A 9 page: "Topology for dummies"

Metric Spaces

The Euclidean distance is:

- 1 dimensional: $d^{(1)}(x,y)=|x-y|$
- 2 dimensional: $d^{(2)}(\mathbf{x},\mathbf{y})=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$
- n dimensional: $d^{(n)}(\mathbf{x},\mathbf{y})=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\cdots(x_n-y_n)^2}$

All these metrics satisfy three conditions (*x*, *y* can also be vectors in $\mathbb{R}^n$):

1. $d(x,y)>0$ if $x\neq y$ and $d(x,y)=0$ if $x=y$
2. $d(x,y)=d(y,x)\ \forall x,y$
3. $d(x,y)\leq d(x,z)+d(y,z)$

Other functions can also be defined that satisfy these three properties:

Taxi-cab metric: $e_1(\mathbf{x},\mathbf{y})=|x_1-y_1|+|x_2-y_2|$

Discrete metric: $d_0(x,y)=\begin{cases}0, & x=y\\ 1, & x\neq y\end{cases}$

Cantor metric: Used for infinite sequences of zeros and ones

Max metric on C[0, 1] (all continuous functions defined on [0, 1]):

$d_{\max}(f,g)=\max\{|g(x)-f(x)|:x\in[0,1]\}$

….

Now, we can combine a set *X* (like $\mathbb{R}^m$) with a metric *d* to have a metric space:

> A set *X* (could be $\mathbb{R}^n$) along with a metric *d* is a metric space (*X*, *d*)

Thus we can define metric spaces more generic than the Euclidean metric space $(\mathbb{R}^m, d^{(m)})$ like $(C[0,1], d_{\max})$ the space of all continuous functions on $C[0,1]$ with the $d_{\max}$ metric.

Obviously we can define more than one metric on the same space X. Then if we can find *m*, *M* for all $x,y\in X$: $md_1(x,y)\leq d_2(x,y)\leq Md_1(x,y)$ then the metrics d_1 and d_2 are called metrically equivalent.

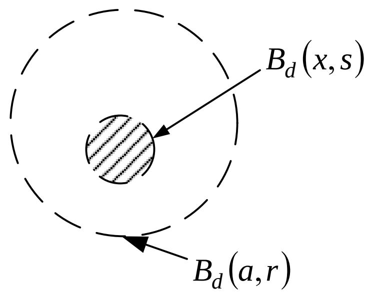
We can also define a metric subspace (A,d_A) when $A\subset X$ and d_A is the same as d but is defined on *A*.

Finally we can use the concept of a generic metric to define an open ball: $B_d(a,r)=\{x:d(a,x)<r\}$, i.e. all the points *x* that are at a distance less than *r* from the point *a*. The concept of distance of course depends on the metric *d*. For example $B_{d^{(1)}}(1,1)=(0,2)$ while $B_{d_0}(1,1)=\mathbb{R}$.

Open sets (1)

In a Euclidian metric space like $\left(\mathbb{R}^1, d^{(1)}\right)$ we can define an open set such as $(2, 3)$. The main property of this set is that we can always define another open set in any of each points. On the other hand we cannot take an open set around 2 for $[2, 3)$.

In a more general case, in any metric space (X, d), we can take an open ball as $B_d(a,r)$. And in any point $x \in B_d(a,r)$ we can take another open ball $B_d(x,s)$. We call this property the fried egg property:



We can use that fried egg property to define an open set as a set where every point has the fried egg property. Of course the idea of the open ball is directly connected with the metric that we use (as in the idea of the open ball). For example in $\mathbb{R}^1$ with the normal Euclidian metric an open ball will look just like the above figure. But if we use the discrete metric then if the radius is greater than 1 the open ball will cover the whole set $\mathbb{R}^1$.

Thus we have to say which metric we use when we define/mention an open set:
d-open set if we are in (X, d).

Also if we have metrically equivalent metrics then they define the same open sets, i.e. a subset of *X* is open wrt both metrics.

Using the above definition of an open set we have three main properties:
1. The union of any number of open sets is also an open set.
2. The intersection of a finite number of open sets is also an open set (if we have an intersection of infinite number of open sets then we could end up with a single point).
3. In a metric space (X, d) the sets $X, \varnothing$ are open.

