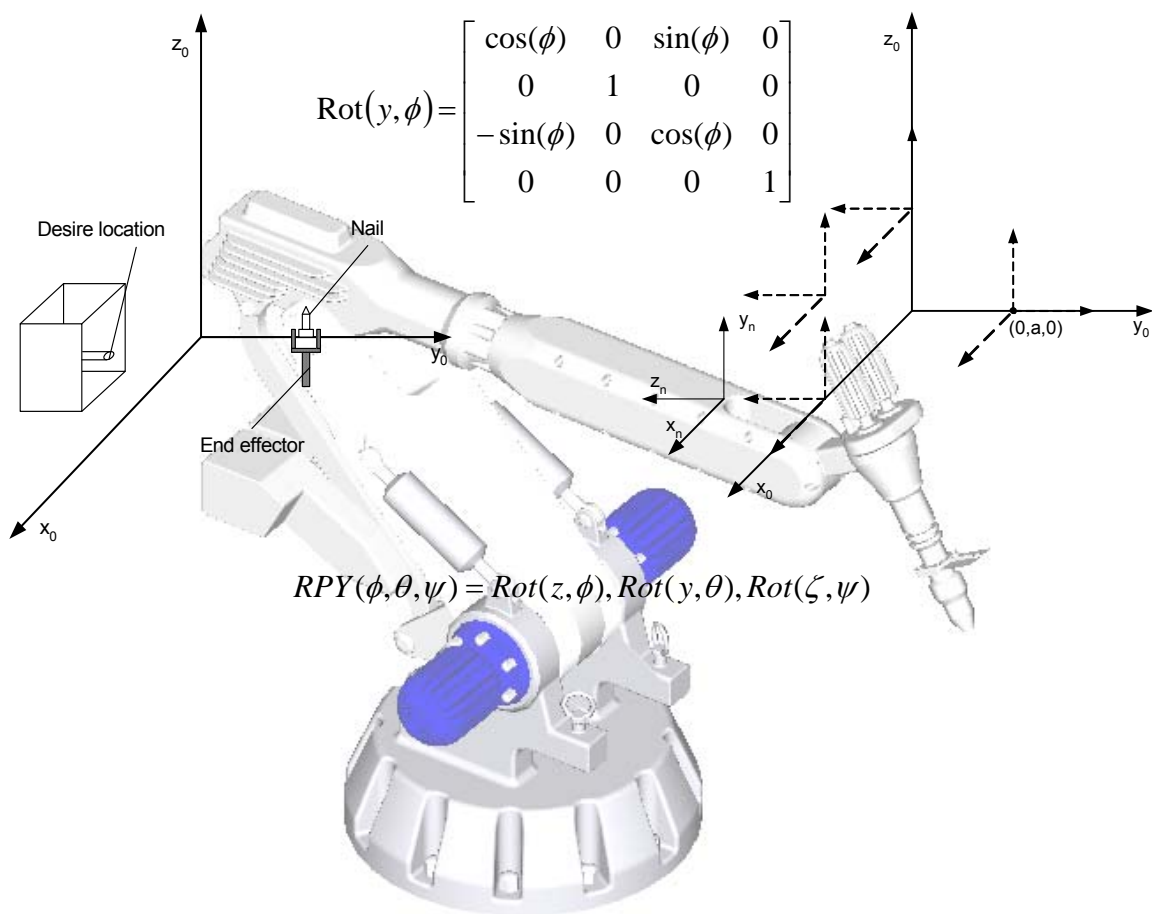




EEE 3008 & EEE 8005 - Industrial Automation, Robotics (and Artificial Intelligence)

Section I: Introduction & Mathematical Background



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Contents

Section 1 Introduction & Mathematical Background.....	3
1.1 Course Contents.....	3
1.2 Recommended Text Books	4
1.3 Introduction to Robotics	5
1.4 Robot Components.....	6
1.4.1 Basic Manipulator Arms.....	6
1.4.2 Wrists	10
1.5 Mathematical background.....	12
1.5.1 Geometrical Vectors	12
1.5.2 Algebraic Vectors	16
1.6 Robotic Motion.....	17
1.6.1 Coordinate frames and objects.....	17
1.6.2 Three and four dimensional transformation matrices.....	21
1.6.3 Example 1	23
1.6.4 Example 2	24
1.7 Orientation matrices	25
1.7.1 Intuitive approach	25
1.7.2 Mathematical Approach.....	27
1.7.3 Tutorial Work	28
1.7.4 Summary	29

Section 1 Introduction & Mathematical Background

1.1 Course Contents

The area of Industrial Automation (IA) extends from simple On-Off control and relays to complicated Programmable Logical Controllers (PLCs) and Artificial Intelligent (AI) controlled robotic wrists and arms. It is therefore impossible in a 12-week module to fully examine and thoroughly understand it. This module, and consequently these notes, will only try to pinpoint the main applications of IA and to examine their principles. The main areas that will be covered are:

1. Robotics
 - a. Robot Anatomy & Geometry
 - b. Object Location
 - c. Forward Kinematics (Position – Velocity – Static Forces?)
2. Programmable Logical Controllers
 - a. On Off Logic
 - b. Siemens S7 PLC
3. Artificial Intelligence (mainly Fuzzy Logic (FL))
 - a. Fuzzy Logic Elements (EEE 8005)
 - b. Fuzzy Logic Algorithms (EEE 8005)
 - c. Fuzzy Logic Control (EEE 8005)
 - d. Artificial Neural Networks (EEE 8005)
 - e. Genetic Algorithms (EEE 8005)

1.2 Recommended Text Books

1. Introduction to robotics (Essential reading)

Author: J. Craig
Notes: Addison-Wesley
ISBN: 0201095289

2. Robot Manipulators - Mathematics, Programming and Control

Author: R.P. Paul
Notes: MIT-Press
ISBN: 026216082x

3. Introduction to Robotics

Author: P.J. McKerrow
Notes: Addison-Wesley
ISBN: 0201182408

4. Modelling and Control of Robot Manipulators

Author: L. Sciavicco and B. Siciliano
Notes: Springer Verlag
ISBN: 1852332212

5. Robot Modelling IFS

Author: P.G. Ranky and C.Y. Ho
Notes: Springer Verlag
ISBN: 0903608723

1.3 Introduction to Robotics

Robot is a Slavonic word for worker and it was popularised in 1921, in a play called Rossum's Universal Robots. The machines in this play revolted and killed their masters, the humans. After that, many films involved robots and their influence on society. There are various formal definitions for the term "Robot" but the most appropriate is the one that is given from the "Robotic Institute of America":

"A programmable multifunctional manipulator designed to move material, parts, or specialised devices through variable programmed motions for the performance of a variety of tasks." (Schlussel, 1985)

Based on this definition the traces of the first robot can be found deep into the 18th century. These were simple mechanical puppets that mimicked human and animal actions and their main goal was to demonstrate the art of mechanical design. This is the so-called "Generation 0".

The second level ("Generation 1") started in the 60s with the help of programmable controllers. They had a Central Processing Unit (CPU) and they were programmed to do a sequence of moves, taught by a human. The program was stored in a paper tape and the overall utility of the robot was limited. Also they were rather expensive devices (just the computer could cost up to £50,000) and were mainly an academic curiosity even if some first industrial applications were found.

"Generation 2" date from the 1980s and their main characteristic is the use of a microprocessor. The impact that the microprocessor had was to fit in a small chip a whole programmable computer, which as well as the size reduction made the overall mechanism relative cheap. The robots of this generation supported high level languages like VAL and BAPS. The use of coordinate transformations, feedback control and PID, vision, networking and sensors made their application attractive to industry and a small revolution began. Soon every large industry began to use robots to replace humans in some applications.

"Generation 3" evolved over the last 15 years with main characteristics involving the use of human-like sensors and AI. The robots can now behave like human beings and their applications are numerous. For example, from Automatic Guided Vehicles (AGVs) to cleaning robots, and from medical applications to nano-robots. The applications of this generation of robots are rapidly expanding and it has been said that in the near future robots will be a piece of everyday life.

1.4 Robot Components

The main parts of a robot are (not all are a compulsory part of a robot):

1. Vehicles
2. Manipulator arms
3. Wrists
4. End effectors
5. Actuators
6. Transmission elements
7. Sensors

Only “Manipulator arms” and “Wrists” will be studied here due to time limits.

