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VANISHING SINGULARITY IN HARD IMPACTING SYSTEMS

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Abstract. It is known that the Jacobian of the discrete-time map of an impact oscillator in the neighborhood of a grazing orbit depends on the square-root of the distance the mass would have gone beyond the position of the wall if the wall were not there. This results in an infinite stretching of the phase space, known as the square-root singularity. In this paper we look closer into the Jacobian matrix and find out the behavior of its two parameters—the trace and the determinant, across the grazing event. We show that the determinant of the matrix remains invariant in the neighborhood of a grazing orbit, and that the singularity appears only in the trace of the matrix. Investigating the character of the trace, we show that the singularity disappears if the damped frequency of the oscillator is an integral multiple of half of the forcing frequency.

1. **Introduction.** This paper concerns the dynamics of such mechanical systems in which the elements of the system may undergo impacts with each other. A large number of experimental and numerical investigations on the dynamics of impacting systems have been reported [1, 2, 3, 4, 5, 6, 7]. It has been observed that in such systems a periodic orbit abruptly loses stability and a large-amplitude chaotic vibration develops as the variation of a parameter drives the system from a non-impacting motion to an impacting motion. The instability is known to be induced by grazing or zero-velocity impacts.

A one-degree-of-freedom oscillator constrained by a wall serves as the archetype in the study of such systems. Nordmark [8, 9] had shown that in the simple impact oscillator, the Poincaré map close to the grazing condition involves the square-root of the penetration depth, i.e., the distance the mass would have gone beyond the position of the wall, if the wall were not there. The square-root term causes the Jacobian to assume infinite values close to the grazing condition, as a result of which there is an infinite local stretching in the state space. This is known as the square-root singularity, which causes the abrupt loss of stability and the transition

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to a large chaotic attractor. Nordmark's work resulted in a surge of interest in the dynamics of the impact oscillator and that of the square-root map.

In continuation of that line of work, we ask the question: Are the elements of the Jacobian subject to some constraints as the system transits from non-impacting motion to impacting motion? It is known that with coordinate transformation, the Jacobian matrix can be expressed as a normal form, in terms of only the trace and the determinant of the matrix. Our numerical investigation with a soft impact oscillator revealed that the trace of the Jacobian undergoes an abrupt change while the determinant does not change [11, 12]. Experimental investigation conducted in collaboration with the researchers at the University of Aberdeen, UK, confirmed this observation [12, 13].

In this paper we analytically show that, in a system undergoing instantaneous impacts, the aforementioned singularity is expressed only in the trace of the Jacobian matrix. Note that in [8], Nordmark argued that the determinant cannot assume very large values and so cannot be singular. In this work we show that the determinant in fact remains invariant in the immediate neighborhood of the grazing orbit. As the impact velocity increases from zero, the determinant changes continuously to r^2 times the value before the impact, where r is the coefficient of restitution. We then probe the form of the trace, which leads to an interesting conclusion: that the singularity should vanish if the damped frequency of the oscillator is an integral multiple of half of the forcing frequency. This implies that the sudden exit from the local orbit can be avoided if the parameters are carefully chosen to satisfy this relation.

The paper is organized as follows. In Section II, we describe the system under consideration, and briefly present the zero-time discontinuity map (ZDM) approach first proposed by Nordmark to obtain the form of the Poincaré map in the neighborhood of the grazing orbit. In Section III, we investigate the character of the determinant and the trace of the Jacobian matrix, and prove the main results of this paper. In Section IV, through numerical tests, we demonstrate that the bifurcation diagrams do not show any abrupt transition to a non-local orbit if the frequency ratio satisfies the condition given in Section III, which implies that the stretching behavior does indeed disappear under this condition.

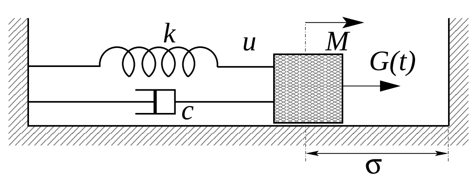

Figure 1. The Hard impact oscillator.

2. **The Hard Impact Oscillator.** As a typical model of an impacting system, we consider a one-degree-of-freedom oscillator, acted upon by a periodic external force $G(t)$, undergoing instantaneous impact with a hard wall (Fig. 1) placed at a distance σ from the equilibrium position of the mass. The evolution of its state vector $x = (u, \dot{u}, t)^T$ is governed by a set of ordinary differential equations (ODEs)

coupled with a set of reset maps as

$$\dot{x} \quad = \quad F(x), \quad \text{if} \quad x \in S^+ \tag{1}$$
$$x \quad \mapsto \quad R(x), \quad \text{if} \quad x \in \Sigma \tag{2}$$

where, $S^+ = \{x : H(x) > 0\}$ and $\Sigma = \{x : H(x) = 0\}$. $H(x)$ is a smooth function, whose zero set defines the hard boundary Σ. The flow given by (1) is restricted only in the region $S^+ \cup \Sigma$. The three cases of nonimpacting orbit, grazing orbit, and an impacting orbit are shown in Fig. 2.

