Contents

Section 4 Velocity Kinematics

4.1 Introduction

The following sections analyse the motion (**velocity and static force components**) of a robotic arm. The method that we use is based on the book of John Craig "Introduction to robotics".

Important note: This is by far the most mathematical component of the module.

4.2 Manipulator Motion

4.2.1 Linear Velocity

Initially we have to define the idea of the velocity vector. Assume that at t=t_0 we have a vector Q defined at a general frame {B} (${}^{B}Q(t_0)$). Then after time the point Q is at ${}^{B}Q(t_0+\Delta t)$ and therefore we can define the difference $\Delta\left({}^{B}Q(t_0)\right) = {}^{B}Q(t_0+\Delta t) - {}^{B}Q(t_0)$. That vector can be scaled to create the rate of change of ${}^{B}Q$: $\frac{\Delta\left({}^{B}Q(t_0)\right)}{\Delta t} = \frac{{}^{B}Q(t_0+\Delta t) - {}^{B}Q(t_0)}{\Delta t}$ which is the average velocity of ${}^{B}Q$ from $t=t_0$ to $t=t_0+\Delta t$. Finally we can impose the restriction that $\Delta t \to 0$ and hence to define the instantaneous velocity:

$$\lim_{\Delta t\to 0}\left(\frac{\Delta\left({}^{B}Q(t_0)\right)}{\Delta t}\right) = \lim_{\Delta t\to 0}\left(\frac{{}^{B}Q(t_0+\Delta t) - {}^{B}Q(t_0)}{\Delta t}\right) = \frac{{}^{B}Q(t_0+dt) - {}^{B}Q(t_0)}{dt} = \frac{d\,{}^{B}Q(t_0)}{dt} = {}^{B}V_Q$$

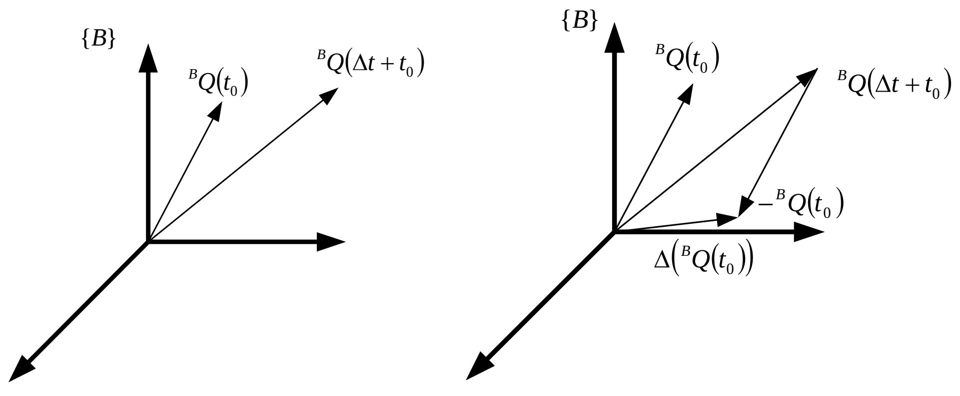

Figure 4.1 Definition of velocity vector

As always/expected the length of that vector describes the magnitude of the velocity and the angle of the vector describes the direction of the velocity.
We can also describe the velocity vector with respect to any general frame: ${}^{A}\left({}^{B}V_Q\right) = {}^{A}R_B\,{}^{B}V_Q$.

Notice that in general ${}^{A}V_{Q} \neq {}^{A}\left({}^{B}V_{Q}\right)$.