1.4.1 Basic Manipulator Arms

A manipulator arm (often also termed a robot arm) consists of **joints and links**. The joints connect different parts (the links) of the robot and they can slide or rotate. Therefore they are called sliding (also called prismatic) and revolute joints. For example the revolute and the sliding joints of Figs. 1.1, 1.2 are joining two different mechanical parts (Link 1 and Link 2) of a robot:

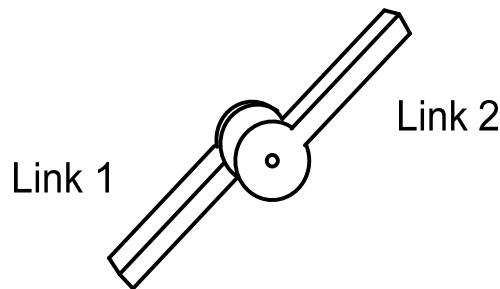


Figure 1.1 Revolute joint

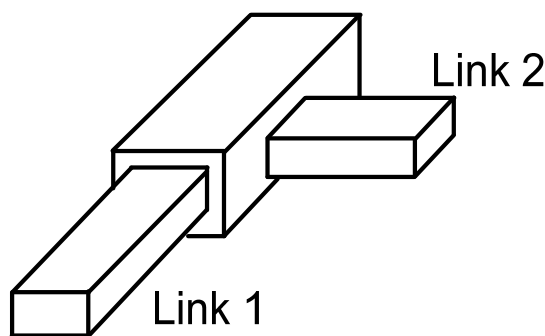


Figure 1.2 Prismatic joint

The links are usually mechanically solid objects that connect two joints. For example the elbow and the shoulder are joints that are linked by the upper arm. The joints give the name to the robot. Hence if a robot has three prismatic joints, is called PPP. Later the joints and the links will be studied in a more detail. A robot arm usually has 3 moving parts (links) and one

immobile, the base. The first part mimics a human torso and is usually the part that is directly connected to the robot base by the waist joint (first joint). The second link, the upper arm, is connected to the torso by the shoulder joint (second joint). The third part is the forearm, which is connected to the upper arm link with elbow joint (third joint).

A robot that has three revolute joints is called “articulated arm” or RRR robot and mimics the human arm. The articulated or revolute arm shown in Figure 1.3 and Figure 1.4 has a rather complicated workspace.

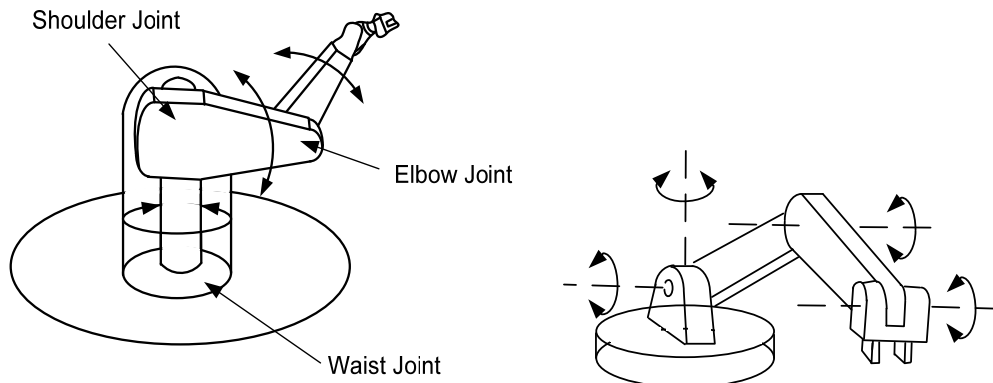


Figure 1.3 Revolute or RRR robot, schematic diagrams

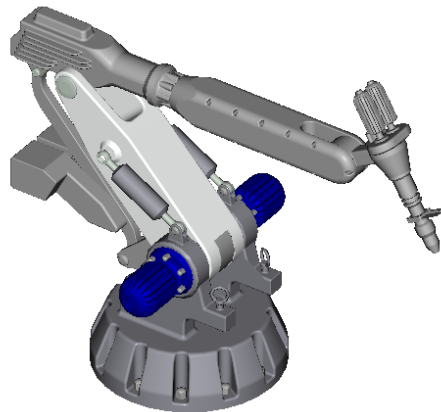


Figure 1.4 Revolute or RRR robot

There are other simple robot geometries that replace one or more revolute joints with prismatic ones. The simplest industrial manipulator is the so called Cartesian or PPP robot. It is extremely simple and it consists of 3 prismatic joints as shown in Figure 1.5 and Figure 1.6. It can reach any position in its rectangular workspace by Cartesian motions of the links.

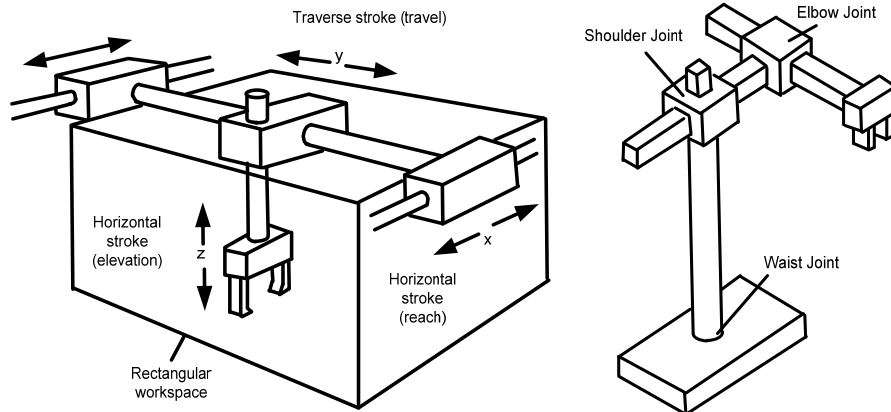


Figure 1.5 Cartesian or PPP robot, schematic diagrams

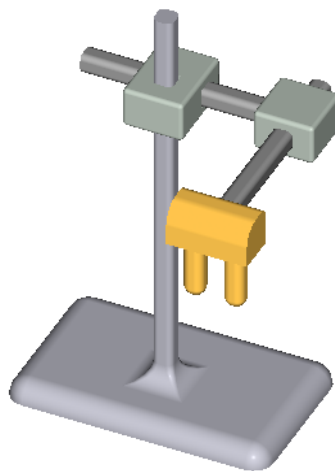


Figure 1.6 Cartesian or PPP robot

By replacing a joint of a Cartesian arm by a revolute joint, a cylindrical geometry arm can be formed. This robot is called Cylindrical or PRP and it has a cylindrical workspace. See Figure 1.7 and Figure 1.8.

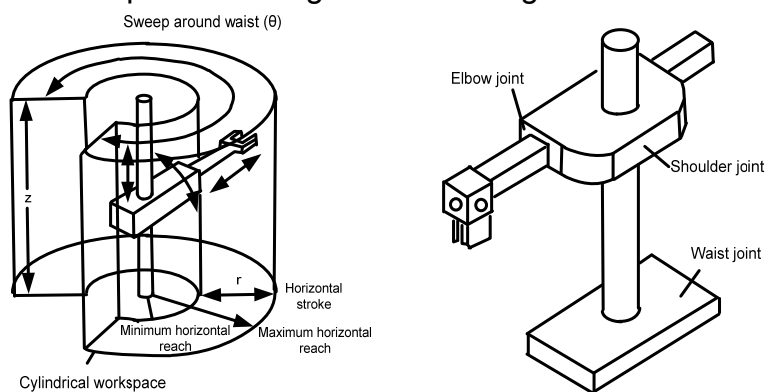


Figure 1.7 Cylindrical or PRP robot, schematics

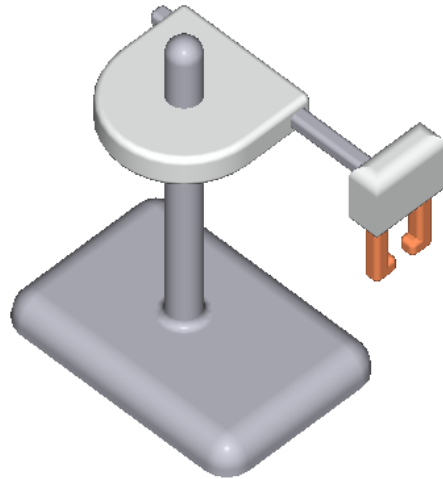


Figure 1.8 Cylindrical or PRP robot

If a revolute joint replaces another prismatic joint then a polar geometry is formed. This robot is called Polar or RRP and it has a spherical workspace. See Figure 1.9 and Figure 1.10.

