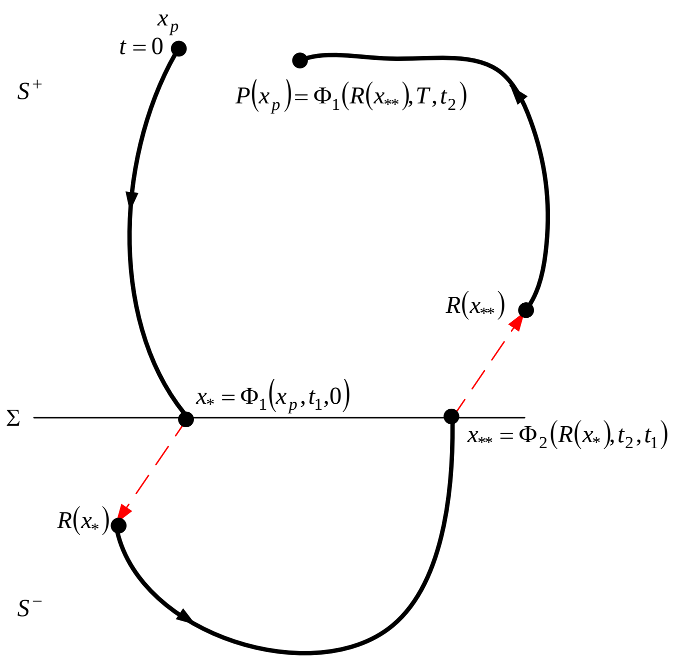
Discontinuity Maps (The Taylor Series' Heaven)

This my understanding of the concept of discontinuity maps as it is described in "*Piecewise-smooth Dynamical Systems: Theory and Applications*" by M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk.

Motivation:

