ZDM

(Another Taylor Series' Heaven!)

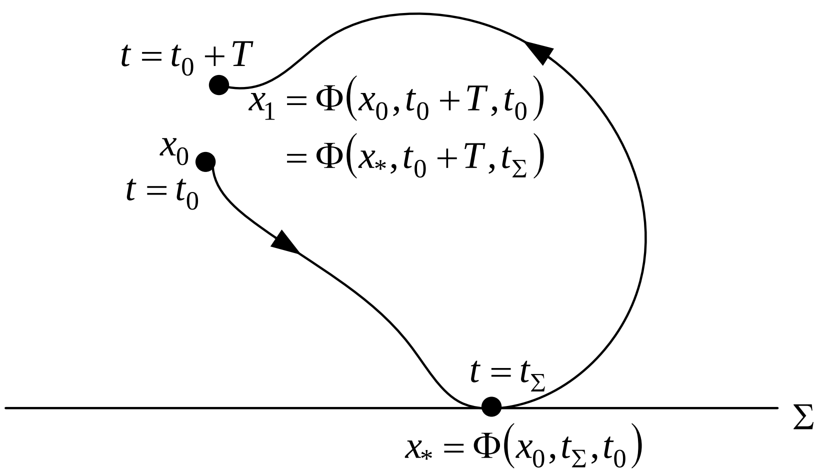
The system is described by:

$$\dot{\mathbf{x}} = F(\mathbf{x}), \quad when\ H(\mathbf{x}) > 0$$
$$\mathbf{x} \to R(\mathbf{x}), \quad when\ H(\mathbf{x}) = 0$$
$$\Sigma = \{x : H(\mathbf{x}) = 0\}$$
$$v(\mathbf{x}) = \frac{dH(\mathbf{x})}{dt} = \frac{dH(\mathbf{x})}{d\mathbf{x}}\frac{d\mathbf{x}}{dt} = H_x(\mathbf{x})F(\mathbf{x})$$
$$a(\mathbf{x}) = \frac{d^2H(\mathbf{x})}{dt^2} = (H_x(\mathbf{x})F(\mathbf{x}))_x F(\mathbf{x})$$

At the grazing point:

$$H(\mathbf{x}_*) = 0, v(\mathbf{x}_*) = 0, a(\mathbf{x}_*) = a_* > 0$$

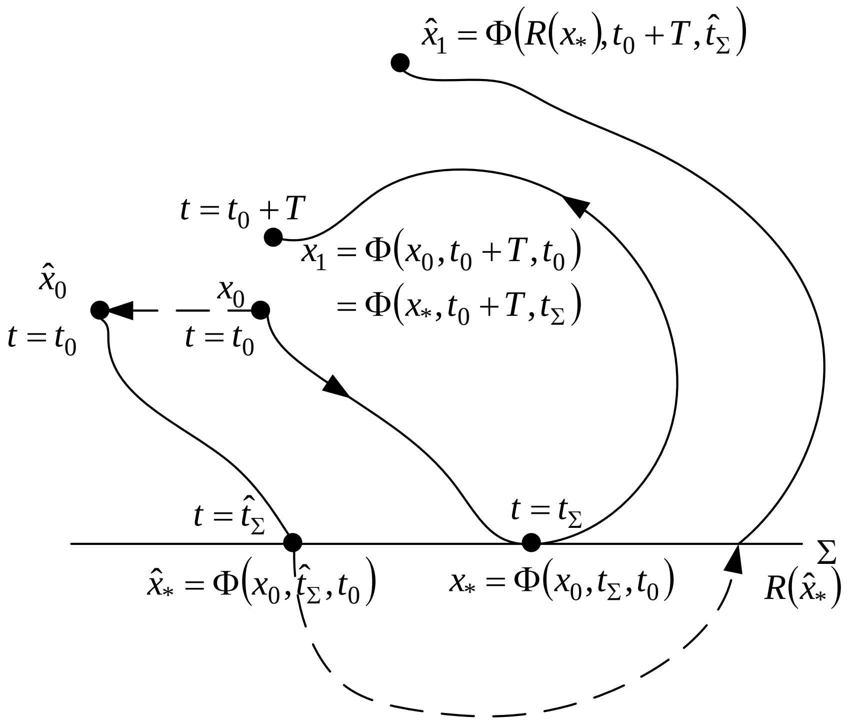
Let's assume an orbit that (near) grazes a switching manifold:



We can define the Poincare map as:

$$x_1 = P(x_0) = \Phi(x_*, T + t_0, t_\Sigma)\Phi(x_0, t_\Sigma, t_0) \tag{1}$$

To study the stability of this orbit we follow the classical Lyapunov method. This means that we add a perturbation and we see how the perturbations behave:

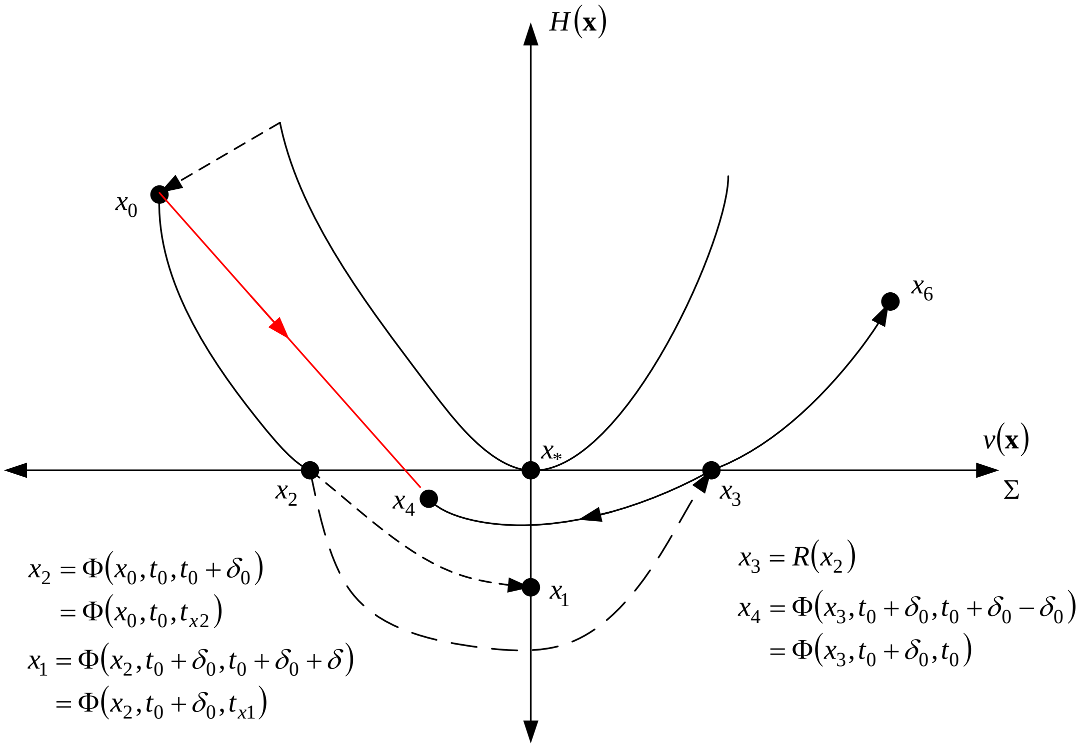


Then if we use the same map as in (1) there is obviously an error as the new orbit is different than the original one.

Thus we need a map that will overcome this problem. This map, that will be applied at a neighbourhood around the grazing point, and will take into account the possibility of crossing the switching manifold. So I expect something like this:

$$x \mapsto \begin{cases} x, & \text{if the orbit that starts in x does not cross } \Sigma \text{ or grazes } \Sigma \\ ZDM(x) & \text{if the orbit that starts in x crosses } \Sigma \text{ or grazes } \Sigma \end{cases} \qquad (2)$$

So lets zoom in an orbit that just grazes Σ (all the previous symbols are re-initialised). Before we do that let's assume that the switching manifold is $h(\mathbf{x}) = x_1$, in that case $v(\mathbf{x}) = x_2$:



The time needed to go from x_0 to x_6 is the same as x_4 to x_6, i.e. by ignoring the impact.