It is also possible to move {B} with respect to {A} so if ${}^{A}P$ is the vector defined on {A} that describes the origin of {B}: ${}^{A}V_{P} = \frac{d\,{}^{A}P}{dt}$

Now we can combine these concepts to define the velocity vector of V_Q when Q is changing with respect to {B} and at the same time {B} also changes with respect to {A}. The two origins are associated with the vector ${}^{A}P$:

$$ {}^{A}V_{Q} = \frac{d\,{}^{A}P}{dt} + {}^{A}\left({}^{B}V_{Q}\right) = {}^{A}V_{P} + {}^{A}R_{B}\,{}^{B}V_{Q} \ . $$

4.2.2 Rotational Velocity

The rotational velocity defines the velocity of an object (and not of a point as the linear velocity does) and hence we have to define an angular velocity vector ${}^{A}\Omega_{B}$:

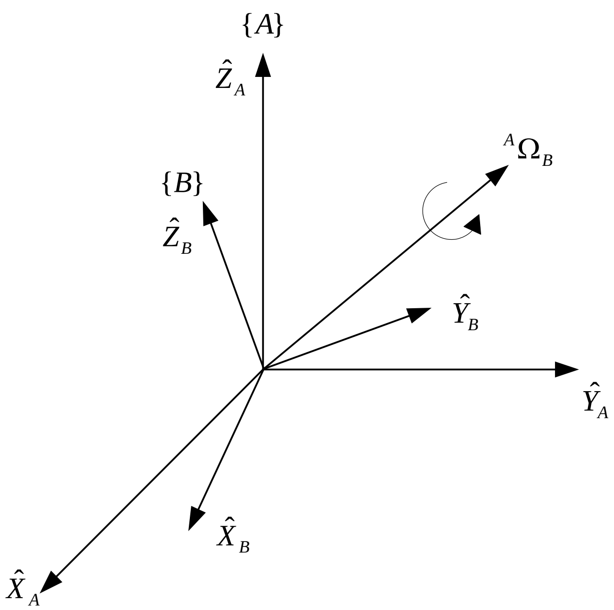

Figure 4.2 Definition of angular velocity vector

Intuitively we can say that this angular velocity vector must have a relationship with the rotational matrices (or with their derivatives). We will now study this relationship:

Remember that we have said that the rotational matrices are orthonormal, hence: $RR^T = I$ and $R^{-1} = R^T$

$$ \Rightarrow \frac{d\left(RR^T\right)}{dt} = 0 \Leftrightarrow \dot{R}\,R^T + R\,\dot{R}^T = 0 \Leftrightarrow \dot{R}\,R^T + \left(\dot{R}\,R^T\right)^T = 0 $$

We define the new matrix $S = \dot{R} R^T$ and hence $S + S^T = 0$. This implies that S is a skew-symmetric matrix (effectively this means that $S = \begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}$).

Assume now that we have a vector ^BQ that remains unchanged with respect to {B}: $^AQ = {}^A({}^BQ) = {}^AR_B {}^BQ$. If {B} is rotating with respect to {A} then: $d\frac{^AQ}{dt} = \frac{d({}^AR_B {}^BQ)}{dt} = d\frac{^AR_B}{dt} {}^BQ$ since ^BQ is not a function of time.

$^AV_Q = {}^A\dot{R}_B {}^BQ$. Now let's express Q with respect to {A}:

$^AV_Q = {}^A\dot{R}_B ({}^AR_B)^{-1} {}^AQ \Leftrightarrow {}^AV_Q = {}^AS_B {}^AQ$. Now we can state that S is the angular velocity matrix (not vector), $S = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}$, where $^A\Omega_B = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix}$ is the angular velocity vector.

A very interesting property of these matrices/vectors is that $SQ = {}^A\Omega_B \times Q$ and hence $^AV_P = {}^A\Omega_B \times {}^AQ$.

If we combine now the linear velocity of a vector with respect to a frame {B}, the linear velocity of {B} with respect to {A} and finally the angular velocity of {B} with respect to {A} we have:

$^AV_Q = {}^AV_P + {}^AR_B {}^BV_Q + {}^A\Omega_B \times {}^AQ$ Or $^AV_Q = {}^AV_P + {}^AR_B {}^BV_Q + {}^A\Omega_B \times {}^AR_B {}^BQ$.

4.2.3 Propagation from link to link

It has been said that a manipulator is a chain of links. Link (n) can have a velocity relative to links (n-1) and (n+1). The velocity now of link (n) will be equal of the velocity (linear and/or angular) of link (n-1) plus any extra components of joint {n}. At this point it must be mentioned that linear velocity of link (n) means linear velocity of the frame {n} while the angular velocity of link (n) is the angular velocity of link (n). Hence the linear velocity is associated with a frame, while an angular velocity is associated with a body (link).