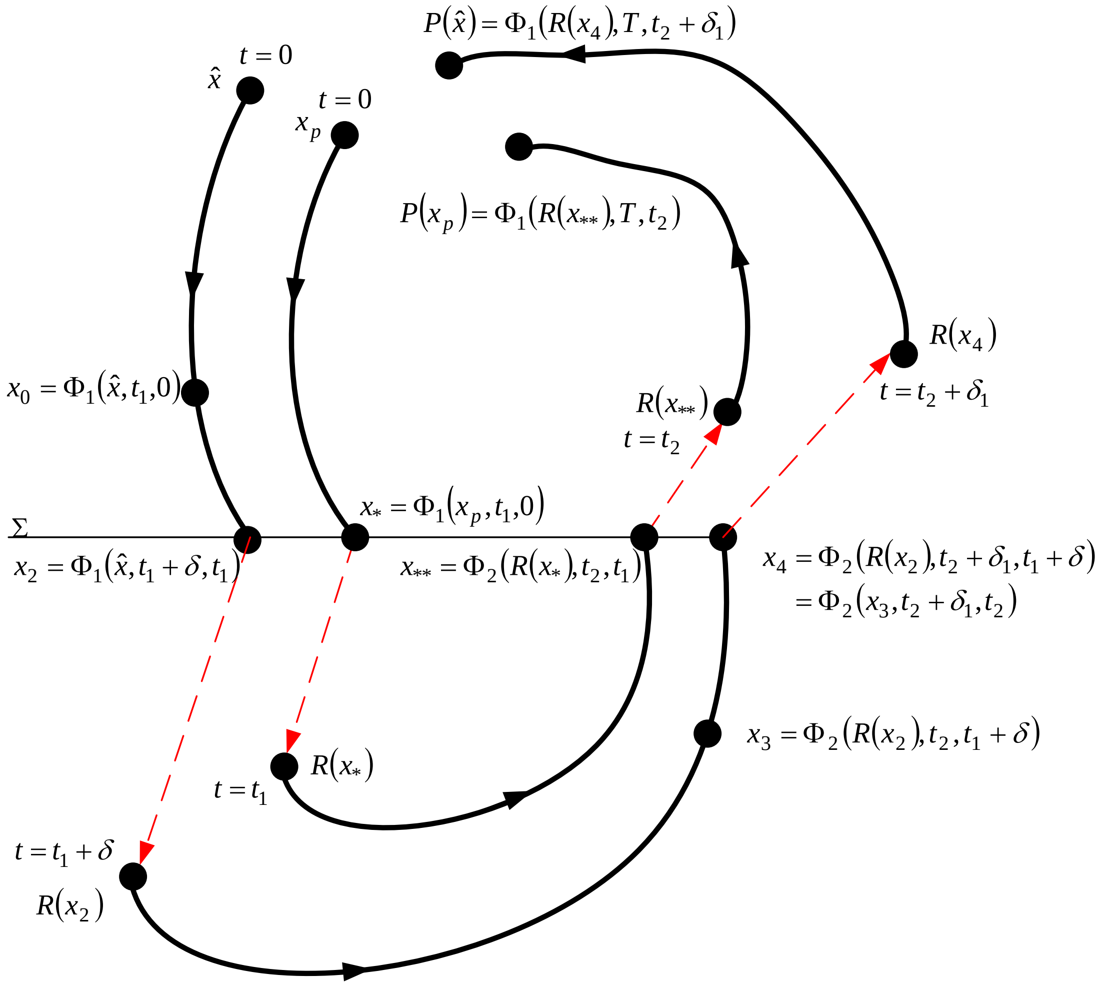


The state space is split into 3 areas: $\mathbb{R}^2 = S^+ \cup \Sigma \cup S^-$. The surface Σ is defined by a smooth scalar function $H(x(t),t)$. Hence $S^+ = \{x \in \mathbb{R}^2 : H(x(t),t) > 0\}$ and $S^- = \{x \in \mathbb{R}^2 : H(x(t),t) < 0\}$ while $\Sigma = \{x \in \mathbb{R}^2 : H(x(t),t) = 0\}$ (in all these cases I could have $H(x(t))$ instead of $H(x(t),t)$).

In that case I start from the point $x_p \in S^+$ at t=0. I evolve through Φ_1 to the point $x_* = \Phi_1(x_p,t_1,0) \in \Sigma$. Then using the jump map we go to $R(x_*) \in S^-$ and we evolve with Φ_2 to $x_{**} = \Phi_2(R(x_*),t_2,t_1) \in \Sigma$. Now, using the jump matrix we go to $R(x_{**}) \in S^+$. The cycle is finished by applying again Φ_1 to $R(x_{**}) \in S^+$: $P(x_p) = \Phi_1(R(x_{**}),T,t_2)$. Thus the T-return or stroboscopic Poincare map is:

$$P(x_p) = \Phi_1\left(R\left(\underbrace{\Phi_2\left(R\left(\underbrace{\Phi_1(x_p,t_1,0)}_{x*} \right), t_2, t_1 \right)}_{x**} \right), T, t_2 \right) \qquad (1)$$

