Variational Equation or Continuous Dependence on Initial Condition or Trajectory Sensitivity & Floquet Theory & Poincaré Map

1. General idea of trajectory sensitivity

- Assume a solution $\phi(t, t_0, x_0)$ to an IVP
- How will the orbit behave if we start very close to it?

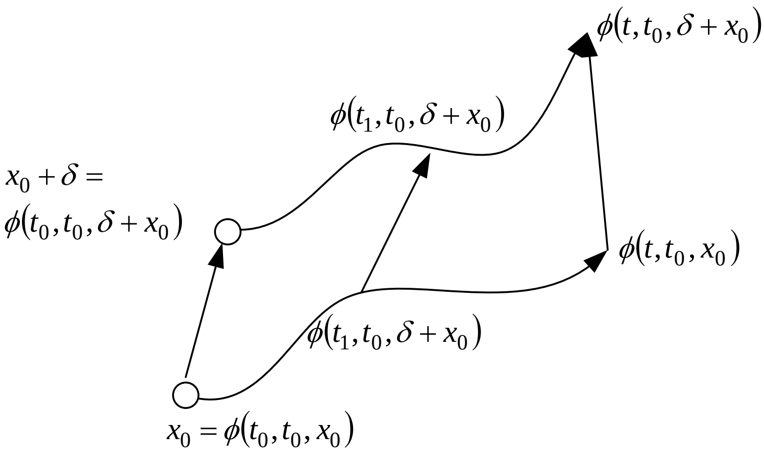


Figure 1

- We define:

  a. $\delta = \phi(t_0, t_0, \delta + x_0) - \phi(t_0, t_0, x_0) = \Delta\phi(t_0, t_0, x_0)$

  b. $\Delta\phi(t, t_0, x_0) = \phi(t, t_0, \delta + x_0) - \phi(t, t_0, x_0)$

- Now we have to express $\Delta\phi(t, t_0, x_0)$ as a function of the original perturbation:

$$\phi(t, t_0, \delta + x_0) \overset{TS}{=} \phi(t, t_0, x_0) + \frac{\partial\phi(t, t_0, x_0)}{\partial x_0}\delta \Leftrightarrow$$

$$\underbrace{\phi(t, t_0, \delta + x_0) - \phi(t, t_0, x_0)}_{\Delta\phi(t, t_0, x_0)} = \frac{\partial\phi(t, t_0, x_0)}{\partial x_0}\delta \Leftrightarrow$$

$$\boxed{\Delta\phi(t, t_0, x_0) = \frac{\partial\phi(t, t_0, x_0)}{\partial x_0}\Delta\phi(t_0, t_0, x_0)} \qquad (1)$$

This is the variational equation.

For $t = t_0$ we have that

$$\Delta\phi(t_0, t_0, x_0) = \frac{\partial\phi(t_0, t_0, x_0)}{\partial x_0}\Delta\phi(t_0, t_0, x_0) \Leftrightarrow \frac{\partial\phi(t_0, t_0, x_0)}{\partial x_0} = I_{n\times n}$$

So if we want to see the sensitivity to the ICs we need to find $\dfrac{\partial \phi(t, t_0, x_0)}{\partial x_0}$

2. Homogeneous Linear Time Invariant System

If we have a linear system $\dot{x}(t) = Ax(t)$ then the solution is:

$\phi(t, t_0, x_0) = e^{A(t-t_0)} x(t_0)$. Then of course: $\boxed{\dfrac{\partial \phi(t, t_0, x_0)}{\partial x_0} = e^{A(t-t_0)}}$ (2)

Notice that through that last equation we can give another definition for the STM (in this exponential matrix): the partial derivative of the solution wrt the initial condition: $\dfrac{\partial \phi(t, t_0, x_0)}{\partial x_0}$

Thus: $\Delta\phi(t, t_0, x_0) = e^{A(t-t_0)} \Delta\phi(t_0, t_0, x_0)$

3. Non - Homogeneous Linear Time Invariant System

If we have a linear system $\dot{x}(t) = Ax(t) + BU(t)$ then the solution is:

$$\phi(t, t_0, x_0) = e^{A(t-t_0)} x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} BU(\tau) d\tau$$

Then of course: $\boxed{\dfrac{\partial \phi(t, t_0, x_0)}{\partial x_0} = e^{A(t-t_0)}}$ (3)

Thus: $\Delta\phi(t, t_0, x_0) = e^{A(t-t_0)} \Delta\phi(t_0, t_0, x_0)$

(see example 1 in the following page)

Example 1 (LTI):