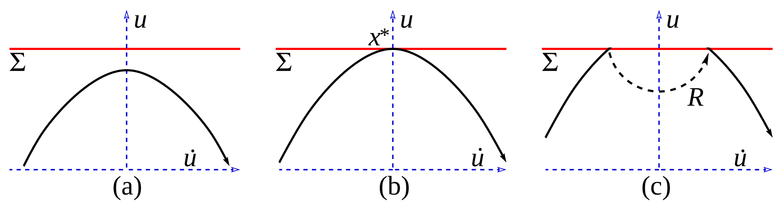

FIGURE 2. Behavior of orbits close to the switching plane Σ: (a) a nonimpacting trajectory, (b) a grazing trajectory, and (c) an impacting trajectory. Upon impact, the velocity instantly reverses while the position remains the same. Thus the state instantly jumps to a new position, given by the mapping $x \mapsto R(x)$.

The normal velocity $v(x)$ is defined as the rate at which the trajectory approaches the impact boundary. It is given by

$$v(x) := \frac{dH(x)}{dt} = \frac{\partial H(x)}{\partial x}\frac{dx}{dt} = H_x F(x). \tag{3}$$

Here the subscript x implies partial derivative with respect to x, i.e., H_x denotes the gradient. The right hand side of (3) gives the normal velocity if $H_x \neq 0$ on Σ. Similarly the normal acceleration $a(x)$ of the flow with respect to the boundary is

$$a(x) := (H_x F)_x F(x).$$

We may now be more specific about the form the reset map $R(x)$ takes. To that end, we observe that the reset map has to be a smooth function of the normal velocity $v(x)$ and furthermore R maps x to itself when grazing occurs. Since at grazing the normal velocity with respect to the boundary becomes zero ($v(x) = 0$), the reset map can be formulated as

$$R(x) = x + W(x)v(x) \tag{4}$$

where W is a smooth 3×1 vector.

Grazing occurs when a trajectory becomes tangent to the discontinuity boundary Σ, as shown in Fig. 2(b). A point $x = x^*$ is called a regular grazing point if it satisfies the conditions

$$H(x^*) \quad = \quad 0 \tag{5}$$
$$v(x^*) \quad = \quad 0 \tag{6}$$
$$a(x^*) \quad = \quad a^* > 0 \tag{7}$$

In addition the scalar function $H(x)$ is assumed to be well defined at $x = x^*$, i.e., $H_x(x^*) \neq 0$.

2.1. **Grazing and Discontinuity Mapping.** In this section we briefly review the current approach in obtaining the discrete-time Poincaré map in the neighborhood of the grazing orbit [8, 9, 10], as a stepping stone for our analysis.

In this approach, one observes the state at every positive-slope zero-crossing of the forcing function, and obtains the function $x_T = f(x_0)$, where T is the period of the forcing function, and the discrete-time state vector is $(u, \dot{u}, t)^T$. For the part of the flow that does not have any impact with the discontinuity boundary the mapping is given by the function F only (see Fig. 3), as $x_T = \varphi(x_0, T)$, where φ is the flow.

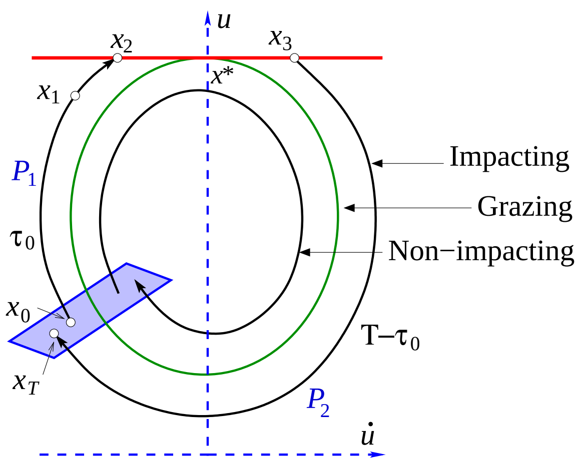

FIGURE 3. Obtaining the map $x_T = f(x_0)$.

Let there be an orbit which grazes the discontinuity boundary Σ at a point x^*. Let us consider an impacting trajectory close to the grazing one. In that case we consider the evolution from $t = 0$ to a point x_1 at a time τ_0, close to but before the impact, such that under small perturbation a perturbed trajectory also does not experience impact at the same time. The trajectory reaches the switching surface at x_2 and then the reset map takes it to x_3 (Fig. 4). After having reached the point x_3, we back-trace the trajectory governed by the ODE (1), to the point x_4 such that the time taken by the trajectory to reach from x_1 to x_3 is the same as would have been taken by the flow to reach from x_4 to x_3. Now, in order to obtain the Poincaré map, one has to move from observation instant (the point x_0 in Fig. 3) to the next (x_T) after the lapse of time T. As per Nordmark's formulation, we can consider this evolution as composed of smooth evolutions from x_0 to x_1 for the interval τ_0, and from x_4 to x_T for the interval $(T - \tau_0)$. But then we would have to consider a correction represented by the instantaneous shift of position from x_1 to x_4. The "Zero-time Discontinuous Mapping" (ZDM) is defined as the mapping $x_1 \mapsto x_4$.

In the literature, the state transition matrix across a switching event is called a *saltation matrix*. Its form has been derived in [14, 10] under the assumption of transversal intersection of the trajectory with the switching manifold in the phase space. The ZDM as described above is basically the saltation matrix close to the grazing condition, where the trajectory becomes tangential to the switching surface.