<u>Nomenclature</u>:

$^n\overline{u}_n$ is the linear velocity of link (n) with respect to frame {n}

$^n\Omega_n$ is the angular velocity of link (n) with respect to frame {n}

Without any further proof we will state the results:

If the joint n is revolute:

$$\left.\begin{aligned} {}^{n}\overline{u}_n &= {}^{n}R_{n-1}\left({}^{n-1}\overline{u}_{n-1} + {}^{n-1}\Omega_{n-1} \times {}^{n-1}Q_n\right) \\ {}^{n}\Omega_n &= {}^{n}R_{n-1}\,{}^{n-1}\Omega_{n-1} + {}^{n}\begin{bmatrix} 0 & 0 & \dot{\theta}_n \end{bmatrix}^T \end{aligned}\right\}$$

And if the joint n is prismatic:

$$\left.\begin{aligned} {}^{n}\overline{u}_n &= {}^{n}R_{n-1}\left({}^{n-1}\overline{u}_{n-1} + {}^{n-1}\Omega_{n-1} \times {}^{n-1}Q_n\right) + {}^{n}\begin{bmatrix} 0 & 0 & \dot{d}_n \end{bmatrix}^T \\ {}^{n}\Omega_n &= {}^{n}R_{n-1}\,{}^{n-1}\Omega_{n-1} \end{aligned}\right\}$$

4.2.4 Example 1: Revolute Joints

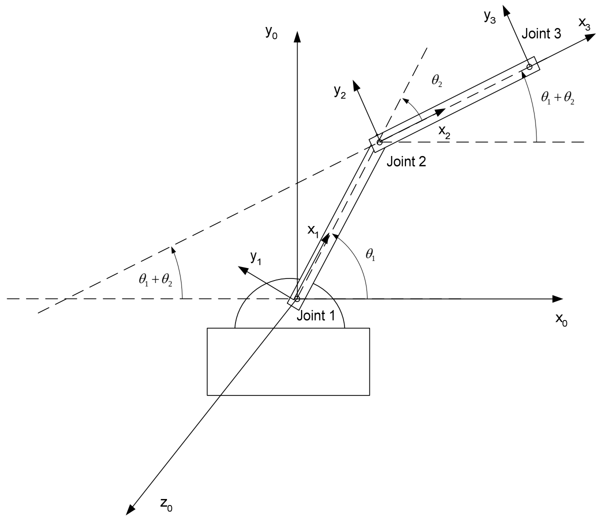

Figure 4.3 2 link revolute arm

Assume the two link RR manipulator in Figure 4.3, find the velocities of the frames {1}, {2} and {3}. Frame {3} has been attached as we wish to express the velocity of the origin of {3} with respect to the frame {3}.

Since all joints are **revolute**:

$$\left.\begin{aligned} {}^{n}\overline{u}_n &= {}^{n}R_{n-1}\left({}^{n-1}\overline{u}_{n-1} + {}^{n-1}\Omega_{n-1} \times {}^{n-1}Q_n\right) \\ {}^{n}\Omega_n &= {}^{n}R_{n-1}\,{}^{n-1}\Omega_{n-1} + {}^{n}\begin{bmatrix} 0 & 0 & \dot{\theta}_n \end{bmatrix}^T \end{aligned}\right\}$$

Furthermore the equations for the **position kinematics** are:

$$^0T_1 = Rot(z,\theta_1) \Rightarrow {}^0R_1 = Rot(z,\theta_1)$$
$$^1T_2 = Trans(l_1,0,0)Rot(z,\theta_2) \Rightarrow {}^1R_2 = Rot(z,\theta_2)$$
$$^2T_3 = Trans(l_2,0,0)$$
$$^0T_3 = Rot(z,\theta_1)Trans(l_1,0,0)Rot(z,\theta_2)Trans(l_2,0,0) \Rightarrow {}^0R_2 = Rot(z,\theta_1)Rot(z,\theta_2)$$