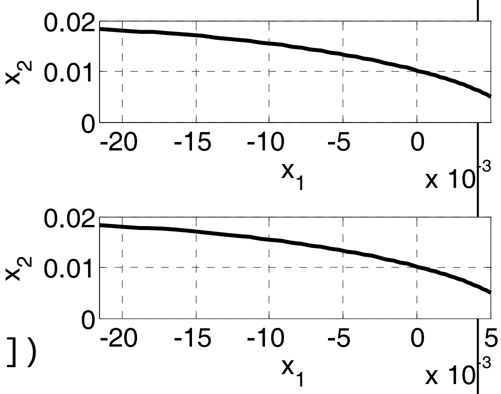
```
% initial
clc, clear, close
syms tau t; t0=0;
A=[1 -4; 2 3]; B=[1;1]; U=sin(t);Utau=sin(tau);

% nominal orbit
x0=[0.01;0.01];
xsol=expm(A*(t-t0))*x0+int(expm(A*(t-
tau))*B*Utau,tau,0,t);
cnt=1;
t2=0:0.01:0.5;
for t1=t2;
xsol1(cnt,1:2)=subs(xsol,t,t1);
cnt=cnt+1;
end


% perturbed orbit
x0=[0.01;0.01]+0.005;
xsol=expm(A*(t-t0))*x0+int(expm(A*(t-
tau))*B*Utau,tau,0,t);
cnt=1;
t2=0:0.01:0.5;
for t1=t2;
xsol2(cnt,1:2)=subs(xsol,t,t1);
cnt=cnt+1;
end

% plot there difference
subplot(2,1,1)
plot(xsol2(:,1)-xsol1(:,1),xsol2(:,2)-xsol1(:,2))
xlim([min(xsol2(:,1)-xsol1(:,1)), max(xsol2(:,1)-
xsol1(:,1))])

% Calculate the diff
Dx0=[0.005; 0.005];
J=expm(A*(t-t0));
cnt=1;
t2=0:0.01:0.5;
for t1=t2;
Dx(cnt,1:2)=subs(J,t,t1)*Dx0;
cnt=cnt+1;
end
subplot(2,1,2)
plot(Dx(:,1),Dx(:,2),'r')
xlim([min(Dx(:,1)), max(Dx(:,1))])
```



4. Homogeneous Linear Time Varying System

If we have a linear system $\dot{x}(t) = A(t)x(t)$

Then the solution is $\phi(t, t_0, x_0) = \Phi_{STM}(t, t_0)x_0$ and hence of course

$$\boxed{\frac{\partial \phi(t, t_0, x_0)}{\partial x_0} = \Phi_{STM}(t, t_0)} \tag{4}$$

To find now this STM we have to numerically solve:

$$\frac{d}{dt}\Phi_{STM}(t, t_0) = A(t)\Phi_{STM}(t, t_0),\ \Phi_{STM}(t_0, t_0) = I_{n \times n}$$

Notice that through that last equation we can give another definition for the STM (in this case: the partial derivative of the solution wrt the initial condition: $\frac{\partial \phi(t, t_0, x_0)}{\partial x_0}$

Example 2 (LTV):

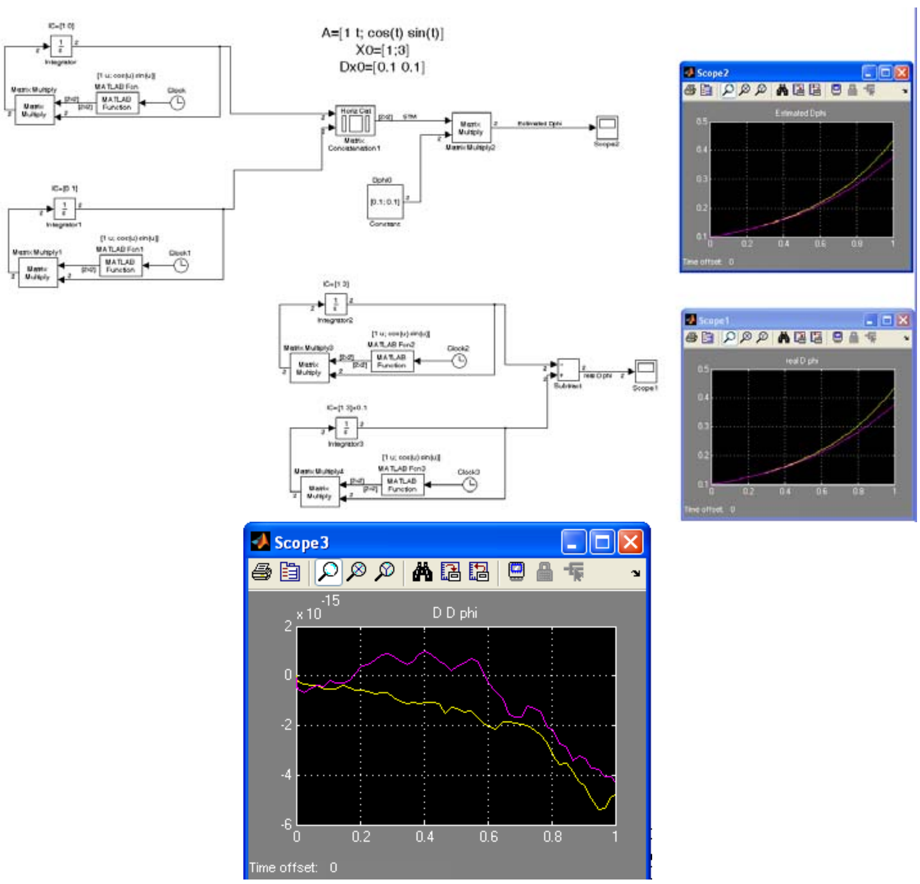


Figure 2

5. Non – Homogeneous Linear Time Varying System

The analysis here is exactly the same as before so no examples are given.