Thus, the complete stroboscopic Poincaré map is composed of three components:

$$P_s = P_2 \circ S \circ P_1. \tag{8}$$

where

$$\begin{aligned} P_1 &= \varphi(x_0, \tau_0), \\ P_2 &= \varphi(x_4, T - \tau_0), \end{aligned}$$

and S is the saltation matrix for this special case, or the ZDM.

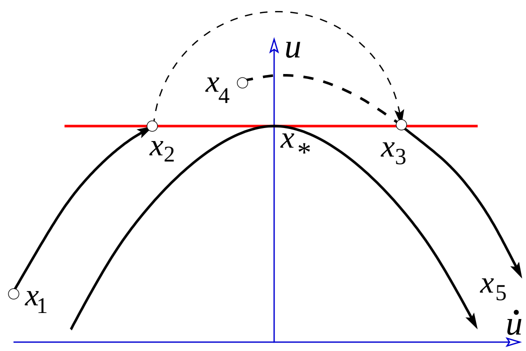

FIGURE 4. Zero-time discontinuity mapping near grazing.

Let $\varphi(x_1, t)$ be the solution of the ODE starting from x_1, i.e., $\varphi(x_1, 0) = x_1$. It has been shown in [9] (see [10] for a detailed treatment) that the form of the ZDM, excluding higher order terms of y (defined after (9)), is

$$x_4 = x_1 - W^*(\sqrt{2a^*})y \tag{9}$$

where, $W^* = W(x^*)$, and $y = \sqrt{-H_{\min}(x_1)}$. $H_{\min}(x_1)$ is defined [10] as the local minimum value of $H(\varphi(x_1, t))$, i.e., the lowest point that the trajectory starting at x_1 would have reached if the switching boundary were not there. Obviously, for the situation shown in Fig. 4, $H_{\min}(x_1)$ will be negative except for the case when x_1 is the same as x^*.

Using (9) the form of the stroboscopic map can be derived, in first order approximation, as:

$$\begin{aligned} P_1 &: \quad x \mapsto N_1 x \\ S \circ P_1 &: \quad x \mapsto N_1 x - \sqrt{2a^*}\sqrt{-H_{\min}(N_1 x)}W^* \\ P_2 \circ S \circ P_1 &: \quad x \mapsto N_2 N_1 x - \sqrt{2a^*}\sqrt{-H_{\min}(N_1 x)}N_2 W^* \end{aligned}$$

where $N_1 := \frac{dP_1}{dx}|_{x=x_0}$, $N_2 := \frac{dP_2}{dx}|_{x=x^*}$.

If x_0 lies on a periodic orbit, its stability depends on how a perturbation $(x - x_0)$ at an observation instant maps to the next observation instant. Since x_0 maps to the point $P_2(S(P_1(x_0)))$, this map takes the form

$$(x - x_0) \quad \mapsto \quad N_2 N_1 (x - x_0) - \sqrt{2a^*}\sqrt{-H_{\min}(N_1 x)}N_2 W^* \tag{10}$$

Also $H_{\min}(N_1 x)$ is linearized about grazing to the form

$$H_{\min}(N_1 x) = H_x N_1 (x - x_0) + H.O.T$$

As can be seen from the expression of the stroboscopic map, and also of the ZDM alone, a square-root term $\sqrt{-H_{\min}}$ is present which accounts for the square-root singularity when the Jacobian of the map is considered.

3. Investigating the Jacobian for Singularity. Now we investigate the constraints on the elements of the Jacobian. It has been shown earlier [15, 16] that such a Jacobian matrix J, through a coordinate transformation, can be expressed in the normal form:

$$J' = \begin{pmatrix} \mathrm{tr}(J) & 1 \\ -\det(J) & 0 \end{pmatrix} \tag{11}$$

where $\mathrm{tr}(J)$ and $\det(J)$ are the trace and the determinant respectively, of the 2-D Jacobian matrix. Since the trace and the determinant are invariant under coordinate transformation, the character of the dynamics around a fixed point is given by just these two numbers. On the basis of this argument, much of the earlier work on border collision bifurcations investigated the dynamics of a piecewise linear map

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{cases} J'_L \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{for } x_n \leq 0 \\ \\ J'_R \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{for } x_n \geq 0 \end{cases} \tag{12}$$

representing the local behavior in the neighborhood of the border. Here the phase space is divided into two halves separated by the borderline $x_n = 0$, and the matrices J'_L and J'_R are the transformed Jacobian matrices in the left half and the right half of the phase space respectively. The variation of the parameter μ from a negative to a positive value causes the border collision bifurcation, and the outcome of the bifurcation depends on the trace and the determinant at the two sides of the border. In tune with this logic, we explore how the trace and the determinant of the Jacobian matrix change across the grazing condition.