From the above diagram we have that:

$$V(x_1) = 0, \quad H(x_2) = 0, \quad H(x_3) = 0 \tag{3}$$

I define as:

$$H_{\min}(x_0) = \min\{\Phi(x_0, t, t_0)\} = \min\{\Phi(x_0, t_0 + \delta, t_0)\} = H(x_1) \equiv -y^2 \tag{6}$$

For a generic point x near x_* (like x_1, x_2, x_3 and x_4) we have:

$$F(x) = F(x_*) + F_x(x_*)\Delta x + HOT \tag{7}$$

As $F(x_*)$ is not zero we can say that this term dominates the RHS of (7) and thus all other terms can be neglected:

$$F(x) = F(x_*) \tag{8}$$

So from now on we will use $F(x_*)$ instead of $F(x_1), F(x_2), F(x_3), \& F(x_4)$.

Similarly for $a(x_*) = a_*$ instead of $a(x_1), a(x_2), a(x_3), \& a(x_4)$

A point near x_1 is given by: $\Phi(x_1, t_{x1}, t_{x1} + t)$. Using a TS wrt time around t=0:

$$\Phi(x_1, t_{x1}, t_{x1} + t) = \Phi(x_1, t_{x1}, t_{x1}) + \left.\frac{\partial \Phi(x_1, t_{x1}, t_{x1} + t)}{\partial t}\right|_{t=0} t + \left.\frac{\partial^2 \Phi(x_1, t_{x1}, t_{x1} + t)}{\partial^2 t}\right|_{t=0} \frac{t^2}{2} + H($$

Now this flow can be evaluated for $t = -\delta$ to get to x_2 and for $t = -\delta_1$ to get to x_0. This will allow us to connect x_0 with x_1 using a simple expression:

$$\Phi(x_1, t_{x1}, t_{x1} + t) \approx x_1 + F(x_1)t \tag{8b}$$

and by setting $t = -\delta$: I have $x_2 \approx x_1 - F(x_1)\delta$ (9)

And using (8) $x_2 \approx x_1 - F(x_*)\delta$ (10)

Similarly: $x_0 \approx x_1 - F(x_*)\delta_1$ (11)

Similarly: $x_0 \approx x_2 - F(x_*)\delta_0$ (12)

Now using the jump map I can relate x_2 with x_3:

$$x_3 = R(x_2) = x_2 + W(x_2)v(x_2) \tag{13}$$

Hence Using (12):

$$x_3 = R(x_2) = x_0 + F(x_*)\delta_0 + W(x_0 + F(x_*)\delta_0)v(x_0 + F(x_*)\delta_0) \tag{14}$$

A point near x_3 is given by: $\Phi(x_3, t_{x2}, t_{x2} + t)$.

Using a TS wrt time around t=0:

$$\Phi(x_3, t_{x2}, t_{x2} + t) = \Phi(x_3, t_{x2}, t_{x2}) + \left.\frac{\partial\Phi(x_3, t_{x2}, t_{x2} + t)}{\partial t}\right|_{t=0} t + HOT \Leftrightarrow$$
$$\Phi(x_3, t_{x2}, t_{x2} + t) = x_3 + F(x_3)t + HOT$$

Evaluating this at $t = -\delta_0$: $x_4 = x_3 - F(x_3)\delta_0 + HOT$ (15)

And by neglecting HOT and using (8): $x_4 = x_3 - F(x_*)\delta_0$ (16)

Hence combining (16) and (14):

$$x_4 = x_0 + W(x_0 + F(x_*)\delta_0)v(x_0 + F(x_*)\delta_0) \tag{17a}$$

Or $x_4 = x_0 + W(x_2)v(x_2)$ (17b)

In (17b) we have 2 unknown factors: $W(x_2)$ and $v(x_2)$. As the point x_2 is near x_* we can replace $W(x_0 + F(x_*)\delta_0)$ with $W(x_*)$. Thus I simply need to find $v(x_2)$.

Using (10): $x_2 = x_1 - F(x_*)\delta$

Thus $v(x_2) = v(x_1 - F(x_*)\delta)$ (18)

Using a TS on (18) wrt x at x_1:

$$v(x_1 - F(x_*)\delta) = v(x_1) - v_x(x_1)F(x_*)\delta + HOT$$

Originally we have seen that $v(x_1) = 0$, hence:

$$v(x_2) \approx -v_x(x_1)F(x_*)\delta \Leftrightarrow$$
$$v(x_2) = -\underbrace{v_x(x_*)F(x_*)}_{a(x*)}\delta \Leftrightarrow v(x_2) = -a_*\,\delta \tag{19}$$