These 3 properties are effectively the corner stone of topology!!!

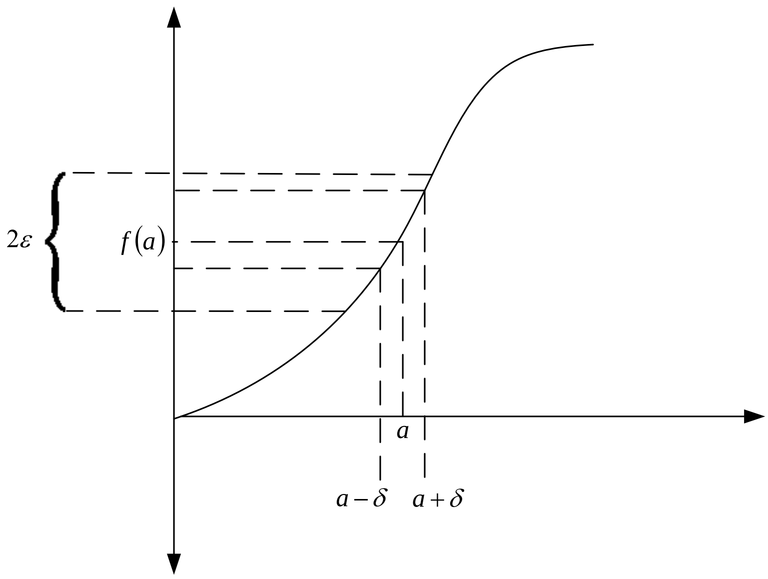
Continuity in metric spaces

The classical $\varepsilon-\delta$ definition of continuity of a function is:

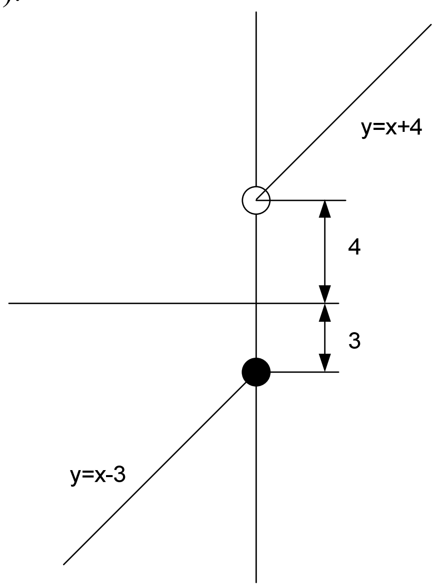
$f:\mathbb{R}^n \to \mathbb{R}^m$ is continuous at $a \in X$ iff:

$\forall \varepsilon > 0\, \exists\, \delta > 0: \quad d^{(m)}(f(\mathbf{x}), f(\mathbf{a})) < \varepsilon \textit{ whenever } d^{(n)}(\mathbf{x},\mathbf{a}) < \delta$

I.e. for any given ε I need to find a range of *x* such that the range of *f*(*x*) will be smaller than 2ε or from a different point of view I need to find a range of *x* around a point *a* such that *f*(*x*) will be in (or just) a predetermined range around *f*(*a*):



For example in 1D ($\mathbf{a}=0$):



$d^{(1)}(f(x), f(0)) = |f(x) - f(0)| = |f(x)+3|$

Hence if $\varepsilon = 8$: $|f(x)+3| < 8 \Leftrightarrow -8 < f(x)+3 < 8 \Leftrightarrow -11 < f(x) < 5$

Hence I need to find a range of *x* around 0 such that *f*(*x*) will get values only into the interval $(-11, 5)$. If $\delta = 1$ then $f(x) \in [-4, -3] \cup (4, 5]$.

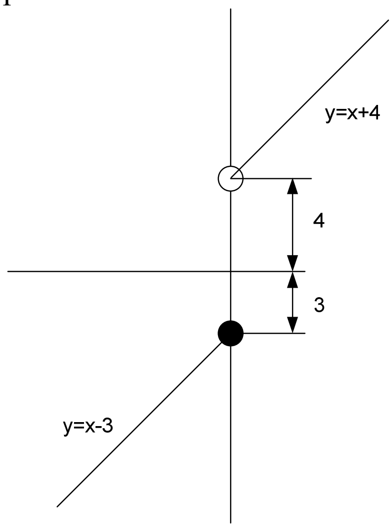
On the other hand if $\varepsilon = 5$ then $|f(x)+3| < 5 \Leftrightarrow -5 < f(x)+3 < 5 \Leftrightarrow -8 < f(x) < 2$

Now whatever the value of *x* when *x*>0 then *f*(*x*)>4 and hence *f* is not continuous.

