
Now let’s try to approximate $\sin(x)$:

$$\begin{align*}
&\text{>> } x = -2\pi : 0.01 : 2\pi; \\
&\text{>> } \text{plot}(x, \sin(x)) \\
&\text{>> } \text{hold on} \\
&\text{>> } \text{plot}(x, x) \\
&\text{>> } \text{plot}(x, x - x^3/6) \\
&\text{>> } \text{plot}(x, x - x^3/6 + x^5/120) \\
&\text{>> } \text{plot}(x, x - x^3/6 + x^5/120 - x^7/5040) \\
&\text{>> } \text{plot}(x, x - x^3/6 + x^5/120 - x^7/5040 + x^9/362880)
\end{align*}$$

![Graphs showing the approximations of $\sin(x)$](image-url)
So by increasing the order of the Taylor Series we get a better approximation. But let’s get back to the first term $y = x$, obviously it is not a good approximation but if we focus close to the origin then:
So we see that close to the origin effectively the two traces coincide, i.e. our approximation is good “close” to the point of expansion. This is exactly how we will study a nonlinear system and the stability of their EPs. By approximating them locally using a Taylor series expansion and by keeping only the linear term:

Assume that we have a nonlinear system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3 \\
\vdots
\end{bmatrix}
= 
\begin{bmatrix}
f_1(x) \\
f_2(x) \\
f_3(x) \\
\vdots
\end{bmatrix}
= f(x)
\]

and a EP at \( x=x_{EP} \):

\[
\dot{x}(t) = f(x_{EP}) + \frac{\partial f(x(t))}{\partial x(t)} \bigg|_{x=x_{EP}} (x(t) - x_{EP})
\]

As \( f(x_{EP}) = 0 \):

\[
\dot{x}(t) = \frac{\partial f(x(t))}{\partial x(t)} \bigg|_{x=x_{EP}} (x(t) - x_{EP})
\]

Obviously we also have that: \( \frac{dx_{EP}}{dt} = 0 \) and hence:

\[
\frac{d(x(t) - x_{EP})}{dt} = \frac{\partial f(x(t))}{\partial x(t)} \bigg|_{x=x_{EP}} (x(t) - x_{EP})
\]

We can define now a new variable: \( \bar{x}(t) = x(t) - x_{EP} \) and we have that:
\[
\frac{d\bar{x}}{dt} = \left. \frac{\partial f(x(t))}{\partial x(t)} \right|_{x=x_{EP}} \bar{x}
\]

Also,
\[
\left. \frac{\partial f(x(t))}{\partial x(t)} \right|_{x=x_{EP}} = \begin{bmatrix}
\frac{\partial f_1(x(t))}{\partial x_1(t)} & \frac{\partial f_1(x(t))}{\partial x_2(t)} & \cdots & \frac{\partial f_i(x(t))}{\partial x_n(t)} \\
\frac{\partial f_2(x(t))}{\partial x_1(t)} & \frac{\partial f_2(x(t))}{\partial x_2(t)} & \cdots & \frac{\partial f_n(x(t))}{\partial x_n(t)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n(x(t))}{\partial x_1(t)} & \frac{\partial f_n(x(t))}{\partial x_2(t)} & \cdots & \frac{\partial f_n(x(t))}{\partial x_n(t)} 
\end{bmatrix}_{x=x_{EP}} = A \text{ (this matrix is called the Jacobian of } f) \text{ and hence:}
\]
\[
\frac{d\bar{x}}{dt} = A\bar{x}
\]

Which is a linear equation that we know how to solve and how to determine its stability.

Similarly if \( \dot{x}(t) = f(x,u) \) then:
\[
\frac{d\bar{x}}{dt} = \left. \frac{\partial f(x(t),u(t))}{\partial x(t)} \right|_{x=x_{EP},u=u_{EP}} \bar{x} + \left. \frac{\partial f(x(t),u(t))}{\partial u(t)} \right|_{x=x_{EP},u=u_{EP}} \bar{u}
\]

with \( \bar{u} = u - u_{EP} \)

which will give us:
\[
\frac{d\bar{x}}{dt} = A\bar{x} + B\bar{u}
\]

which is the form for a linear state space model.
Example: 1st order

\[ \dot{x} = -x^2 + 1 \Rightarrow x_{EP} = \pm 1 \]

\[ A = \frac{\partial f(x)}{\partial x} = \frac{\partial (-x^2 + 1)}{\partial x} = -2x \]

The “matrix” \( A \) evaluated at the 2 EPs: \( A = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \)

Hence, the 2 state space models are:

\[ \dot{x} = -2x \text{ for } x_{EP} = 1 \]

\[ \dot{x} = +2x \text{ for } x_{EP} = -1 \]

This implies that the eigenvalue of the 1st EP is -2 and hence this EP is stable, while the eigenvalue of the 2nd EP is +2 and hence that EP is unstable. SO we expect the following 1D state space:

For

\[ x_1' = x_1 - x_2 \]
\[ x_2' = x_1^2 + x_2^2 - 2 \]