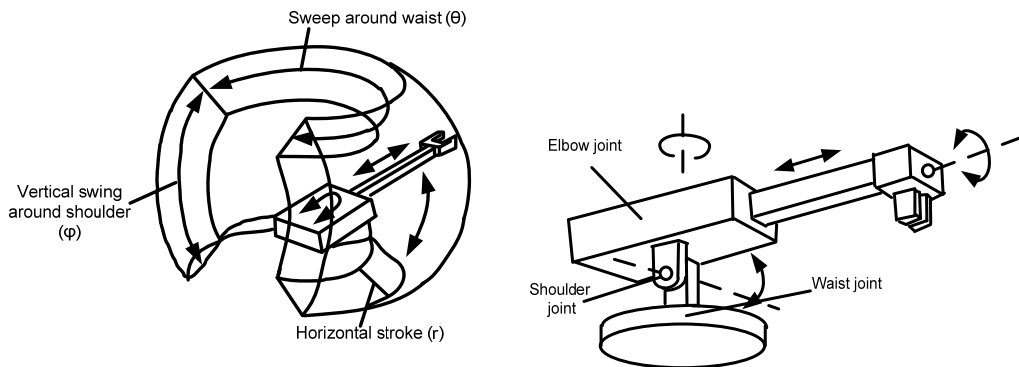


Figure 1.9 Spherical or RRP Root, schematic diagrams

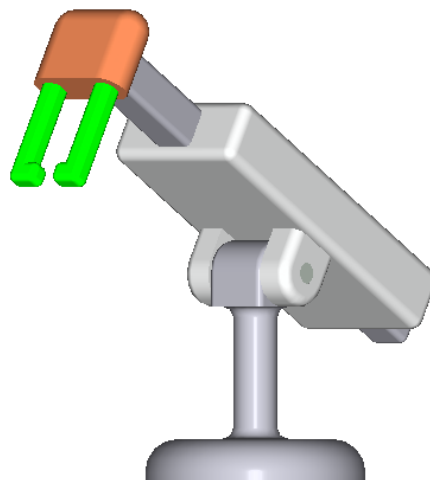


Figure 1.10 Spherical or RRP robot

Hence there are various robot types with different geometries, which can be used in different applications. Each model has its own advantages and disadvantages, which are shown in Table 1.1.

Robot		Coordinates	Pro and Cons
PPP Cartesian	or	x y z	Linear Motion in 3D Simple kinematics model Rigid structure Easy to visualise Can use inexpensive pneumatic drives for pick and place operation Requires large volume to operate in Workspace is smaller than robot volume Unable to reach areas under objects Guiding surfaces of prismatic joints must be covered to prevent ingress of dust.
PRP Cylindrical	or	z θ r	Simple kinematic model Easy to visualise Good access into cavities and machine openings Very powerful when hydraulic drives are used Restricted workspace Prismatic guides difficult to seal from dust and liquids Back of robot can overlap work volume
RRP Spherical	or	θ ϕ r	Covers a large area from a central support Can bend down to pick objects off the floor Complex kinematic model Difficult to visualise
RRR Articulated	or	θ_1 θ_2 θ_3	Maximum flexibility Covers large area of work relative to volume of robots Revolute joints are easy to seal Suits electrical motors Can reach over and under objects Complex kinematics Difficult to visualise Control of linear motions is difficult Structure not very rigid at full reach

Table 1.1 Advantages and disadvantages of the robots, PPP, PRP, RRP and RRR

1.4.2 Wrists

A robot arm has three main joints (waist, shoulder and elbow) and three links (torso, upper arm and forearm). The human body also has a wrist and then end effectors (fingers). Likewise a robot arm has a wrist to orient the end effectors. Usually the wrist has three degrees of freedom; see Figure 1.11, and the joints are revolute and not prismatic. The rotations of the wrists are commonly described as roll (z axis rotation), pitch (y axis rotation) and yaw (x

axis rotation). Simplified wrists may have two degrees of freedom (roll & pitch).

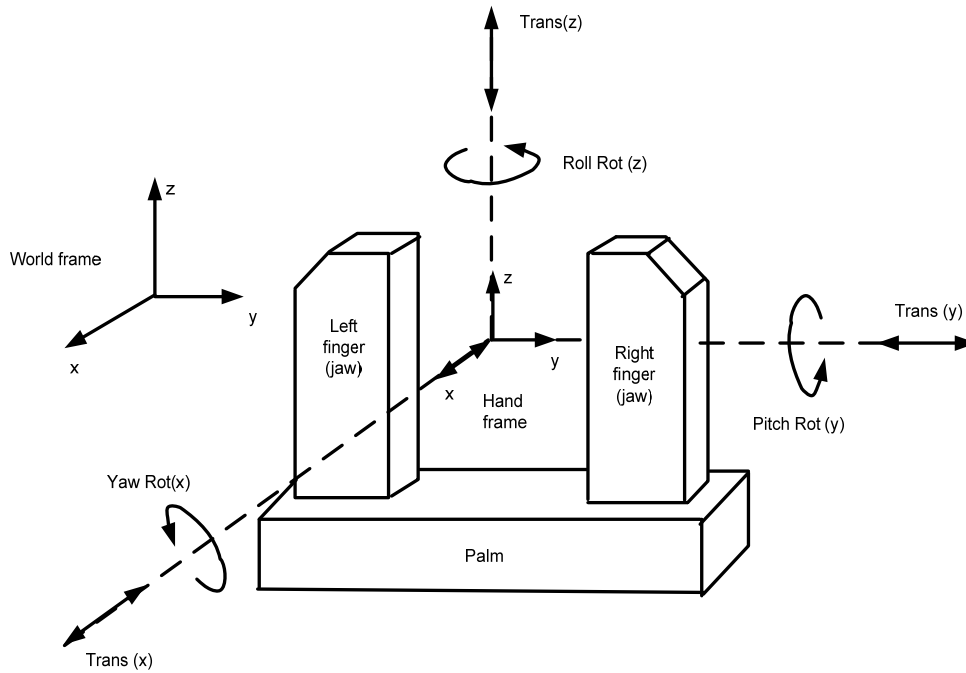


Figure 1.11 Six motions that a wrist must make to place the gripper in any point and in any orientation

1.5 Mathematical background

1.5.1 Geometrical Vectors

To define the position of the end effector with respect to the base or to any other reference object the concept of the “**vector**” has to be analysed and fully defined. The next paragraph briefly describes these ideas. This material is not directly part of the syllabus, but is essential for proper understanding.

Many students will have some knowledge of vectors. The need for the vector notation rises because a simple number (or scalar) cannot define some quantities. For example the mass of an object can be fully defined by saying that it is 5 kg. But if it is stated that its velocity is 10 m/s then this information is not adequate. The direction of the velocity must also be mentioned. These quantities are termed vectorial and arrows can represent them. For a vector in a plane (2 dimensional, or 2-D) the length of the arrow can represent the magnitude of the quantity and the angle represents its direction. For example, see Figure 1.12 below:

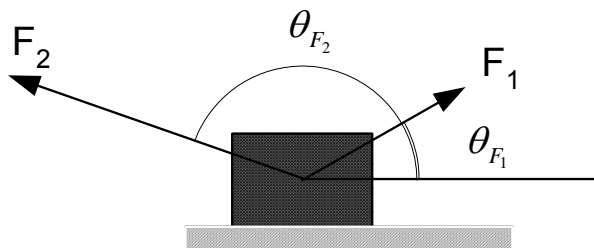


Figure 1.12 Force F2 has a different direction and magnitude to F1

Therefore a vector is an arrow with a specific direction and length. One symbolic notation for vectors is to use small case, bold letters like **a**, **b**, **c**.... In some books other symbols may be used like \vec{a} or \bar{a} or even as \underline{a} . Two vectors are called equal if they have the same length and direction but they do not have to be on the same axis (Figure 1.13). If two vectors have the same length but opposite directions then they are called opposite vectors and obviously $\mathbf{b} = -\mathbf{a}$.