We can use this definition for more generic metric spaces:
Let (*X*, *d*) and (*Y*, *e*) be 2 metric spaces $f : X \to Y$ is continuous at $a \in X$ iff:
$\forall \varepsilon > 0 \, \exists \, \delta > 0: \quad e(f(x), f(a)) < \varepsilon \text{ whenever } d(x,a) < \delta$.

As we use metrics to define the conditions $e(x,y) < \varepsilon \,\&\, d(x,y) < \delta$ the continuity depends on the specific metric that we use.

For example if we have the previous function:



and in its domain we use the discrete metric (i.e. $d = d_0$) then for $\varepsilon = 5$ (or for any $\varepsilon > 0$): $|f(x) - f(0)| < 5 \Leftrightarrow |f(x)+3| < 5 \Leftrightarrow -8 < f(x) < 2$. Hence we are looking for a nonzero value of δ such as $8 < f(x) < 2$ when $d_0(x,0) < \delta$. But if we choose $\delta = 0.5$ (or any other number less than 1) the only point that satisfies $d_0(x,0) < 0.5$ is $x = 0$ and thus $8 < f(x) < 2$.

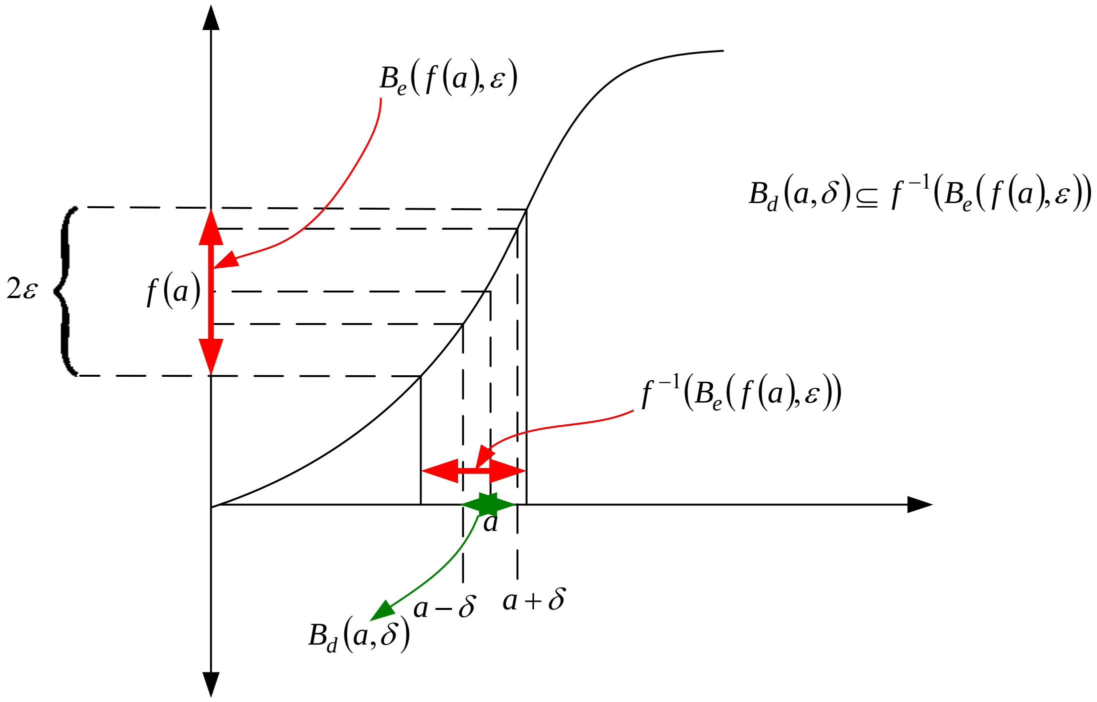
If we have 2 metrics on the same set that are metrically equivalent then if a function is continuous wrt one it is also wrt the other.