So:

$$n=1 \Rightarrow \left\{ \begin{matrix} {}^1\bar{u}_1 = {}^1R_0\left({}^0\bar{u}_0 + {}^0\Omega_0 \times {}^0Q_1\right) \\ {}^1\Omega_1 = {}^1R_0\,{}^0\Omega_0 + \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T \end{matrix} \right\} \Rightarrow \left\{ \begin{matrix} {}^1\bar{u}_1 = \left(Rot(z,\theta_1)\right)^{-1}(0+0) \\ {}^1\Omega_1 = {}^1R_0\,0 + \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T \end{matrix} \right\}$$

$$\Rightarrow \left\{ \begin{matrix} {}^1\bar{u}_1 = 0 \\ {}^1\Omega_1 = \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T \end{matrix} \right\}$$

$$\left. \begin{matrix} {}^2\bar{u}_2 = {}^2R_1\left({}^1\bar{u}_1 + {}^1\Omega_1 \times {}^1Q_2\right) \\ {}^2\Omega_2 = {}^2R_1\,{}^1\Omega_1 + \begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T \end{matrix} \right\} \Rightarrow \left. \begin{matrix} {}^2\bar{u}_2 = \left(Rot(z,\theta_2)\right)^{-1}\left(0 + \begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right) \\ {}^2\Omega_2 = Rot(z,\theta_2)\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 & \dot{\theta}_2 \end{bmatrix}^T \end{matrix} \right\} \Rightarrow$$

$$\left. \begin{matrix} {}^2\bar{u}_2 = \left(Rot(z,\theta_2)\right)^{-1}\left(0 + \begin{bmatrix} 0 \\ \dot{\theta}_1 l_1 \\ 0 \end{bmatrix}\right) \\ {}^2\Omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \end{matrix} \right\} \Rightarrow \left. \begin{matrix} {}^2\bar{u}_2 = \begin{bmatrix} l_1 \sin(\theta_2)\dot{\theta}_1 \\ l_1 \cos(\theta_2)\dot{\theta}_1 \\ 0 \end{bmatrix} \\ {}^2\Omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \end{matrix} \right\}$$

$$\left. \begin{matrix} {}^3\bar{u}_3 = {}^3R_2\left({}^2\bar{u}_2 + {}^2\Omega_2 \times {}^2Q_3\right) \\ {}^3\Omega_3 = {}^3R_2\,{}^2\Omega_2 + {}^3\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} \end{matrix} \right\} \Rightarrow \left. \begin{matrix} {}^3\bar{u}_3 = I\left(\begin{bmatrix} l_1 \sin(\theta_2)\dot{\theta}_1 \\ l_1 \cos(\theta_2)\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l_2 \\ 0 \\ 0 \end{bmatrix}\right) \\ {}^3\Omega_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} + 0 \end{matrix} \right\} \Rightarrow$$

$$\left.\begin{array}{l} {}^{3}\overline{u}_3 = I\left(\begin{bmatrix} l_1 \sin(\theta_2)\dot{\theta}_1 \\ l_1 \cos(\theta_2)\dot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\dot{\theta}_2 + \dot{\theta}_1\right) l_2 \\ 0 \end{bmatrix}\right) \\ {}^{3}\Omega_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} {}^{3}\overline{u}_3 = \begin{bmatrix} l_1 \sin(\theta_2)\dot{\theta}_1 \\ l_1 \cos(\theta_2)\dot{\theta}_1 + \left(\dot{\theta}_2 + \dot{\theta}_1\right) l_2 \\ 0 \end{bmatrix} \\ {}^{3}\Omega_3 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \end{array}\right.$$

Hence the linear velocity of the joint 3 is: $${}^{3}\overline{u}_3 = \begin{bmatrix} l_1 \sin(\theta_2)\dot{\theta}_1 \\ l_1 \cos(\theta_2)\dot{\theta}_1 + \left(\dot{\theta}_2 + \dot{\theta}_1\right) l_2 \\ 0 \end{bmatrix}.$$