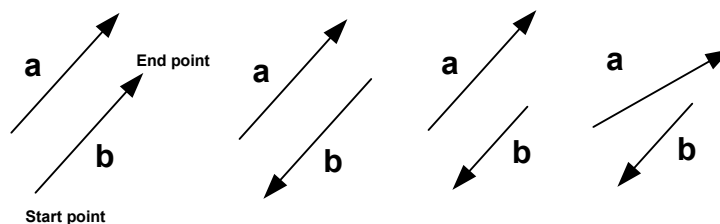


Figure 1.13 Vectors

Vector addition can be done with the use of the parallelogram rule. For example if the addition of two vectors (**a** & **b**) is required, see Figure 1.14, then create a vector equal to the vector **b** but with a starting point at the end point of **a**, then simply connect the starting point of **a** with the end point of the replaced vector **b**.

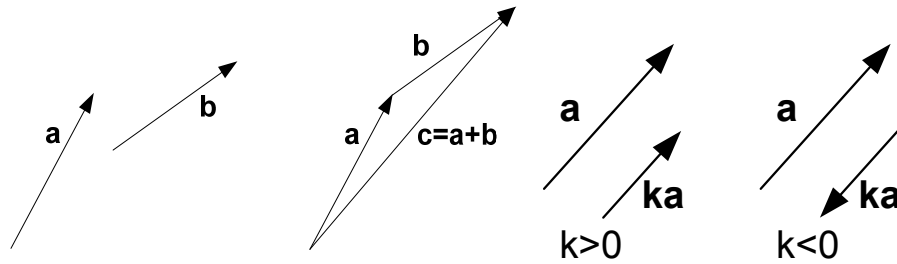


Figure 1.14 Vector addition and Scalar vector multiplication

The multiplication of a vector with a scalar number will only change its length and not its direction (note if the scalar number is negative then the direction will be reversed). Now the vector difference can be defined as: $a-b=a+(-1)b$.

To simplify the manipulation of vectors **coordinate frames** can be used. The most common is the so-called Cartesian reference frame (Figure 1.15):

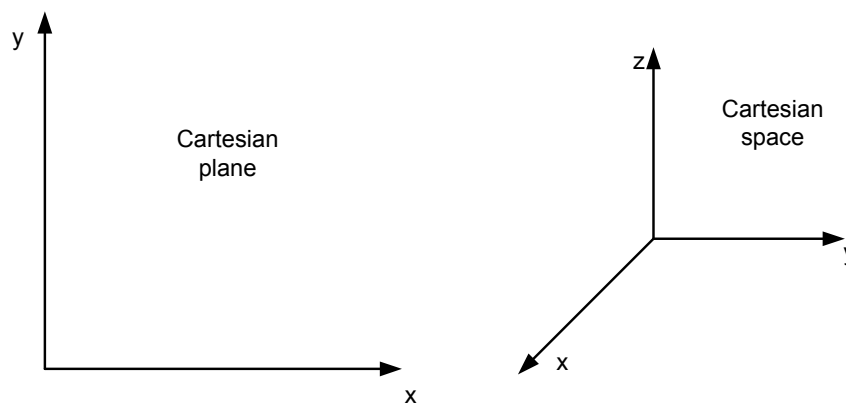


Figure 1.15 Cartesian plane and space

By using the previous definition for equal vectors, any vector in the Cartesian reference frame can be considered to have as a starting point the origin i.e. $0(0,0)$ for a 2-D plane, or $0(0,0,0)$, for a 3-D space.

In a 3D space three elementary vectors can be defined that will coincide with the axes and they will have a length of 1. The elementary vector for the x-axis is **i**, for the y axis is **j** and for the z axis is **k**.

Then every point on an axis will easily be defined by a multiplication of the appropriate vector and a scalar number. The same concept can be used for any vector that does not lie on an axis. The end point now, of a vector will have specific coordinates, and these will fully describe the vector.

For example, consider Figure 1.16:

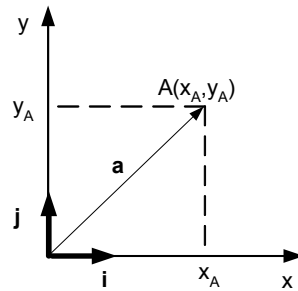


Figure 1.16 A vector in a Cartesian plane

The point A, which is the end point of the vector **a**, fully defines the vector **a**. Hence it can be said that the coordinates of the vector **a** are (x_A, y_A) and is usually written as:

$$\mathbf{a} = x_A \bar{i} + y_A \bar{j} = \begin{bmatrix} x_A \\ y_A \end{bmatrix}$$

The **vector addition** can be simplified to:

$$\mathbf{c} = (x_A + x_B) \bar{i} + (y_A + y_B) \bar{j} = \begin{bmatrix} x_A + x_B \\ y_A + y_B \end{bmatrix}$$

The **multiplication of a vector** will be:

$$k\mathbf{a} = k x_A \bar{i} + k y_A \bar{j} = \begin{bmatrix} kx_A \\ ky_A \end{bmatrix}$$

The **length of a vector** can be found by its **Euclidian norm**:

$$\|\bar{a}\|_2 = \sqrt{x_A^2 + y_A^2}$$

Of course other norms can also be used to define the length of the vector:

$$\|\bar{a}\|_p = (x_A^p + y_A^p)^{1/p}$$

Note: For simplification, the notation of the 2-norm is:

$$\|\bar{a}\|_2 = |\bar{a}|$$

There are two vector multiplications, the **inner or scalar or dot product** and the **outer or vector or cross product**. As it is apparent from the names the outcome of the first product is a scalar number while the outcome of the second is a vector.

Consider Figure 1.17. The inner product is defined as:

$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\theta) = x_A x_B + y_A y_B$, where θ is the angle between the two vectors:

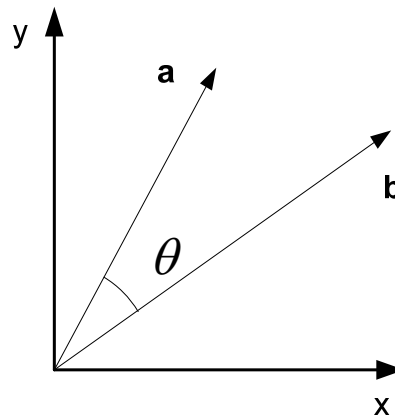


Figure 1.17 Vector angle

Now the angle between two vectors can be defined as:

$$\cos(\theta) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{x_A x_B + y_A y_B}{\sqrt{x_A^2 + y_A^2} \sqrt{x_B^2 + y_B^2}}$$

Orthogonal or perpendicular vectors are those vectors whose inner products are zero.

With the use of the dot or inner product we can define the projection of a vector \vec{x} onto a vector \vec{y} as:

$$proj_{\vec{y}} \vec{x} = \frac{\langle \vec{x}, \vec{y} \rangle}{|\vec{y}|} \vec{y}$$

The cross product of two vectors (in a three dimensional space) can be defined as:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix}$$

For example the cross product of the x and y axis is the z axis:

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} + \vec{j} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \vec{i}0 + \vec{j}0 + \vec{k}1 = \vec{k}$$

Finally two vectors (or lines) in a 2-dimensional space can have either one common point or to be parallel. In a 3-dimensional space there is a third possibility and then these lines are called skew lines. The mutual perpendicular to these lines is called common normal and effectively is the distance between the two skew lines.

1.5.2 Algebraic Vectors

Until now a vector had a direction and a length and was easily visualised in a 2 or 3 dimensional space. But what can be said about vectors that are defined in a 4 or even higher dimensional space? Therefore a more general definition is needed. A vector is an order n-tuple:

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ or } x = [x_1 \quad x_2 \quad \cdots \quad x_n] .$$

Assume now a set of n-dimensional vectors $V = \{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_m\}$ that:

- $\bar{v}_i + \bar{v}_j \in V, \forall i, j \in [1, m]$
- $\lambda \bar{v}_i \in V, \forall i \in [1, m], \lambda \in (-\infty, +\infty)$

There are the zero and the identity vectors:

- a. $\bar{0}\bar{v}_i = \bar{0} \in V, \bar{0} + \bar{v}_i = \bar{v}_i \in V \forall i \in [1, m]$
- b. $1\bar{v}_i = \bar{v}_i \in V \in V \forall i \in [1, m]$

Then V is called vector space.