Another way to see this is to see the expressions $d(x,y) < 0$ and $e(x,y) < 0$ as open balls (around *a* and *f*(*a*) respectively):
Let (*X*, *d*) and (*Y*, *e*) be 2 metric spaces $f : X \to Y$ is continuous at $a \in X$ iff:
$\forall \varepsilon > 0 \, \exists \, \delta > 0: \quad f(x) \in B_e(f(a), \varepsilon) \text{ whenever } x \in B_d(a, \delta)$:

Using the previous example we see that if $\varepsilon = 5$ then for every $\delta > 0$
$\exists x \in B_d(a, \delta): f(x) \notin B_e(f(a), \varepsilon)$

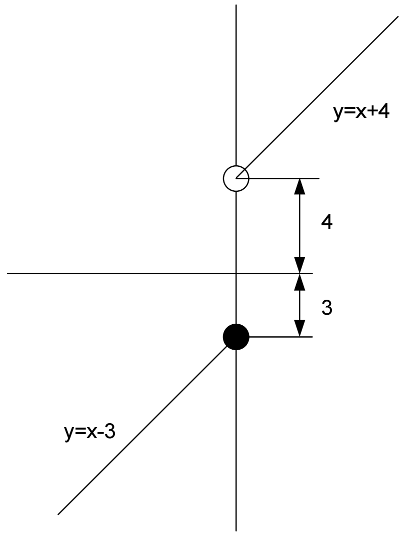
Hence now we can define continuity in more abstract spaces like $(C[0,1], d_{\max})$.

We can also define the continuity using the inverse images of functions:
Let (*X*, *d*) and (*Y*, *e*) be 2 metric spaces $f : X \to Y$ is continuous at $a \in X$ iff:
$\forall \varepsilon > 0 \, \exists \, \delta > 0: B_d(a, \delta) \subseteq f^{-1}(B_e(f(a), \varepsilon))$:



Finally we can define continuity in terms of open sets. The main property here is that if we take the inverse image of an open ball then we have an open set (it could be a ball or a union of balls). Hence, let (*X*, *d*) and (*Y*, *e*) be 2 metric spaces $f: X \to Y$ is continuous at $a \in X$ iff: $f^{-1}(U)$ is *d*-open subset of *X* when *U* is an *e*-open set of *Y*.

For example if we have $f(x) = 0.5x$ ($\mathbb{R}^1 \to \mathbb{R}^1$) and we take an open set $(2, 4)$ in the co-domain which will result to $f^{-1}(2, 4) = (4, 8)$, i.e. another open set. On the other hand if we have something like this:



Then if we take the open set $U = (-1, -0.5)$ we have that $f^{-1}(U) = \varnothing$ which is open! But if $U = (-4, -2)$ we have that $f^{-1}(U) = [0, -1)$ which is not open!

Topological Spaces

We have seen that a Metric Space (MS) is a set X with a metric d (X, d). Into that MS we define the concept of Open Sets (OS) using the metric d and the idea of the "fried egg property". We have also seen there that the OS have three important properties.

Now, let's try to expand/generalise the concept of the MS. In order to do that we abolish the use of a metric, thus the concept of the "fried egg property" is meaningless now. In our underlying set X we take a collection of subsets T that have the property that if we take the union of any number of subsets from that collection the result will be another subset in that collection and if we take the intersection of a finite number of subsets then again the result will belong to this collection T. We also need to take the set X (as the union may result in X) and the empty set (for the intersection of 2 disjoint sets) to be in T. Now, these subsets as called open wrt T for obvious reasons. It is important to note that these OS are very different that the OS of a metric space. These sets are simply elements of T that satisfy a couple of specific properties. For example it is likely that the same subset may be open in T_1 while it is not open in T_2, i.e. it belongs to T_1 but not in T_2.

The set X with the collection T define a topological space (X, T_X) and T is called a topology on X.

A special type of a topology in X is formed if we do not take all the sets but we take only those that satisfy the "fried egg property" i.e. those that are d-open. If the underline set X is $\mathbb{R}^n$ then this topology is called the Euclidean Topology.

Finally it does not make sense to talk about OS before we define a topology. The OS that we have learned in high school are effectively based on the Euclidean topology on $\mathbb{R}^n$.

Obviously there are many examples of topologies.