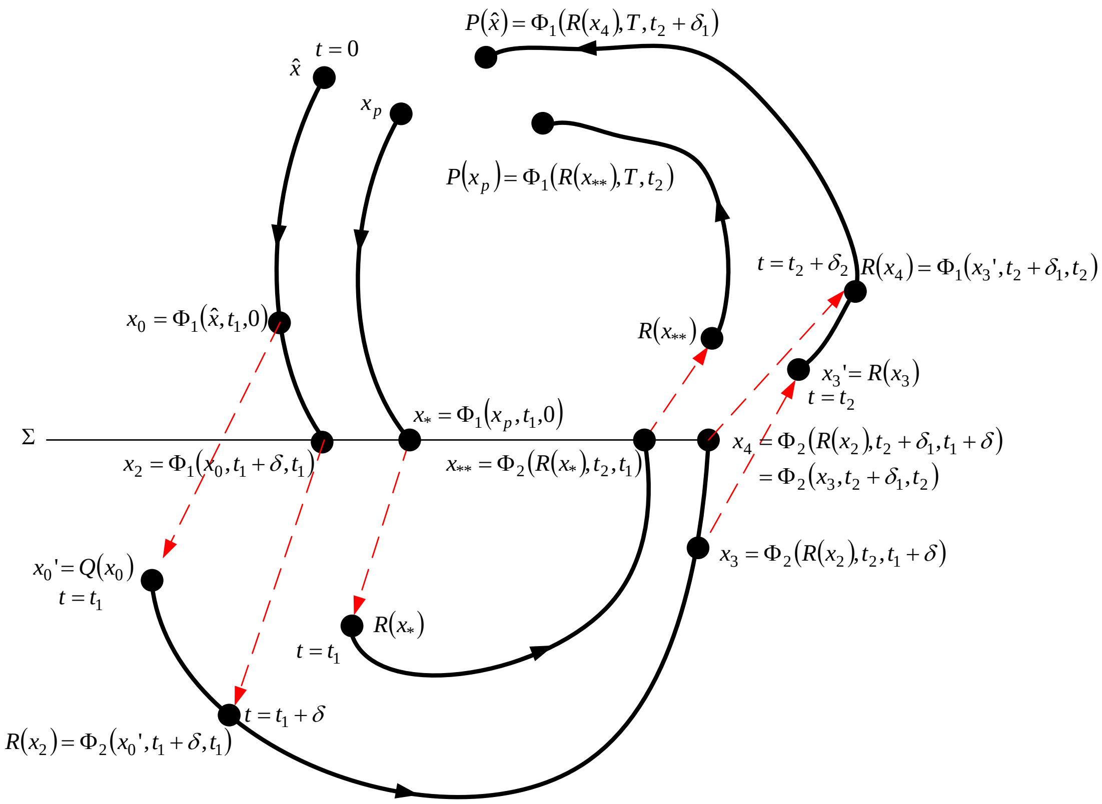
But if we start close to x_p:



Hence if we use the same sequence of matrices $P(\hat{x}) = \Phi_1(R(\Phi_2(R(\Phi_1(\hat{x}, t_1, 0)), t_2, t_1)), T, t_2)$ there is an error as at $t = t_1$ the new orbit is not on Σ (it is on x_0) and it needs an extra time δ (δ can be positive or negative). This is obvious as we must apply the jump matrix to x_2 and not to x_0.

Solution:

So in order to overcome this problem we will still apply a map at x_0 but this map Q will have to take into account that we are not on Σ. This map will take us to another point x_0'. Then using again the flow Φ_2 (for time δ) we will go to $R(x_2)$ and from that point the flow will continue as before. The same correction must be applied on the second intersection at x_4. This map must also insure that if $x_0 \in \Sigma$ then $Q(x_0) = R(x_0)$ (i.e. if $\delta = 0$ then $Q(x_0) = R(x_0)$).



So that the new Poincare map is $P(\hat{x}) = \Phi_1(Q(\Phi_2(Q(\Phi_1(\hat{x}, t_1, 0)), t_2, t_1)), T, t_2)$ and by using the fact that $Q(x(t)) = R(x(t))$ if $x(t) \in \Sigma$:

$$P(x_p) = \Phi_1(Q(\Phi_2(Q(\Phi_1(x_p, t_1, 0)), t_2, t_1)), T, t_2) \tag{2}$$

and hence the Jacobian:

$$P_x(x_p) = \Phi_{1_X}(R(x_{**}), T, t_2) \times Q_x(x_{**}) \Phi_{2_X}(R(x_*), t_2, t_1) \times Q_x(x_*) \times \Phi_{1_X}(x_p, t_1, 0) \tag{3}$$

Calculation of the correction map *Q*:

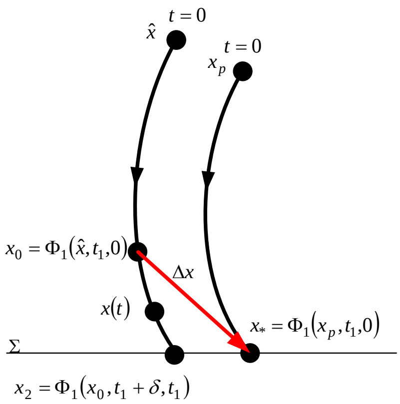
First we need to find where the point x_2 is and hence through that the time δ.