Assume a subset of vectors in V such that the scalar equation $a_1\bar{x}_1 + a_2\bar{x}_2 + \dots + a_n\bar{x}_n = 0$ is satisfied only for $a_1 = a_2 = \dots = a_n = 0$. Then these vectors are called linear independent and they define a basis for the V, $V_B = \{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n\}$. All other vectors in V can be expressed as a linear combination of the basis vectors VB: $\bar{y} = b_1\bar{x}_1 + b_2\bar{x}_2 + \dots + b_n\bar{x}_n, \bar{y} \in V, b_j \neq 0 j \in [1, n]$.

The inner product is defined as:

$$\langle \bar{x}, \bar{y} \rangle = x_1y_1 + x_2y_2 + \dots + x_ny_n = \sum_{k=1}^n x_k y_k$$

And the length of a vector can be defined as:

$$\|\bar{x}\|_p = (x_1^p + x_2^p + \dots + x_n^p)^{1/p} \xrightarrow{p=2} \|\bar{x}\| = \sqrt{(x_1^2 + x_2^2 + \dots + x_n^2)} \Rightarrow \|\bar{x}\|^2 = x_1^2 + x_2^2 + \dots + x_n^2 = \langle \bar{x}, \bar{x} \rangle$$

As it has been said the vector space is a set of vectors that is closed under the operations of addition and multiplication. This concept can be extended to sets that consist of functions and matrices (function and matrix spaces). For example the set that contains all the functions that are defined in $[0, 1]$...

1.6 Robotic Motion

1.6.1 Coordinate frames and objects

The motion of a robot can be described in four different frames of reference. These are:

- Motor Frame
- Joint Frame
- Tool Frame
- World Frame

In this course, only the world and the joint reference frames will be studied.

The world reference frame is attached to a point in the overall environment while in the joint frame can be attached to any joint and its coordinates are the link angles (if the joint is revolute) and the distance between the links (if the joint is prismatic). In robotics, the chosen frame forms a right-handed set of vectors and the positive rotation is shown in Figure 1.18, i.e. from x to y, and from x to y to z, respectively:

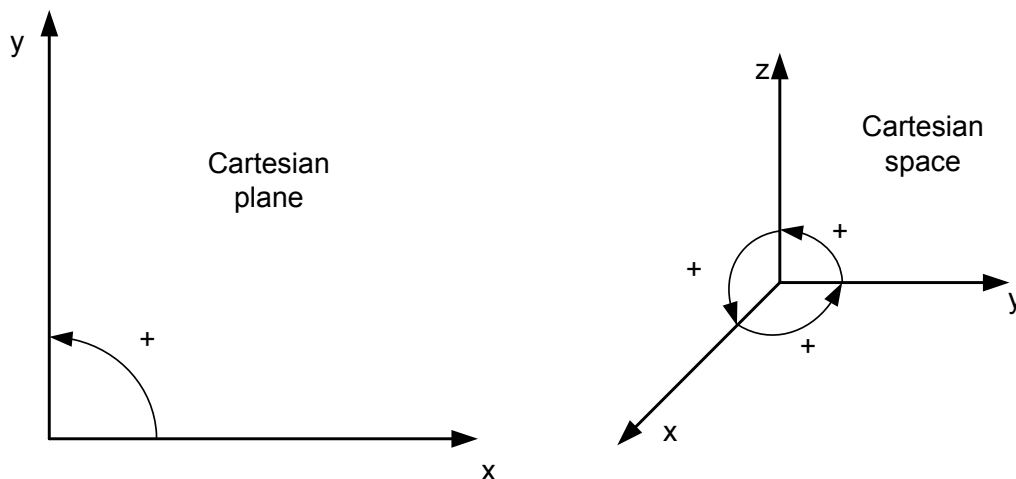


Figure 1.18 Positive angles in a Cartesian plane/space

The location of a point in a 2D frame is described by its two coordinates or by a vector that connects the origin with this point. For example in Figure 1.19 the point A is described either from the pair (x_A, y_A) or from the vector \mathbf{a} . If this point is moved to B $((x_B, y_B), \mathbf{b})$ then the translation is the difference of the two vectors, i.e. $\mathbf{b}-\mathbf{a}$. By using the coordinates of these vectors it can be found that the translation is:

$$\begin{bmatrix} x_B - x_A \\ y_B - y_A \end{bmatrix}$$

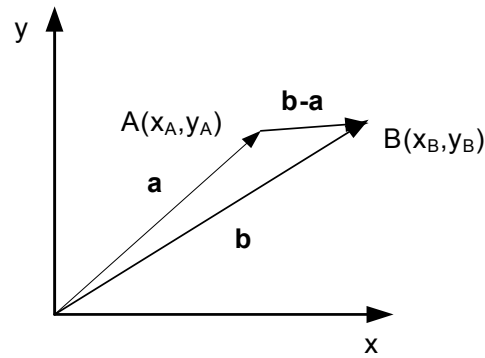


Figure 1.19 Point translation

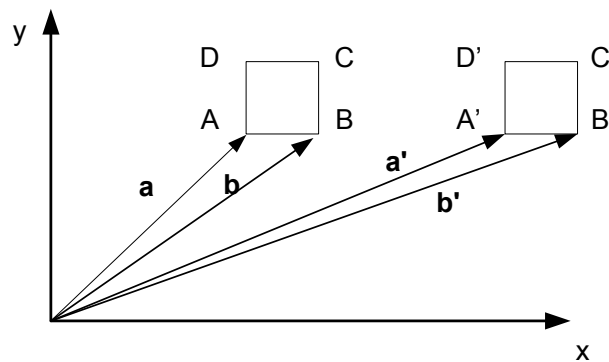


Figure 1.20 2D object translation (only two vectors are shown)

Assume now that there is a square object in a plane (Figure 1.20). This object is fully described by the coordinates of its corners, i.e. $A(x_A, y_A)$, $B(x_B, y_B)$, $C(x_C, y_C)$, $D(x_D, y_D)$ and hence four vectors, i.e. a , b , c , d . If this object is moved to another location then four new vectors will be assigned to describe the new location. Four differences, like the one above, will give the translation.

If the plane now is transformed to a 3D space then eight vectors will be needed for the initial description and eight vectors for the new location. As it can be easily understood a more complicated object would need a large number of vectors and at every translation or rotation would require numerous new vectors.

The solution to this problem is to create a new reference frame on the object, for example on point A, and then another four vectors will characterise the object (Figure 1.21) (or 8 in a 3D space – see Figure 1.22). Then the new location of the object will require only one new vector which characterises the translation of the origin:

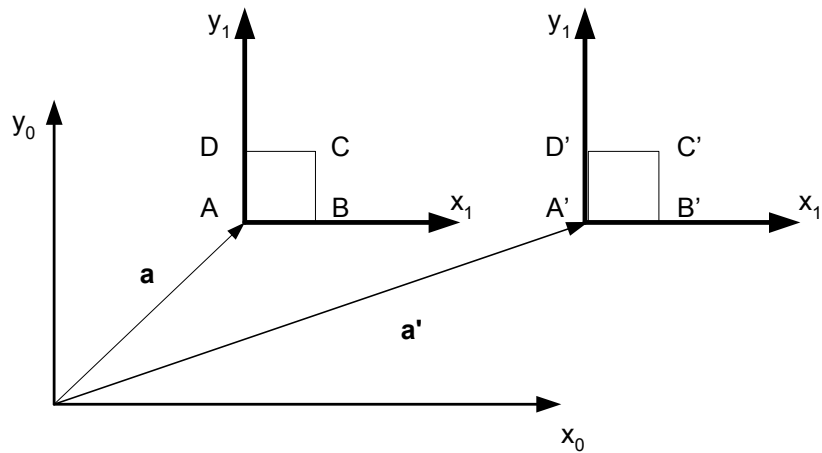


Figure 1.21 Object 2D translation

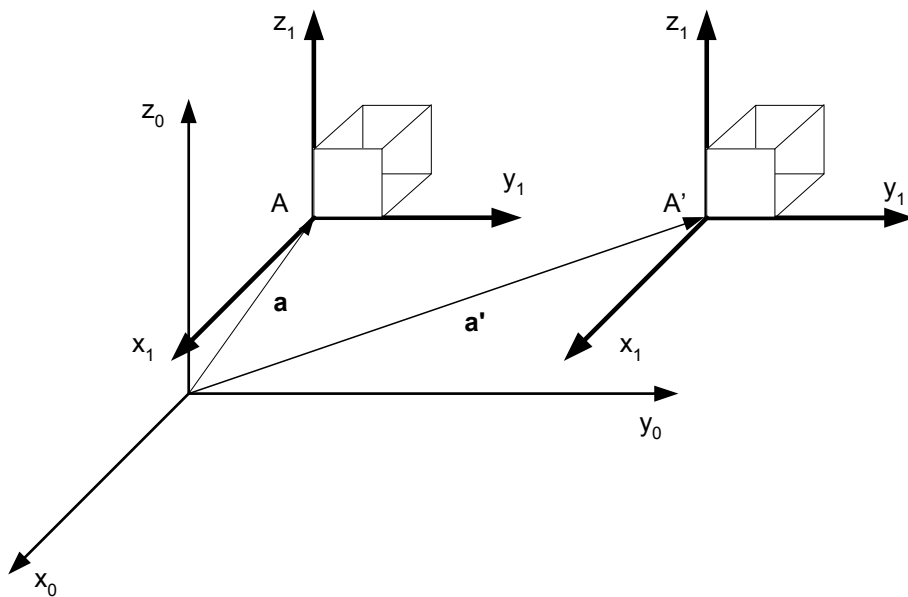


Figure 1.22 Object 3D translation

Generally speaking, an object like the one of Figure 1.22 has 6 degrees of freedom. It can **translate** along the three axis (x,y,z) and it can **rotate** around the three axis: (Figure 1.23)

Note: Figure 1.23 shows the positive rotation. Using the right hand it can be found by putting the thumb in the positive direction of the axis. Then the direction of the fingers will give the positive angle.

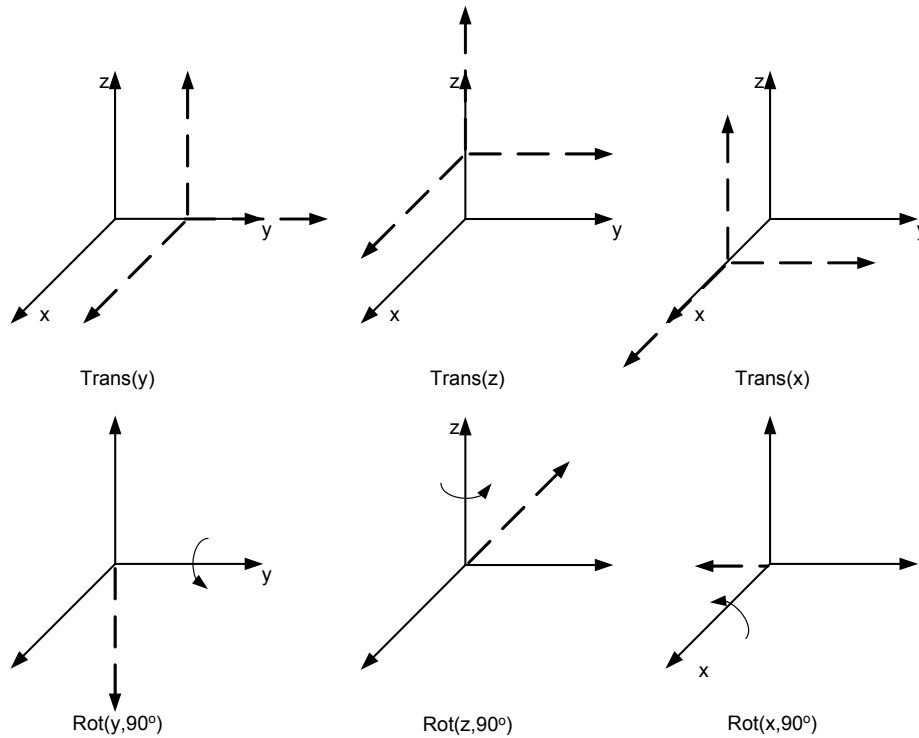


Figure 1.23 Six degrees of freedom of a 3D object

A more complicated movement of the reference frame can be separated to successive translations and revolutions. For example the displacement that is shown in Figure 1.24 can be considered to be a translation according to y-axis, then a translation according to x-axis, then according to z-axis and finally a 90° rotation around z-axis.

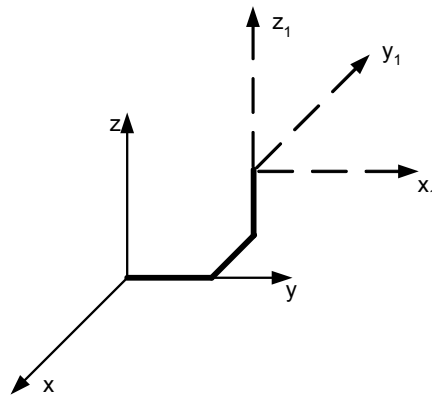


Figure 1.24 Frame displacement

A “stricter” notation regarding the previous reference frames is to name the reference frame with a capital letter in curly brackets (eg. {A}) and to use the symbol $\hat{}$ to denote the unit vectors of the frame (eg. $\hat{Z}_A, \hat{Y}_A, \hat{X}_A$, i.e. the unit vectors along the directions of Z, Y, X of the frame A). A point P that is

defined with respect to {A} is written as: ${}^A P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$. This is shown in Figure

1.25

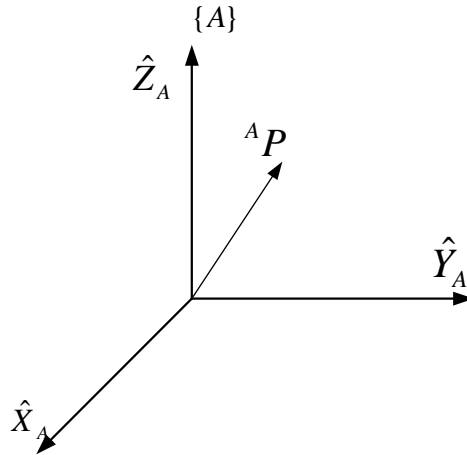


Figure 1.25 Vector notation

Obviously P can be defined with respect to any reference frame, lets say {B}, ${}^B P$: (Figure 1.26)

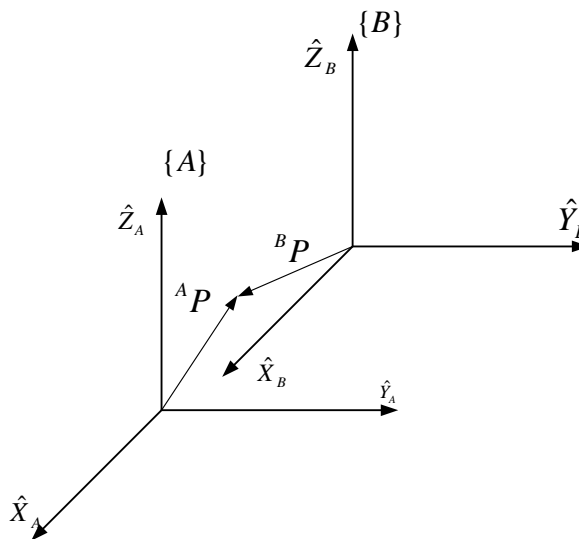


Figure 1.26 Two frames defining the same point

From the above it can be seen that the notation used to describe a vector with respect to {A} is a left superscript. The same can be applied for the unit vectors of other reference frames. For example, assume that we have 2 frames, {A} and {B}, then we can refer or define the unit vectors of {B} with respect to {A} as ${}^A \hat{X}_B, {}^A \hat{Y}_B, {}^A \hat{Z}_B$.