As the continuity was previously defined in terms of open sets (i.e. sets in a Euclidean space that have the fried egg property) we can define continuity in more general topological spaces:
Let (X,T_X) and (Y,T_Y) be topological spaces, a function $f : X \to Y$ is continuous if $f^{-1}(U) \in T_X$ when $U \in T_Y$.

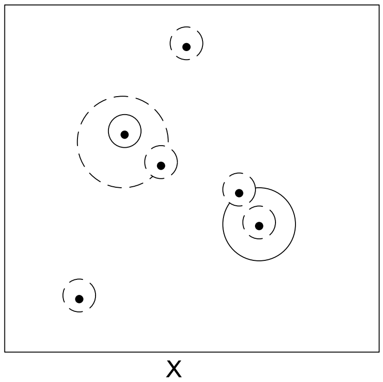
A homeomorphism between 2 topological spaces (X,T_X) and (Y,T_Y) is an onto and one-one function such as $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous. Then the topological spaces (X,T_X) and (Y,T_Y) are called homeomorphic. Effectively this means that $\forall U \in T_Y$, $f^{-1}(U) \in T_X$ and $\forall V \in T_X$, $f^{-1}(V) \in T_Y$, i.e. there is a one-one correspondence between the 2 topological spaces. Then we say that (X,T_X) and (Y,T_Y) are topologically equivalent, i.e. we can study (X,T_X) for some specific properties of each T_X-open sets and export these results to sets in T_Y.

Closed Sets

Let (X,T) be a topological space then a set *D* is called closed (wrt to *T*) if $D^C \in T$.

A neighbourhood of a point *x* in *X* is an *T*-open set *U* that contains *x*. Of course the neighbourhood depends on the topology used!

A closure point $x \in X$ of a set *A* is a point where all of its neighbourhoods intersect A. This effectively means that if we have a *A* in *X* and a generic point *x* in *X* then I take all the possible open sets around that point *x*. If all of them intersect *A* then this point is a closure point. Note that an "internal" point is also a closure point:



The **closure** of a set *A* is the collection of all of its closure points. I.e. the closure of *A* is the smallest possible closed set that contains *A*. Again the closure depends on the topology.

A set *A* is called **dense** in *X* if its closure is *X*, i.e. all points in *x* have all of their neighbourhoods intersecting *A*. If we can find a point *x* in *X* that has just one neighbourhood which does not intersect *A* then *A* is called "nowhere dense".

If I have a set *x* in *A* (not in *X*) and I take a neighbourhood around it and this does not have contain any other element of *A* then this point is called **isolated** point.

A point *x* in *A* is an **interior** point of *A* if it has a (even just one) neighbourhood *U* such as $U \subseteq A$.
A point *x* in *A* is an **exterior** point of *A* if it has a (even just one) neighbourhood *U* such as $U \cap A = \varnothing$.
A point *x* in *A* is a **boundary** point of *A* if all of its neighbourhoods intersect both *A* and A^c.

Properties of Topological Spaces

If we have a topological space (*X*, *T*) and we can find 2 sets *U* & *V* in *T* such as $U \cup V = X$ then (*X*, *T*) is called **disconnected**.
The Euclidean space $\mathbb{R}^n$ is connected and more specificly a subset in $\mathbb{R}$ is connected if it is an interval.

The **component** of *x* in *X* is the largest connected subset of *X* that is connect (wrt *T*).

A topological space is **path connected** if it is connected and there is a path that connects each point *x* in *X* with each point *y* in *X*. A path is a continuous function from $[0, 1]$ (wrt Euclidean topology) to *X*.
Note: in most cases (but not always a connected space is also a path connected).

Assume the set *A*=[0, 1] and a collection $S = \{(-1, 0.1), (0, 0.8), (0.5, 5]\}$ then obviously $\bigcup\limits_{U \in S} U \supseteq [0, 1]$. This collection *S* is called a **cover** of the set [0, 1]. If the sets in *S* are open then we have an **open cover**. It is possible to have infinite sets in *S* to cover the same set *A*: $S = \{(x - 0.1, x + 0.1) : x[0,1]\}$.
As there is an overlapping to the above cover we can take a **subcover**, i.e. a collection of sets that cover *A* that has some elements of *S*. If this subcover has a finite number of sets then it is called a **finite subcover**.