Now, to transform this with respect to the base, simply multiply with $^{0}R_3$:

$${}^{0}\overline{u}_3 = \begin{bmatrix} -l_1 \sin(\theta_1)\dot{\theta}_1 - l_2 \sin(\theta_1 + \theta_2)\left(\dot{\theta}_2 + \dot{\theta}_1\right) \\ l_1 \cos(\theta_1)\dot{\theta}_1 + l_2 \cos(\theta_1 + \theta_2)\left(\dot{\theta}_2 + \dot{\theta}_1\right) \\ 0 \end{bmatrix}$$

4.2.5 Example 2: Prismatic Joints

For the PP robot find the linear velocity of joint 2:

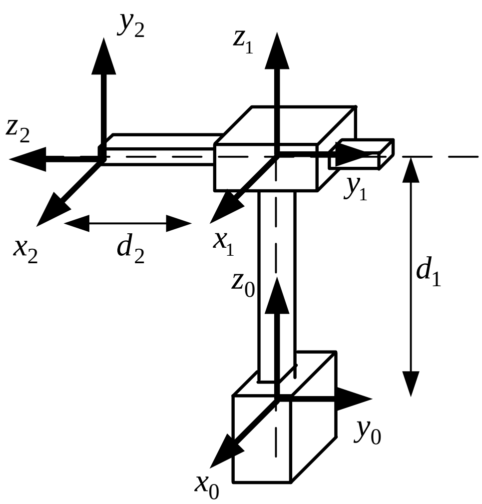

Figure 4.4 2 link prismatic arm

Since the robot has **two prismatic joints**:

$$\left.\begin{aligned} {}^{n}\bar{u}_n &= {}^{n}R_{n-1}\left({}^{n-1}\bar{u}_{n-1} + {}^{n-1}\Omega_{n-1} \times {}^{n-1}Q_n\right) + {}^{n}\begin{bmatrix} 0 & 0 & \dot{d}_n \end{bmatrix}^T \\ {}^{n}\Omega_n &= {}^{n}R_{n-1}\,{}^{n-1}\Omega_{n-1} \end{aligned}\right\}$$

Furthermore the equations for the **position kinematics** are:

$${}^{0}T_1 = Rot(x, a_0)Trans(l_0,0,0)Rot(z,\theta_1)Trans(0,0,d_1) = Trans(0,0,d_1) \Rightarrow {}^{0}R_1 = I$$

$${}^{1}T_2 = Rot(x,90)Trans(l_1,0,0)Rot(z,\theta_2)Trans(0,0,d_2) = Rot(x,90)Trans(0,0,d_2) \Rightarrow {}^{1}R_2 = Rot(x,90)$$

$${}^{0}T_2 = Trans(0,0,d_1)Rot(x,a_1)Trans(0,0,d_2)$$

$${}^{1}\Omega_1 = {}^{1}R_0\,{}^{0}\Omega_0 = 0 \text{ and } {}^{1}u_1 = {}^{1}R_0\left({}^{0}u_0 + {}^{0}\Omega_0 \times {}^{0}Q_1\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix}$$

$${}^{2}\Omega_2 = {}^{2}R_1\,{}^{1}\Omega_1 = 0$$

$${}^{2}u_2 = {}^{2}R_1\left({}^{1}u_1 + {}^{1}\Omega_1 \times {}^{1}Q_2\right) + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} = {}^{2}R_1\,{}^{1}u_1 + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} = \left(Rot(x,90)\right)^{-1}\begin{bmatrix} 0 \\ 0 \\ \dot{d}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \dot{d}_1 \\ \dot{d}_2 \end{bmatrix}$$

To transform this with respect to the base, simply multiply with ${}^{0}R_2$:

$${}^{0}u_2 = {}^{0}R_2\,{}^{2}u_2 = \begin{bmatrix} 0 \\ -\dot{d}_2 \\ \dot{d}_1 \end{bmatrix}$$

Which is the correct vector since the z_2 points at the negative y_0.