Using a TS on $x(t) = \Phi_1(x_0, t_1 + t, t_1)$ wrt *t* around 0 (in this case $x(t) \in [x_0, x_2]$):

$$x(t) = \Phi_1(x_0, t_1 + t, t_1) = \Phi_1(x_0, t_1, t_1) + \left.\frac{\partial \Phi_1(x_0, t_1 + t, t_1)}{\partial t}\right|_{t=0} (t - 0) + O(t^2) \Rightarrow$$

$$x(t) = x_0 + F_1(x_0)t + O(t^2) \tag{4}$$

By setting $x_0 = x_* + \Delta x$:



$$x(t) = x_* + \Delta x + F_1(x_* + \Delta x)t + O(t^2) \tag{5}$$

Using a TS at $F_1(x_* + \Delta x)$ wrt *x* around x_*:

$$F_1(x_* + \Delta x) = F_1(x_*) + F_{1x}(x_*)\Delta x + O(\Delta x^2) \tag{6}$$

$$(5) \overset{(6)}{\Rightarrow} x(t) = x_* + \Delta x + \left(F_1(x_*) + F_{1x}(x_*)\Delta x + O(\Delta x^2)\right) t + O(t^2)$$

$$x(t) = x_* + \Delta x + F_1(x_*)t + O(\Delta x^2, t^2) \tag{7}$$

Or $x(t) = x_* + \Delta x + F_1(x_*)t$ (8)

Also using the TS on $H(x(t),t)$ wrt x, t around x_*, t_1:

$$H(x(t),t) = \underbrace{H(x_*,t_1)}_{0} + H_x(x_*,t_1)(\Delta x + F_1(x_*)t) + H_t(x_*,t_1)t \Leftrightarrow$$
$$H(x(t),t) = H_x(x_*,t_1)(\Delta x + F_1(x_*,t_1)t) + H_t(x_*,t_1)t \Leftrightarrow$$
$$H(x(t),t) = H_x(x_*,t_1)\Delta x + H_x(x_*,t_1)F_1(x_*)t + H_t(x_*,t_1)t \Leftrightarrow$$

Also $H(x(\delta),\delta) = 0$:

$$H(x(\delta),\delta) = H_x(x_*,t_1)\Delta x + H_x(x_*,t_1)F_1(x_*)\delta + H_t(x_*,t_1)\delta \Leftrightarrow$$
$$0 = H_x(x_*,t_1)\Delta x + (H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1))\delta \Leftrightarrow$$

$$\boxed{\delta = -\frac{H_x(x_*,t_1)\Delta x}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)}} \qquad (9)$$

Now, $x(t) = \Phi_2(R(x_2), t_1 + \delta - t, t_1 + \delta) \in S_-$

Using a TS on $x = \Phi_2(R(x_2), t_1 + \delta - t, t_1 + \delta)$ wrt t around 0:

$$x(t) = \Phi_2(R(x_2), t_1 - t + \delta, t_1 + \delta) \Leftrightarrow$$
$$x(t) = \Phi_2(R(x_2), t_1 + \delta - 0, t_1 + \delta) + \left.\frac{\partial \Phi_2(R(x_2), t_1 + \delta - t, t_1 + \delta)}{\partial t}\right|_{t=0} (t - 0) + O(t^2) \Leftrightarrow$$
$$x(t) = \Phi_2(R(x_2), t_1 + \delta - 0, t_1 + \delta) + \left.\frac{\partial \Phi_2(R(x_2), t_1 + \delta - t, t_1 + \delta)}{\partial t}\right|_{t=0} t$$

$$x(t) = R(x_2) + F_2(R(x_2))t \qquad (11)$$

By setting $t = -\delta$:

$$x(-\delta) = Q(x_0) = R(x_2) - F_2(R(x_2))\delta \qquad (12)$$

But using (8) at $t = \delta$, $x_2 = x_* + \Delta x + F_1(x_*)\delta$ and hence (12) can be written as

$$Q(x_0) = R(x_* + \Delta x + F_1(x_*)\delta) - F_2(R(x_* + \Delta x + F_1(x_*)\delta))\delta \qquad (13)$$