1.6.2 Three and four dimensional transformation matrices

Describing the motion of an object is easier if a frame is attached to a point on the object and then every translation or rotation of the object is transformed to the position and orientation of the new frame. The basic problem is to find the

new position and orientation of the new frame and hence the position and orientation of any vector that was attached to the moving frame. This is the concept of the **forward transformation**, i.e. the translations and rotations are given and the question is to find the new position and orientation of the object. This is the main topic of the next section.

To simplify the mathematics, initially the object, and hence the moving or new frame, is placed on the origin of the original frame. **A four by four matrix that consists of four vectors can describe a new coordinate frame with respect to the old. The first three vectors represent the direction of the new axes and the fourth vector represents the translation of the origin.** Hence a new frame is fully described by knowing the original frame and this transformation matrix.

To visualise this assume a simple 2D frame, a point P (x_1, y_1) on it with an associated vector, p (see Figure 1.27). If this point is multiplied by a 2 by 2 general transformation matrix, the result is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ cx_1 + dy_1 \end{bmatrix}, \text{ which is another vector (say } \mathbf{q}\text{).}$$

The values of a, b, c, d characterise the transformation of the point. For example, if $a = 2, b = c = 0, d = 1$, then the new vector is $\begin{bmatrix} 2x_1 \\ y_1 \end{bmatrix}$ and is clearly a translation parallel to the x-axis:

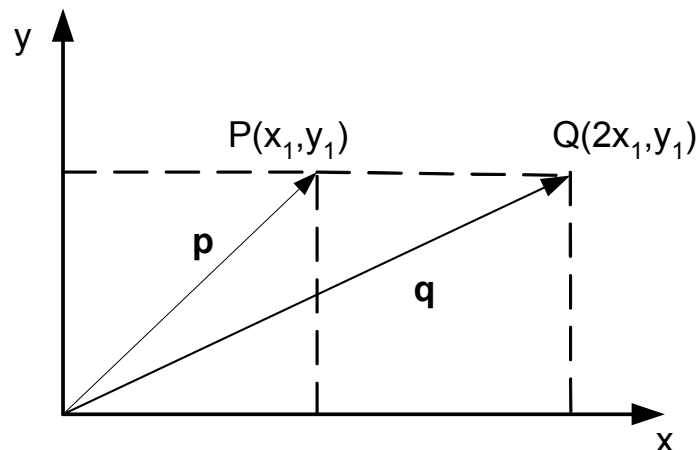


Figure 1.27 x-axis translation

This translation could have been the translation of a reference frame assuming that the initial origin of this frame was on the point P. In the same way a 3 by 3 matrix can describe translations in a 3D space:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

Another way to illustrate the matrix representation of a translation is to say that the general transformation matrix is now:

$$\begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \end{bmatrix} \quad \text{Or simply,} \quad \begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \end{bmatrix}$$

where the vector $\mathbf{r} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}$ is the difference of the two vectors, \mathbf{p} and \mathbf{q} and

hence the translation of the vector \mathbf{p} to the vector \mathbf{q} .

As it will be shown later the transformation matrix $\begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \end{bmatrix}$ will need to

be inverted and therefore it has to be square:

$$\begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The elements 4,1 to 4,3 correspond to the projection of the object and is used mainly in the area of graphic and image processing. The last element is for the scaling of the object and for the purposes of this module will always considered to be 1. Therefore the vector dimension will not be three but four.

For example, the vector $\bar{\mathbf{p}} = [10 \ 0.1 \ -3]^T$ will be written as $\bar{\mathbf{p}} = [10 \ 0.1 \ -3 \ 1]^T$

1.6.3 Example 1

Assume a vector \mathbf{p} that was translated from $\bar{\mathbf{p}}_{old} = [1 \ 1 \ 1 \ 1]^T$ to $\bar{\mathbf{p}}_{new} = [2 \ 1 \ 1 \ 1]^T$. It is obvious that the vector was translated along the x axis for 1 unit, while the translations along the z and y axes are zero. By using the previous notation:

$$\begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} r_x + 1 \\ r_y + 1 \\ r_z + 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \begin{cases} r_x = 1 \\ r_y = 0 \\ r_z = 0 \end{cases}$$

1.6.4 Example 2

Assume an object like the one of Figure 1.28, the task now is to translate it along the y-axis for “a” units and to find the coordinates after the translation by using a transformation matrix. As it has been said a new frame will be attached on this frame and for simplification reasons the world (or original) frame is considered to coincide with the new one:

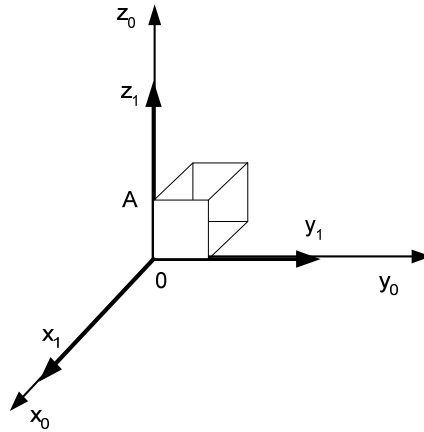


Figure 1.28 New and old frames

The object is then translated for “a” units along the y-axis:

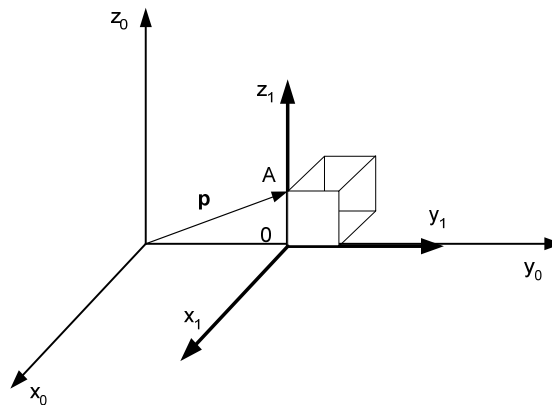


Figure 1.29 New and old frames after the translation

The new location of point A after the translation, with respect to the original frame, is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 \\ a \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

By comparing this outcome with Figure 1.29, it can be seen that the above calculation was correct. The same can be done for the other 7 points of the object.

1.7 Orientation matrices

Until now, we have seen that the transformation matrix can describe the translation (even though this can easily be done by using just one vector). The next logical step is to find an expression (obviously of the transformation matrix) that will define the relative orientation of two frames.

1.7.1 Intuitive approach

Let's start with a more "intuitive" method but mathematically less "sound". A 90° rotation around the y-axis:

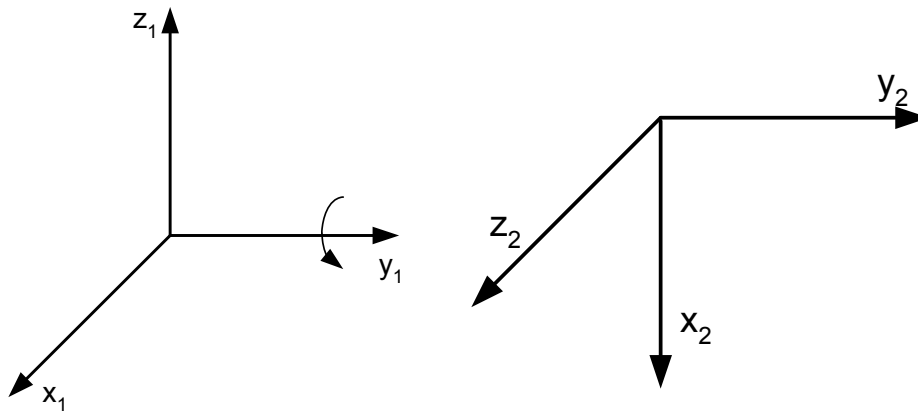


Figure 1.30 Result of a y-axis 90° rotation

Assume now that there is a vector attached on the z axis: $\mathbf{a} = [0 \ 0 \ 1 \ 1]^T$, after the rotation this vector will have the orientation as shown in Figure 1.31.