The Jacobian of the stroboscopic map of an impacting orbit near grazing would be

$$J = N_2 \circ J_{ZDM} \circ N_1 \tag{13}$$

where J_{ZDM} is the Jacobian of the ZDM given by

$$J_{ZDM} = I + \sqrt{2a^*}\frac{W^* H_x}{2\sqrt{-H_{\min}}}. \tag{14}$$

To arrive at the particular forms W^* and H_x would take, let us concentrate on the one degree-of-freedom impact oscillator of Fig. 1. For $u < \sigma$ the motion of the mass is governed by the differential equation (1), which can be expressed in the form

$$\frac{d^2u}{dt^2} + 2\zeta\omega_n\frac{du}{dt} + {\omega_n}^2 u = g(t), \tag{15}$$

where ζ is the damping factor and ω_n is the natural frequency of oscillation. For the system shown in Fig. 1, $\zeta = c/2\sqrt{kM}$, $\omega_n = \sqrt{k/M}$ and $g(t) = G(t)/M$. At $u = \sigma$ the reset map R is applied. The equation of the discontinuity boundary Σ in the present case is

$$H(x) = H(u, \dot{u}, t) = \sigma - u.$$

Thus we have $\frac{\partial H}{\partial \dot{u}} = 0$, $\frac{\partial H}{\partial t} = 0$ and hence,

$$H_x = (h_1 \;\; 0 \;\; 0). \tag{16}$$

where $h_1 = \frac{\partial H}{\partial u}$. In this particular case $h_1 = -1$, but for our purpose it suffices to note that the second and third terms are zero.

The reset map is $R : \Sigma \mapsto \Sigma$, where $R(x)$ has the form given in (4). Since the position of the mass just before the impact, u^-, is same as that just after the impact, u^+, we can write (4)

$$W = (0 \;\; 1{+}r \;\; 0)^T; \tag{17}$$

where r is the constant of restitution.
Using (16) and (17) in (14) we obtain

$$\begin{aligned}
J_{ZDM} &= I + \sqrt{2a^*}\frac{W^* H_x}{2\sqrt{-H_{\min}}} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sqrt{2a^*}}{2\sqrt{-H_{\min}}} \begin{pmatrix} 0 \\ 1{+}r \\ 0 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{\sqrt{2a^*}}{2\sqrt{-H_{\min}}} \begin{pmatrix} 0 & 0 & 0 \\ (1{+}r)h_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned} \tag{18}$$

where

$$\alpha = \frac{(1{+}r)h_1\sqrt{2a^*}}{2\sqrt{-H_{\min}}}.$$

3.1. **Investigating the Determinant for Singularity.** From (13), the determinant of the normal form map near grazing is

$$|J| = |N_2||J_{ZDM}||N_1| = |N_2||N_1|$$

since $|J_{ZDM}| = 1$, from (18).

Since the singularity is only in the ZDM, and not in the maps N_1 and N_2, we conclude that the determinant of the normal form map does not contain the square-root singularity, and remains invariant in the immediate neighborhood of the grazing orbit.

However, as shown earlier, the ZDM formulation is applicable only in the immediate neighborhood of the grazing orbit. For impacting orbits away from the grazing condition (where impact occurs with a non-zero velocity), one has to consider the state transition matrix across the switching event, which is called the saltation matrix. The form of the saltation matrix was shown to be [10]

$$S = R_x + \frac{[F(R(x^*)) - R_x R(x^*)]}{H_x(x^*)F(x^*)} H_x(x^*)$$

In our case, after simple algebraic manipulation, its form comes out to be

$$\begin{aligned}
S &= \begin{pmatrix} -r & 0 & 0 \\ -\frac{(1+r)(ku-mg(t))}{mv} & -r & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow |S| &= r^2
\end{aligned}$$

Therefore, as the change of a parameter drives the system away from grazing well into the impacting mode, the determinant of the Jacobian matrix is multiplied by a factor r^2.

Fig. 5 shows the variation of the determinant of the Jacobian matrix, obtained numerically with the value of r taken to be 0.9. For each parameter value, the periodic orbit was located, and the Jacobian was obtained by observing the subsequent iterations starting from a perturbed initial state. To further explain the situation, we note that for a periodic orbit an initial state x_{i0} is mapped to itself by the normal form map. We perturb the initial state to x_{i1} so that it is mapped to x_{e1} by the normal form map. Then from the definition of the Jacobian, $(x_{e1} - x_{i0}) = J(x_{i1} - x_{i0})$. We can iterate the map for many such perturbed initial states and note down the final states on the Poincaré plane and compute the Jacobian from the initial and final states. The graph shows a steep but continuous variation, finally attaining a value that is r^2 times the value of the determinant before impact.

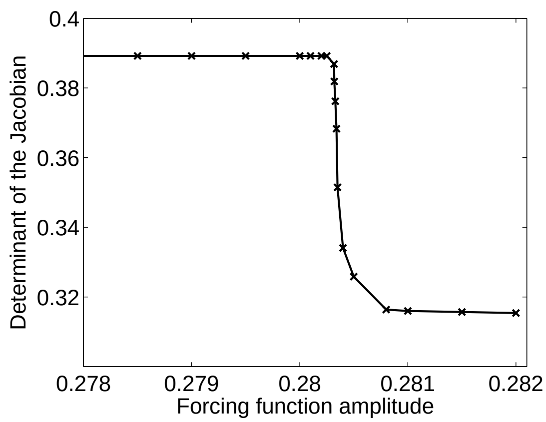

FIGURE 5. The variation of the determinant of the Jacobian matrix with change in the amplitude of the forcing function ($r = 0.9$).