Using a TS on $R(x_* + \Delta x + F_1(x_*)\delta)$ wrt x at $x=x_*$:

$$R(x_* + \Delta x + F_1(x_*)\delta) = R(x_*) + R_x(x_*)(\Delta x + F_1(x_*)\delta) \qquad (14)$$

Using a TS on $F_2(R(x_* + \Delta x + F_1(x_*)\delta))$ wrt to $R(x)$ at $R(x) = R(x_*)$:

$$
\begin{aligned}
F_2\left(R\left(x_* + \Delta x + F_1(x_*)\delta\right)\right) &= F_2\left(R(x_*) + R_x(x_*)(\Delta x + F_1(x_*)\delta)\right) \\
&= F_2\left(R(x_*)\right) + \underbrace{\left.\frac{\partial F_2(R(x))}{\partial R(x)}\right|_{R(x)=R(x*)}}_{0} R_x(x_*)(\Delta x + F_1(x_*)\delta) \\
&= F_2\left(R(x_*)\right)
\end{aligned}
$$

Thus $Q(x_0) = R(x_*) + R_x(x_*)(\Delta x + F_1(x_*)\delta) - F_2(R(x_*))\delta \Rightarrow$

$$
Q(x_0) = R(x_*) + R_x(x_*)\Delta x + \left(R_x(x_*)F_1(x_*) - F_2(R(x_*))\right)\delta \tag{15}
$$

Using (10) in (15):

$$
Q(x_0) = R(x_*) + R_x(x_*)\Delta x + \left(-R_x(x_*)F_1(x_*) + F_2(R(x_*))\right)\frac{H_x(x_*)\Delta x}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)} \Leftrightarrow
$$

$$
Q(x_0) = R(x_*) + R_x(x_*)\Delta x + \left(F_2(R(x_*)) - R_x(x_*)F_1(x_*)\right)\frac{H_x(x_*)\Delta x}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)} \Leftrightarrow
$$

$$
Q(x_0) = R(x_*) + \left(R_x(x_*) + \left(F_2(R(x_*)) - R_x(x_*)F_1(x_*)\right)\frac{H_x(x_*)}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)}\right)\Delta x \Leftrightarrow
$$

$$
\boxed{Q(x_0) = R(x_*) + \left(R_x(x_*) + \left(F_2(R(x_*)) - R_x(x_*)F_1(x_*)\right)\frac{H_x(x_*)}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)}\right)(x_0 - x_*)} \tag{16}
$$

Thus the Jacobian is:

$$
\boxed{\frac{\partial Q(x_0)}{\partial x_0} = Q_x(x_*) = R_x(x_*) + \left(F_2(R(x_*)) - R_x(x_*)F_1(x_*)\right)\frac{H_x(x_*)}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)}} \tag{17}
$$

This is the **saltation** matrix.

A similar case is when we have a pure impact. In that case we expect F_1=F_2:

$$
\boxed{Q(x_0) = R(x_*) + \left(R_x(x_*) + \frac{\left(F(R(x_*)) - R_x(x_*)F(x_*)\right)H_x(x_*,t_1)}{H_x(x_*,t_1)F(x_*) + H_t(x_*,t_1)}\right)(x_0 - x_*)} \tag{18}
$$

A similar case is when we have a pure transversal intersection. In that case we expect *R(x)=x*:

$$
\boxed{Q(x_0) = x_* + \left(I + \left(F_2(x_*) - F_1(x_*)\right)\frac{H_x(x_*)}{H_x(x_*,t_1)F_1(x_*) + H_t(x_*,t_1)}\right)(x_0 - x_*)} \tag{19}
$$