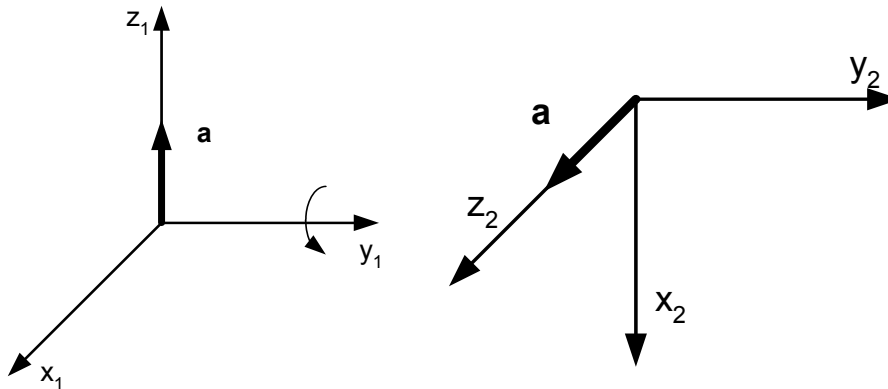


Figure 1.31 A vector on the z-axis before and after a y-axis 90° rotation

Now let's name this new vector as \mathbf{a}' . This vector with respect to the original frame is:

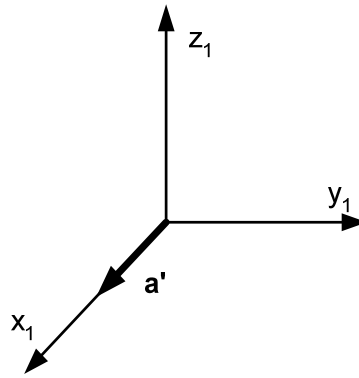


Figure 1.32 The new vector with respect to the old frame.

It can be seen that the new coordinates (x_2, y_2, z_2) of \mathbf{a} are associated with the original as $(z_1, y_1, -x_1)$, or:

$$\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ y_1 \\ -x_1 \end{bmatrix}$$

Therefore the transformation matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ can simply be: $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$,

or $\begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix}$ where $\phi=90^\circ$.

Hence a simple matrix can describe the rotations or translations of a general frame. Again the above transformation matrix is extended to a four by four matrix as:

$$\begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

With the same manipulation as before the rotation around the x-axis is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

And the rotation around the z-axis:

$$\begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

1.7.2 Mathematical Approach

Now that we know the result that we want to establish lets try to propose a more rigorous/proper mathematical proof. Assume two reference frames {A} and {B}:

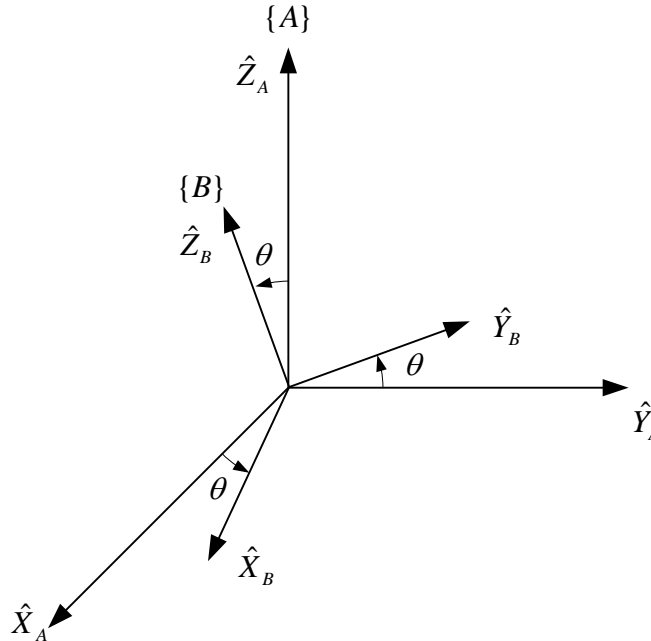


Figure 1.33 The new vector with respect to the old frame.

The rotation of {B} with respect to {A} will be described by using the projections of the unit vectors ${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, ${}^A\hat{Z}_B$ onto \hat{X}_A , \hat{Y}_A , \hat{Z}_A . It can easily be seen that ${}^A\hat{X}_B$ is nothing else than the projection of the vector \hat{X}_B onto {A} and therefore:

$${}^A\hat{X}_B = \text{proj}_{\{A\}} \hat{X}_B = \begin{bmatrix} \text{proj}_{\hat{X}_A} \hat{X}_B \\ \text{proj}_{\hat{Y}_A} \hat{X}_B \\ \text{proj}_{\hat{Z}_A} \hat{X}_B \end{bmatrix} = \begin{bmatrix} \frac{\hat{X}_B \hat{X}_A}{|\hat{X}_A|^2} \\ \frac{\hat{X}_B \hat{Y}_A}{|\hat{Y}_A|^2} \\ \frac{\hat{X}_B \hat{Z}_A}{|\hat{Z}_A|^2} \end{bmatrix} = \begin{bmatrix} \hat{X}_B \hat{X}_A \\ \hat{X}_B \hat{Y}_A \\ \hat{X}_B \hat{Z}_A \end{bmatrix}$$

Similarly we can define all 3 vectors (${}^A\hat{X}_B$, ${}^A\hat{Y}_B$, ${}^A\hat{Z}_B$) which play a very important role in the theory of robotics and therefore they are placed together in such way that they create a 3 by 3 matrix:

$${}^A R_B = \begin{bmatrix} {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \end{bmatrix} = \begin{bmatrix} \hat{X}_B \hat{X}_A & \hat{Y}_B \hat{X}_A & \hat{Z}_B \hat{X}_A \\ \hat{X}_B \hat{Y}_A & \hat{Y}_B \hat{Y}_A & \hat{Z}_B \hat{Y}_A \\ \hat{X}_B \hat{Z}_A & \hat{Y}_B \hat{Z}_A & \hat{Z}_B \hat{Z}_A \end{bmatrix}$$

(Notice that this matrix has orthonormal columns (length one and perpendicular to each other)). This matrix describes the relative rotation of {B} with respect to {A} and hence the name ${}^A R_B$.

1.7.3 Tutorial Work

Prove that: $\text{Rot}(y, \phi) = {}^A R_B = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Prove that: $\text{Rot}(x, \phi) = {}^A R_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Prove that: $\text{Rot}(z, \phi) = {}^A R_B = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Comment: Based on the previous analysis it can be said that while the position of a point requires a vector to fully define the orientation of an object we need a 3 by 3 matrix or simply 9 elements.

Since the inner product is given as $\bar{a} \cdot \bar{b} = |a||b|\cos(\theta)$ and the lengths of the unit vectors are 1 then it is clear that the previous rotation matrix will consist of cosines. This method of describing the rotation is called direction cosines. Later we will encounter other methods.

With the same process the unit vectors of {A} can be expressed in {B}:

$${}^B R_A = \begin{bmatrix} {}^B \hat{X}_A & {}^B \hat{Y}_A & {}^B \hat{Z}_A \end{bmatrix} = \begin{bmatrix} \hat{X}_B \hat{X}_A & \hat{X}_B \hat{Y}_A & \hat{X}_B \hat{Z}_A \\ \hat{Y}_B \hat{X}_A & \hat{Y}_B \hat{Y}_A & \hat{Y}_B \hat{Z}_A \\ \hat{Z}_B \hat{X}_A & \hat{Z}_B \hat{Y}_A & \hat{Z}_B \hat{Z}_A \end{bmatrix} = {}^A R_B^T$$

We can easily prove that ${}^A R_B^T {}^B R_A = I_{3 \times 3}$ and therefore ${}^A R_B^{-1} = {}^A R_B = {}^A R_B^T$.

1.7.4 Summary

To summarise

- The transformation matrix that associates the original reference frame with the new when there is a translation by a vector $r(r_x, r_y, r_z)$ is:

$$\text{Trans}(r_x, r_y, r_z) = \begin{bmatrix} 1 & 0 & 0 & r_x \\ 0 & 1 & 0 & r_y \\ 0 & 0 & 1 & r_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix that associates the original reference frame (eg. {A}) with the new (eg. {B}) when there is rotation around y-axis is :

$$\text{Rot}(y, \phi) = {}^A R_B = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix that associates the original reference frame with the new when there is rotation around x-axis:

$$\text{Rot}(x, \phi) = {}^A R_B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\phi) & -\sin(\phi) & 0 \\ 0 & \sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The transformation matrix that associates the original reference frame with the new when there is rotation around z-axis:

$$\text{Rot}(z, \phi) = {}^A R_B = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$