We have seen that the EPs are:
\[
\begin{align*}
(x_1, x_2) &= (1, 1) \\
(x_1, x_2) &= (-1, -1)
\end{align*}
\]

So we have:

\[
x_1' = x_1 - x_2 \\
x_2' = x_1^2 + x_2^2 - 2
\]

\[
A = \begin{bmatrix}
\frac{\partial (x_1 - x_2)}{\partial x_1} & \frac{\partial (x_1 - x_2)}{\partial x_2} \\
\frac{\partial (x_1^2 + x_2^2 - 2)}{\partial x_1} & \frac{\partial (x_1^2 + x_2^2 - 2)}{\partial x_2}
\end{bmatrix} = \begin{bmatrix}
1 & -1 \\
2x_1 & 2x_2
\end{bmatrix}
\]

Now the Jacobian evaluated at the EPs:

\[
A_i = \begin{bmatrix}
1 & -1 \\
2 & 2
\end{bmatrix} \Rightarrow \text{eigs} = \frac{3 \pm \sqrt{17}i}{2} \text{ which implies that (1,1) is an unstable focus.}
\]

\[
A_2 = \begin{bmatrix}
1 & -1 \\
-2 & -2
\end{bmatrix} \Rightarrow \text{eigs} = \frac{-1 \pm \sqrt{17}}{2} \text{ which implies that (1,1) is a saddle.}
\]

Now in order to understand what happens we have to draw the eigenvectors of the saddle (of the focus do not give us anything as we have seen). The saddle has 2 eigenvalues at 1.56 and -2.56 with the corresponding eigenvectors being:

\[
e_1 = \begin{bmatrix}
1 \\
-0.56
\end{bmatrix} \text{ and } e_2 = \begin{bmatrix}
0.28 \\
1
\end{bmatrix}
\]

Hence, the state space is:
More examples in class.

4. Limit cycles

Apart from the previous difference of having multiple equilibria in nonlinear systems, another feature of these systems is that it is possible to have a persistence periodic motion, a limit cycle. Assume the Van der Pol system \( \ddot{x} + (\alpha^2 - 1) \dot{x} + x = 0 \). The time response is:
5. Bifurcations

Another difference between linear and nonlinear systems is the phenomenon of bifurcation, which (very briefly) is the creation (or destruction) of equilibria and fixed points. Assume\(^1\) the system \(\dot{x} = r + x^2\) for various values of \(r\). The equilibria are:

\[
\dot{x} = r + x^2 \iff x^2 = -r
\]

So if \(r > 0\) there are no equilibrium points

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\(^1\) Example taken from https://en.wikipedia.org/wiki/Saddle-node_bifurcation
While if $r<0$:

$$\dot{x} = r + x^2 \iff x^2 = \pm \sqrt{-r}$$

i.e. 2 equilibria which for various values of $r$ are:

```matlab
hold on
for r=-10:0.1:0
    plot(r,sqrt(-r), 'k. ')
    plot(r,-sqrt(-r), 'k. ')
end
```

The linearised system closed to these 2 points is:

$$\begin{cases}
    \dot{x} = 2\sqrt{-r}\tilde{x} \\
    \dot{\tilde{x}} = -2\sqrt{-r}\tilde{x}
\end{cases} \Rightarrow \tilde{x} = \tilde{x}(t_0)e^{2\sqrt{-r}}$$

The response for $r=-1$: 

![Graph showing the response for r=-1]
6. Stability

In linear systems we called a system stable if the orbit was not diverging to infinity (exponentially or with oscillations). In nonlinear systems, we have multiple equilibria and we can check the stability “locally” around each equilibrium point. But we need a more general definition that will allow us to focus on “global” stability. Assume that we have a general nonlinear ODE and a specific solution \( x(t) \). Obviously as this will depend on the initial conditions we can denote it as \( \phi(x(t_0),t) \). Now we want to test the stability of the general orbit \( \phi \). To do that, we add a perturbation (say at \( t=t_0 \)) and then we observe the perturbed orbit. If it stays “close” to the nominal then it is called stable, Fig. 1. Effectively this means that for each required \( \Delta \) we can find a suitable \( \delta \).

![Fig. 1](image)

As we are focused on equilibria we have the following definition:

An equilibrium point \( x_E \) is stable if for each \( R>0 \) there exists an \( r>0 \) such that

\[
\|x_E - x(t_0)\| < r, \text{ then } \|x_E - x(t)\| < R
\]

An importance difference with respect to linear systems is that it is possible to have the case where the perturbed trajectory does not diverge to infinity,
nor converges to the equilibrium point (asymptotic stability) but it remains close to it at a radius $R$:

![Diagram showing the behavior of a system near an equilibrium point](image)

Fig. 2

The proper definition for the asymptotic stability is: an equilibrium point is stable and $\lim_{t \to \infty} \| x_E - x(t) \| = 0$

Lyapunov Stability Theorem

Let function $V$ of the states $x$, such as:

- $V(x) > 0$
- $\dot{V}(x) < 0$
- $\lim_{\|x\| \to \infty} V(x) = \infty$

then the equilibrium point is globally asymptotic stable.