Also, wrt the Euclidean topology every open cover of [0, 1] has a finite subcover.
The finite subcover property (wrt Euclidean topology) applies only to closed and bounded sets.

Using the property of the finite subcover we can prove that a continuous function $f : [a, b] \to \mathbb{R}$ is bounded and attains its least upper and greatest lower bounds. This means that the image of $[a, b]$ under *f* is closed and bounded: **Extreme value theorem**.

We have seen that the finite subcover property (wrt Euclidean topology) applies only to closed and bounded sets. These sets are called **compact** and can be extended to more generic topologies: A topological space is called compact if each open cover of X contains a finite subcover of X. Note that the concept that compact sets are closed and bounded sets does not hold for topologies other than the Euclidean.
Also, every closed subset of a compact set is also compact.

A **Hausdorff space** is a topological space with some specific properties where the compact sets have some nice properties. More specifically a Hausdorff space is a topological space where each pair of distinct points *x*, *y* in *X* can have disjoint neighbourhoods. For example every metric space is a Hausdorff space. Also in a Hausdorff space every set {x} is closed. Previously we said that in the Euclidean topologies the compact sets are those that are closed and bounded. In order to have this connection for a topological space then the space must be Hausdorff, i.e. in a Hausdorff space a compact set is closed and bounded and vice versa.

Sequences in Topological Spaces

A sequence (a_n) converges to l in $\mathbb{R}$ if (a_n-l) is null, i.e. for each ε>0 there exists N such as $d^{(1)}(a_n, l) < \varepsilon \Leftrightarrow |a_n - l| < \varepsilon \Leftrightarrow -\varepsilon < a_n - l < \varepsilon \Leftrightarrow l - \varepsilon < a_n < \varepsilon + l$ when $n>N$. I.e. there is an open set $(l - \varepsilon, l + \varepsilon)$ where (a_n) is when $n>N$.

Now we can expand this definition to any topological space (T, X): a sequence (a_n) converges to a in X if for each neighbourhood U of a there is an N such as a_n is in U when $n>N$.
The interesting property here is **that a sequence may converge to two or more points** if these points have the same neighbourhoods. For example if we use the indiscrete topology then all points have only X as a neighbourhood and hence each sequence converges to all points in X.
Of course if we go back to a metric space then again we need a null sequence to define a convergence.

An interesting property is that a point a is a **closure** of $A \subseteq X$ iff there is a sequence (a_n) in A that converges to a.

We can also define a sequence of functions $f_n : A \to \mathbb{R}$, for example $f_n(x) = x_n$, $x \in [0, 1]$. Then we say that the sequence f_n converges to the function f if for each $x \in A$ then $f_n(x) = f(x)$ as $n \to \infty$. This is the concept of **pointwise convergences**. We can have a more conservative definition if we impose the function that the function f is continuous. Then we have the concept of **uniform convergence**.

It is very important to know when a sequence converges in a metric space. This comes out to be that a sequence converges when its terms become arbitrary close for large n. Hence we have that a sequence converges when for each $\varepsilon > 0$ there exits a $N \in \mathbb{N}$ such $d(a_n, a_m) < \varepsilon$ when $n, m > N$. These sequences are called **Cauchy Sequences**. I.e. every convergent sequence in a metric space is a Cauchy sequence. If we also have the inverse i.e. every Cauchy sequence converges then this metric space is called complete.
The metric spaces $(\mathbb{R}^m, d^{(m)})$ are complete. Also every compact space is complete. But the inverse does not hold as $(\mathbb{R}^m, d^{(m)})$ is complete but is not compact (not closed and bounded).

Contraction mapping

In a metric space (X, d) a function $f : X \to X$ is called a **contraction mapping** if for all points $x, y \in X$ there exist $\lambda \in [0, 1)$ such as $d(f(x), f(y)) \leq \lambda d(x, y)$, i.e. f is Lipschitz function (map) with a Lipschitz constant strictly less than 1. Every contraction map has at least one fixed point. In order to ensure that we have exactly one point we must impose that the given metric space is complete. Then every sequence $x,\ f(x),\ f(f(x)), \ldots$ converges to the fixed point: $d(f^n(x), x_{FP}) \to 0$ as $n \to \infty$.