Thus my only unknown now is δ. To find that I will use (8b) and I will evaluate the function H on that general point: $H\left(x_1 + F(x_1)t + F_x(x_1)F(x_1)\frac{t^2}{2} \right)$

Now, by taking the TS wrt x around x_1

$$H\left(x_1 + F(x_1)t + F_x(x_1)F(x_1)\frac{t^2}{2} \right) = H(x_1) + H_x(x_1)\left(F(x_1)t + F_x(x_1)F(x_1)\frac{t^2}{2} \right) +$$

$$+ H_{xx}(x_1)\frac{\left(F(x_1)t + F_x(x_1)F(x_1)\frac{t^2}{2} \right)^2}{2} + HOT$$

$$= H_{\min}(x_0) + \underbrace{H_x(x_1)F(x_1)}_{v(x_1)=0}t + H_x(x_1)F_x(x_1)F(x_1)\frac{t}{2}$$

$$+ H_{xx}(x_1)F(x_1)F(x_1)\frac{t^2}{2} + HOT$$

$$\approx -y^2 + (H_x(x_1)F_x(x_1)F(x_1) + H_{xx}(x_1)F(x_1)F(x_1))$$

$$= -y^2 + a(x_1)\frac{t^2}{2} \Leftrightarrow$$

$$H(\Phi(x_1, t_{x1}, t_{x1} + t)) = -y^2 + a_* \frac{t^2}{2} \qquad (20)$$

By evaluating (20) at $t = -\delta$ I have that

$$H(\Phi(x_1, t_{x1}, t_{x1} - \delta)) = 0 = -y^2 + a_* \frac{\delta^2}{2} \Rightarrow \delta = \sqrt{\frac{2}{a_*}}y \qquad (21)$$

Thus using (17b), (19) and (21):

$$x_4 = x_0 - W(x_*)a_* \sqrt{\frac{2}{a_*}}y \Leftrightarrow$$

$$x_4 = x_0 - W(x_*)\sqrt{2a_*}y \qquad (22)$$

Thus we have the ZDM map:

$$ZDM(x) = x - W(x_*)\sqrt{2a_*}\,y$$

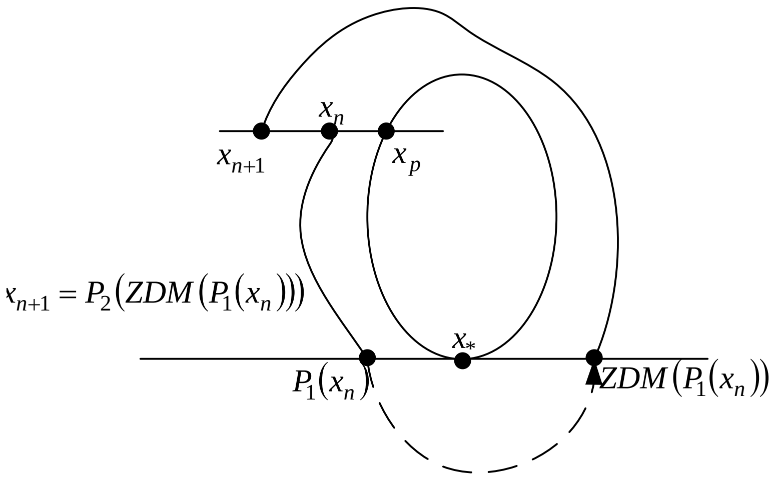
It is interesting to point that the Jacobian of that map is singular at the point of grazing as:

$$\frac{\partial ZDM(x)}{\partial x} = I - W(x_*)\sqrt{2a_*}\,\frac{\partial y}{\partial x} \tag{23}$$

But $\frac{\partial y}{\partial x} = \frac{\partial\sqrt{-H_{\min}(x)}}{\partial x} \approx \frac{\partial\sqrt{-H(x_1)}}{\partial x} = -\frac{\partial H(x_1)}{\partial x}\frac{1}{2\sqrt{-H(x_1)}}$ (24)

And as $x \to x_*$ I have $H(x_1) \to 0$ and thus: $\frac{\partial y}{\partial x} \to \pm\infty$

Now let's see how we can use this map.



The overall map is $x_{n+1} = P_2(ZDM(P_1(x_n)))$, in the case where $x_n = x_p$ I have:

$$x_{n+1} = P_2\left(\underbrace{ZDM\left(\underbrace{P_1(x_p)}_{x^*}\right)}_{x^*}\right) = x_p$$, i.e. indeed $x = x_p$ is a fixed point of our map.