3.2. **Investigating the Trace for Singularity.** Since the Jacobian matrix is singular but the determinant of the matrix is nonsingular, the normal form (11) shows that the singularity must be contained only in the trace of the matrix. We now probe the form of the trace.

To obtain the expression for the trace of the Jacobian J in (13), we need to obtain the expressions for the maps P_1 and P_2 first. Let us consider a periodic solution to the equation given in (15) as

$$p(t) = (u(t), v(t), t)^T.$$

Let $(p(t) + \delta p(t))$ be a perturbed orbit, where δu satisfies the variational equation

$$\delta\ddot{u} + 2\zeta\omega_n\delta\dot{u} + {\omega_n}^2\delta u = 0. \tag{19}$$

The variational equation needs to be solved to obtain the perturbed flow $\delta p(\tau) = (\delta u(\tau), \delta v(\tau), 0)^T$. Solving the variational equation amounts to solving the first-order differential equations

$$\frac{d}{dt}\begin{pmatrix} \delta u(t) \\ \delta v(t) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -{\omega_n}^2 & -2\zeta\omega_n & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \delta u(t) \\ \delta v(t) \\ 0 \end{pmatrix},$$

with $\delta u(0) = \delta u_0, \delta v(0) = \delta v_0$.

For $\zeta < 1$, the solution of the above problem for a time of evolution τ can be expressed as

$$\begin{pmatrix} \delta u(\tau) \\ \delta v(\tau) \\ 0 \end{pmatrix} = N_\tau \begin{pmatrix} \delta u_0 \\ \delta v_0 \\ 0 \end{pmatrix} \tag{20}$$

where

$$N_\tau = e^{-\zeta\omega_n\tau} \begin{pmatrix} \cos(\omega_0\tau) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_0\tau) & \sin(\omega_0\tau)/\omega_0 & 0 \\ -\frac{1}{\sqrt{1-\zeta^2}}\omega_0\sin(\omega_0\tau) & \cos(\omega_0\tau) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_0\tau) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $\omega_0 = \omega_n\sqrt{1-\zeta^2}$.

Now we can proceed to obtain the expression of the trace of the Jacobian in (13). In the situation shown in Fig. 3, using the notation in (20), we get

$$N_1 = N_{\tau_0} = e^{-\zeta\omega_n\tau_0} \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{13} & n_{14} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{21}$$

$$N_2 = N_{(T-\tau_0)} = e^{-\zeta\omega_n(T-\tau_0)} \begin{pmatrix} n_{21} & n_{22} & 0 \\ n_{23} & n_{24} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{22}$$

where n_{ij} represent the shorthand form of the terms in (20).
Using (18), (21) and (22) in (13), we get the Jacobian as

$$\begin{aligned} J &= e^{-\zeta\omega_n T} \begin{pmatrix} n_{21} & n_{22} & 0 \\ n_{23} & n_{24} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{13} & n_{14} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{-\zeta\omega_n T} \begin{pmatrix} n_{21} + \alpha n_{22} & n_{22} & 0 \\ n_{23} + \alpha n_{24} & n_{24} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} & 0 \\ n_{13} & n_{14} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= e^{-\zeta\omega_n T} \begin{pmatrix} n_{21}n_{11} + n_{22}n_{13} + \alpha n_{22}n_{11} & * & 0 \\ * & n_{23}n_{12} + n_{24}n_{14} + \alpha n_{24}n_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Simple algebraic manipulation yields the trace as

$$\begin{aligned} Tr(J) &= e^{-\zeta\omega_n T}\{n_{21}n_{11} + n_{22}n_{13} + n_{23}n_{12} + n_{24}n_{14} \\ &\quad + \alpha(n_{22}n_{11} + n_{24}n_{12}) + 1\} \end{aligned} \tag{23}$$

The expression for the trace of the Jacobian $Tr(J)$, in (23), shows that the singularity term α has a coefficient $(n_{22}n_{11} + n_{24}n_{12})$. Let us take a closer look at this coefficient.

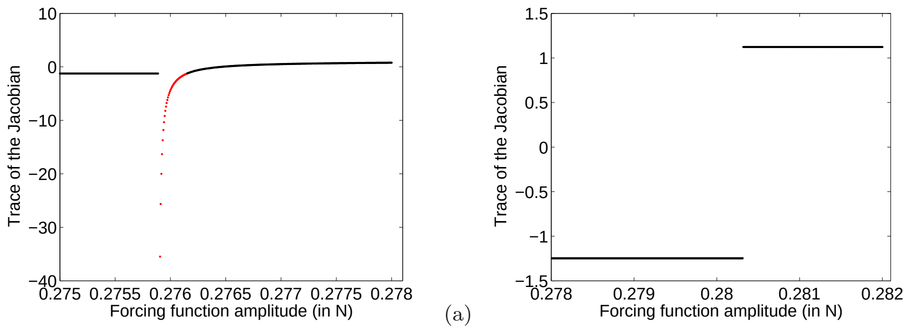

FIGURE 6. The graphs show the variation of the trace of the Jacobian matrix with the variation of the parameter: (a) for m =2.97, and (b) for m =3.