Going back to the definition of the map:

$$x_{n+1} = P_2(ZDM(P_1(x_n))) \qquad (25)$$

In order to simplify the analysis I will semi-linearise that expression. We will still not get an analytic form which we can use to find eigenvalues but it will give us an easier form.

$$P_1(x_n) = P_1(x_p) + \frac{\partial P_1(x_n)}{x_n}(x_n - x_p) = x_* + N_1(x_n - x_p) \qquad (26)$$

Now let's apply the ZDM on that point:

$$ZDM(x_* + N_1(x_n - x_p)) = x_* + N_1(x_n - x_p) - W(x_*)\sqrt{2a_*}\sqrt{-H_{\min}(x_* + N_1(x_n - x_p))}$$

Now the map P_2:

$$P_2\left(x_* + N_1\left(x_n - x_p\right) - W\left(x_*\right)\sqrt{2a_*}\sqrt{-H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right)}\right) \tag{27}$$

And using a TS wrt x around x_*:

$$\begin{aligned}
&P_2\left(x_*\right) + \left.\frac{\partial P_2(x)}{\partial x}\right|_{x=x*}\left(N_1\left(x_n - x_p\right) - W\left(x_*\right)\sqrt{2a_*}\sqrt{-H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right)}\right) \\
&= P_2\left(x_*\right) + N_2\left(N_1\left(x_n - x_p\right) - W\left(x_*\right)\sqrt{2a_*}\sqrt{-H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right)}\right) \\
&= x_p + N_2 N_1\left(x_n - x_p\right) - N_2 W\left(x_*\right)\sqrt{2a_*}\sqrt{-H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right)}
\end{aligned}$$

We can simplify this expression even more by using a TS on $H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right)$:

$$\begin{aligned}
H_{\min}\left(x_* + N_1\left(x_n - x_p\right)\right) &= \underbrace{H_{\min}\left(x_*\right)}_{0} + \left.\frac{\partial H_{\min}(x)}{\partial x}\right|_{x=x*} N_1\left(x_n - x_p\right) \\
&= H_{x\min}\left(x^*\right) N_1\left(x_n - x_p\right) \\
&= H_x\left(x^*\right) N_1\left(x_n - x_p\right)
\end{aligned}$$

The TS expansion of (25) wrt x_n around x_p will give:

$$x_{n+1} = P_2\left(ZDM\left(P_1\left(x_p\right)\right)\right) + \left.\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial x_n}\right|_{x_n=x_p}\left(x_n - x_p\right) \Leftrightarrow$$

$$x_{n+1} = x_p + \left.\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial x_n}\right|_{x_n=x_p}\left(x_n - x_p\right) \tag{26}$$

The Jacobian of this map:

$$\left.\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial x_n}\right|_{x_n=x_p} = \left.\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial ZDM\left(P_1\left(x_n\right)\right)}\right|_{x_n=x_p}\left.\frac{\partial ZDM\left(P_1\left(x_n\right)\right)}{\partial P_1\left(x_n\right)}\right|_{x_n=x_p}\left.\frac{P_1\left(x_n\right)}{\partial x_n}\right|_{x_n=x_p}$$

Now $\left.\frac{P_1\left(x_n\right)}{\partial x_n}\right|_{x_n=x_p} = N_1$ and if $x_n = x_p$ I have that $P_1\left(x_n\right) = x_*$ and thus

$ZDM\left(x_*\right) = x_*$. Hence $\left.\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial ZDM\left(P_1\left(x_n\right)\right)}\right|_{x_n=x_p} = \left.\frac{\partial P_2(\hat{x})}{\partial \hat{x}}\right|_{\hat{x}=x*} = N_2$

Similarly, $\left.\frac{\partial ZDM\left(P_1\left(x_n\right)\right)}{\partial P_1\left(x_n\right)}\right|_{x_n=x_p} = \left.\frac{\partial ZDM\left(\hat{\hat{x}}\right)}{\partial P_1\left(\hat{\hat{x}}\right)}\right|_{\hat{\hat{x}}=x_p} = N_1$

Thus: $\frac{\partial P_2\left(ZDM\left(P_1\left(x_n\right)\right)\right)}{\partial x_n} = N_2 J_{ZDM} N_1$ (27)