Since

$$\begin{aligned} n_{11} &= \cos(\omega_0\tau_0) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_0\tau_0) \\ n_{12} &= \frac{\sin(\omega_0\tau_0)}{\omega_0} \\ n_{22} &= \frac{\sin\{\omega_0(T-\tau_0)\}}{\omega_0} \\ n_{24} &= \cos\{\omega_0(T-\tau_0)\} - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\{\omega_0(T-\tau_0)\}, \end{aligned} \tag{24}$$

after simple manipulation we get

$$n_{22}n_{11} + n_{24}n_{12} = \frac{\sin(\omega_0 T)}{\omega_0}. \tag{25}$$

Let a number m be defined as

$$m = \frac{2\omega_0}{\omega_{\text{forcing}}}$$

where ω_{forcing} is the angular frequency of the periodic forcing function $g(t)$, i.e., $\omega_{\text{forcing}}T = 2\pi$. It follows that

$$n_{22}n_{11} + n_{24}n_{12} \neq 0, \ \ \forall \ \text{non-integer} \ \ m \tag{26}$$

Equation (26) implies that when m is non-integer, the coeffcient of α in the expression of the trace (23) of the Jacobian of the stroboscopic map must be a non-zero entity. Thus the singularity in α survives, and hence a square-root singularity must occur in the trace of the Jacobian.

This result also has a very interesting implication: that the singularity must vanish for

$$\omega_{\text{forcing}} = \frac{2\omega_0}{m}, \tag{27}$$

where m is an integer. This implies that if the frequency is chosen to satisfy (27), the singularity will disappear, and there will be no stretching of the phase space in the neighborhood of a grazing orbit. Fig. 6 shows the variation of the trace from the non-impacting condition to the impacting condition, obtained from simulation.

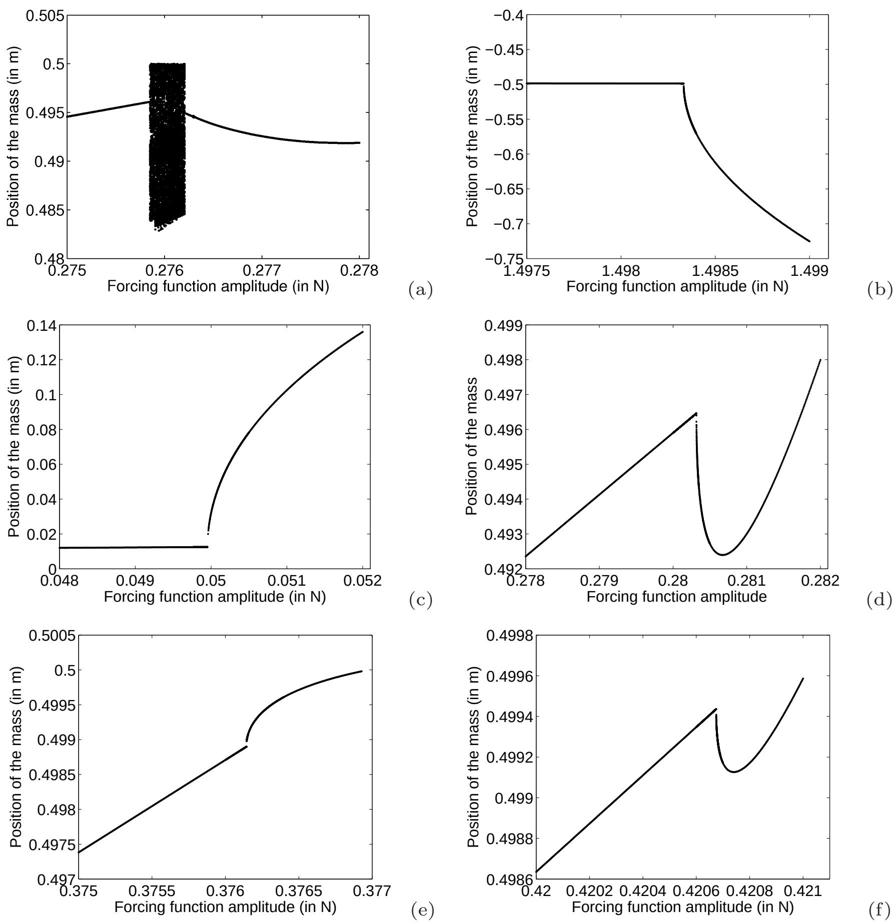

FIGURE 7. (a) The bifurcation diagram showing the effect of infinite local stretching at $m = 2.97$. (b-f) The bifurcation diagrams for integer values of m: (b) $m = 1$, (c) $m = 2$, (d) $m = 3$, (e) $m = 4$, (f) $m = 5$.

It clearly shows that for a non-integer value of m the trace exhibits a square-root singularity, while for an integer value of m it does not.

4. **Numerical test of vanishing singularity.** It is known that close to grazing in an impact oscillator, there is a sudden transition to a much larger chaotic orbit [17, 10] owing to the stretching in the phase space. As a typical example, the bifurcation diagram for $m = 2.97$ is shown in Fig. 7(a) for $M = 1$ kg, $c = 0.5$ N-s/m, and $k = 1$ N/m. At an excitation amplitude of 0.2759 N, the orbit experiences grazing, resulting in the disappearance of the local behavior and the onset of a much larger chaotic orbit.

In contrast, when m is taken to be an integer, the bifurcation diagrams shown in Fig. 7(b-f) do not display any abrupt disappearance of the local orbit close to grazing. This shows that the singularity does indeed disappear if the damped frequency of the oscillator is an integral multiple of half of the forcing frequency.

This observation may have important technological consequences. In engineering systems where there can be nonimpacting as well as impacting motion (for example, in bearings with clearance) it has been observed that, as an increase in the oscillation amplitude drives it into the grazing condition, the system goes into a violent chaotic vibration. This is recognized to be a major problem in such systems, and some research effort has been directed towards developing control methods to suppress the oscillation [17]. Our result shows that such unwanted vibration can be avoided if the excitation frequency is chosen to satisfy the above condition in relation to the natural frequency of the system.

The plots also show that, for odd m values, the position coordinate of the fixed point decreases after grazing, while it increases at even values of m. To understand what is causing this different behavior for odd and even values of m, we consider the normal form map given by (11) and (12), and solve for the fixed points before the impact (x_L^*, y_L^*) and after the impact (x_R^*, y_R^*). Combining (11) and (12), we get

$$\begin{aligned}\begin{pmatrix} x_L^* \\ y_L^* \end{pmatrix} &= \frac{1}{(1-\tau_L+\delta_L)}\begin{pmatrix} \mu \\ -\mu\delta_L \end{pmatrix}\\ \begin{pmatrix} x_R^* \\ y_R^* \end{pmatrix} &= \frac{1}{(1-\tau_R+\delta_R)}\begin{pmatrix} \mu \\ -\mu\delta_R \end{pmatrix}\end{aligned} \tag{28}$$

where, τ_L is the trace and δ_L is the determinant of the Jacobian matrix for a non-impacting orbit, and τ_R and δ_R are those for an impacting orbit, evaluated close to the grazing condition. Since the determinant does not change through the impact, we have $\delta_L = \delta_R = \delta$.

$$\begin{pmatrix} x_R^* \\ y_R^* \end{pmatrix} - \begin{pmatrix} x_L^* \\ y_L^* \end{pmatrix} = \frac{(\tau_R-\tau_L)}{(1-\tau_L+\delta)(1-\tau_R+\delta)}\begin{pmatrix} \mu \\ -\mu\delta \end{pmatrix}. \tag{29}$$

Thus with the help of a 2D normal form map in (12), we see that the change in the location of the fixed point due to impact depends on the change in the trace of the Jacobian matrix. With this observation we now revert back to our system of one degree-of-freedom oscillator. While calculating the Jacobian, we consider the saltation matrix S [10] in order to account for non-zero-velocity impacts as well. Thus,

$$\begin{aligned} J_R &= N_2 S N_1; \quad S = \begin{pmatrix} -r & 0 & 0 \\ -\frac{(1+r)(ku-mg(t))}{mv} & -r & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ J_L &= N_2 N_1; \end{aligned} \tag{30}$$

Following simliar algebra as in (23), we obtain the traces

$$\begin{aligned} \tau_R &= e^{-\zeta\omega_n T}\{-r(n_{21}n_{11}+n_{22}n_{13}+n_{23}n_{12}+n_{24}n_{14})+1\} \\ \tau_L &= e^{-\zeta\omega_n T}\{n_{21}n_{11}+n_{22}n_{13}+n_{23}n_{12}+n_{24}n_{14} \\ &\qquad +\beta(n_{22}n_{11}+n_{24}n_{12})+1\}; \quad \beta = -\frac{(1+r)(ku-mg(t))}{mv}. \end{aligned} \tag{31}$$

For integral values of m in (27), using (25) we have

$$\tau_R - \tau_L = e^{-\zeta\omega_n T}\{-(1+r)(n_{21}n_{11}+n_{22}n_{13}+n_{23}n_{12}+n_{24}n_{14})\} \tag{32}$$

since $n_{22}n_{11} + n_{24}n_{12} = 0$. Using (24), and noting that

$$\begin{aligned} n_{21} &= \cos\{\omega_0(T-\tau_0)\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\{\omega_0(T-\tau_0)\} \\ n_{23} &= \frac{-\omega_0\sin\{\omega_0(T-\tau_0)\}}{\sqrt{1-\zeta^2}} \\ n_{13} &= \frac{-\omega_0\sin(\omega_0\tau_0)}{\sqrt{1-\zeta^2}} \\ n_{14} &= \cos(\omega_0\tau_0) - \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_0\tau_0), \end{aligned} \tag{33}$$

we can write,

$$\begin{aligned} & (n_{21}n_{11} + n_{22}n_{13} + n_{23}n_{12} + n_{24}n_{14}) \\ = & \left(\frac{1-2\zeta^2}{1-\zeta^2} + \frac{1}{\sqrt{1-\zeta^2}}\right)\cos(\omega_0 T) + \left(\frac{1}{1-\zeta^2} - \frac{1}{\sqrt{1-\zeta^2}}\right)\cos\{\omega_0(2\tau_0 - T)\} \\ \approx & \; 2\cos(\omega_0 T); \quad \zeta << 1 \\ = & \; 2\cos(m\pi) \\ = & \left\{ \begin{array}{ll} +2; & m = even \\ -2; & m = odd \end{array} \right. \end{aligned}$$

Thus we see that the sign of $(\tau_R - \tau_L)$ alternates as m takes even and odd values. Hence, from (29), for even and odd values of m, we see that the fixed point moves in opposite directions (Fig. 7).

5. **Conclusions.** In this paper we have analyzed the character of the map function of an impact oscillator, as the variation of a parameter drives the system from a non-impacting orbit to an impacting orbit. It is known that in such condition the Jacobian matrix undergoes an abrupt change and has a square root singularity. We probed if the changes in the elements of the Jacobian matrix are subject to some constraints.

From the earlier literature one could conclude that the determinant of the Jacobian matrix cannot contain a singularity. In this paper we have demonstrated that the determinant of the Jacobian matrix must *remain invariant* across the grazing condition. Moreover, as the velocity of impact increases, it must change smoothly to a value r^2 times the value for a nonimpacting orbit.

We have shown that, if the Jacobian matrix is moulded in the normal form by a coordinate transformation, only the trace (which is invariant across coordinate transformation) contains the singularity. Thus the square-root singularity is expressed only in the trace of the Jacobian matrix, i.e., the trace of the Jacobian matrix assumes an infinite value immediately following grazing.

We have shown that the singularity itself is not an invariant property of the system. The singularity vanishes if the parameters of the oscillator and the forcing function are chosen such that $\omega_{\text{forcing}} = 2\omega_0/m$, where m is an integer. It is known that the singularity is a source of many problems in practical systems because of the sudden transition to a large-amplitude chaotic motion following grazing, and people have been thinking of ways to control the oscillations resulting from the singularity. Our observation on the disappearance of the singularity points to aspects in system design that can avoid this problem.

The above theoretical prediction has been validated using simulated bifurcation diagrams. The numerically obtained bifurcation diagrams showed different behaviors not only for noninteger and integer values of m, but also for even and odd values of m. We analytically explained why this must be so.

REFERENCES

[1] S. W. Shaw and P. J. Holmes. *A periodically forced piecewise linear oscillator.* Journal of Sound & Vibration, **90** (1983), 129–155.
[2] F. Peterka and J. Vacik. *Transition to chaotic motion in mechanical systems with impacts.* Journal of Sound and Vibration, **154** (1992), 95–115.
[3] B. Blazejczyk-Okolewska and T. Kapitaniak. *Co-existing attractors of impact oscillator.* Chaos, Solitons & Fractals, **9** (1998), 1439–1443.
[4] S. Lenci and G. Rega. *A procedure for reducing the chaotic response region in an impact mechanical system.* Nonlinear Dynamics, **15** (1998), 391–409.
[5] D. J. Wagg, G. Karpodinis, and S. R. Bishop. *An experimental study of the impulse response of a vibro-impacting cantilever beam.* Journal of Sound & Vibration, **228** (1999), 243–264.
[6] E. K. Ervin and J. A. Wickert. *Experiments on a beam-rigid body structure repetitively impacting a rod.* Nonlinear Dynamics, **50** (2007), 701–716.
[7] J. Ing, E. Pavlovskaia, and M. Wiercigroch. *An experimental study into the bilinear oscillator close to grazing.* In International Symposium on Nonlinear Dynamics, Journal of Physics: Conference Series, **96**, (2007), 012119.
[8] A. B. Nordmark. *Non-periodic motion caused by grazing incidence in an impact oscillator.* Journal of Sound and Vibration, **145** (1991), 279–297.
[9] A. B. Nordmark. *Universal limit mapping in grazing bifurcations.* Phys. Rev. E, **55** (1997), 266–270.
[10] M. di Bernardo, C. J. Budd, A. R. Champneys, and P. Kowalczyk. Piecewise-smooth Dynamical Systems: Theory and Applications. Springer Verlag (Applied Mathematical Sciences), London, 2008.
[11] Y. Ma, M. Agarwal, and S. Banerjee. *Border collision bifurcations in a soft impact system.* Physics Letters A, **354** (2006), 281–287.
[12] Y. Ma, J. Ing, S. Banerjee, M. Wiercigroch, and E. Pavlovskaia. *The nature of the normal form map for soft impacting systems.* International Journal of Nonlinear Mechanics, **43** (2008) 504–513.
[13] J. Ing, E. Pavlovskaia, M. Wiercigroch, and S. Banerjee. *Experimental study of impact oscillator with one-sided elastic constraint.* Philosophical Transactions of the Royal Society of London, Part A, **366** (2008), 679–704.
[14] R. I. Leine and H. Nijmeijer. Dynamics and Bifurcations in Non-Smooth Mechanical Systems. Springer Verlag, Berlin, 2004.
[15] S. Banerjee and C. Grebogi. *Border collision bifurcations in two-dimensional piecewise smooth maps.* Physical Review E, **59** (1999), 4052–4061.
[16] S. Banerjee, P. Ranjan, and C. Grebogi. *Bifurcations in two-dimensional piecewise smooth maps — theory and applications in switching circuits.* IEEE Transactions on Circuits and Systems-I, **47** (2000), 633–643.
[17] H. Dankowicz and F. Svahn. *On the stabilizability of near-grazing dynamics in impact oscillators.* Int. J. Robust & Nonlinear Control, **17** (2007), 14051429.

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