6. Nonlinear system

If we have a nonlinear system:

$$\phi(t,t_0,x_0)=x_0+\int_0^t f(\phi(\tau,t_0,x_0),\tau)d\tau \Leftrightarrow$$

$$\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}=I+\int_0^t \frac{\partial f(\phi(\tau,t_0,x_0),\tau)}{\partial x_0}d\tau \Leftrightarrow$$

$$\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}=I+\int_0^t \frac{\partial f(\phi(\tau,t_0,x_0),\tau)}{\partial \phi(\tau,t_0,x_0)}\frac{\partial\phi(\tau,t_0,x_0)}{\partial x_0}d\tau \Leftrightarrow$$

$$\frac{d}{dt}\left(\underbrace{\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}}_{Y(t)}\right)=\underbrace{\frac{\partial f(\phi(t,t_0,x_0),t)}{\partial\phi(t,t_0,x_0)}}_{Jacobian}\frac{\partial\phi(\tau,t_0,x_0)}{\partial x_0}\Leftrightarrow$$

$$\frac{d}{dt}\left(\phi_{x_0}(t,t_0,x_0)\right)=\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_0)}\phi_{x_0}(t,t_0,x_0)$$

Now, the RHS must be equal to the LHS for $t=t_0$ which means that $\phi_{x_0}(t_0,t_0,x_0)=I_{n\times n}$. Also, in a linear system $\frac{\partial f(x,t)}{\partial x}=A(t)$ which means again that $\phi_{x_0}(t,t_0,x_0)$ is nothing more than the STM.

Example 3 (NL):

Consider the system $\left.\begin{aligned}\frac{dx}{dt}&=-y+x\left(1-x^2-y^2\right)\\ \frac{dy}{dt}&=x+y\left(1-x^2-y^2\right)\end{aligned}\right\}$. The numerical solution for $x(0)=1$ and $y(0)=0$ is:

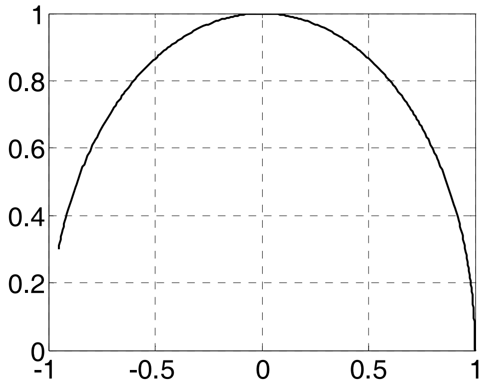


Figure 3

The Jacobian is: $\frac{\partial f(x)}{\partial x} = \begin{bmatrix} 1-3x^2-y^2 & -1-2xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$ which needs to be evaluated along the orbit $\phi(t, t_0, x_0)$ in order to find the STM:

$$\frac{d}{dt}\left(\phi_{x_0}(t, t_0, x_0)\right) = \left.\frac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_0)} \phi_{x_0}(t, t_0, x_0),\ \phi_{x_0}(t_0, t_0, x_0) = I_{n\times n}$$

The Jacobian is:

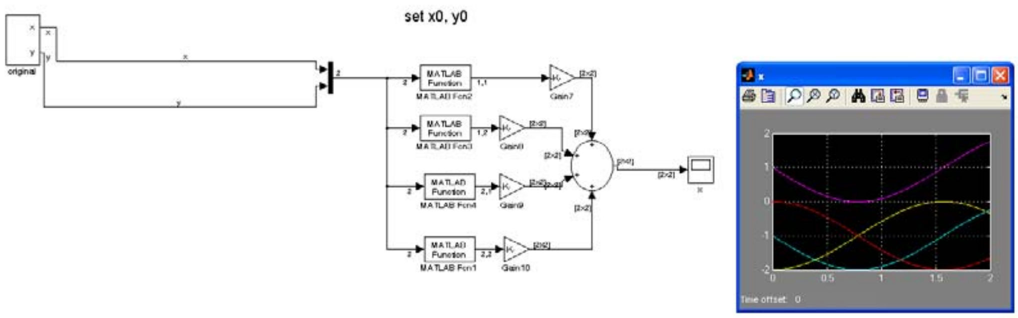


Figure 4

Hence, after 2s the STM has the value:

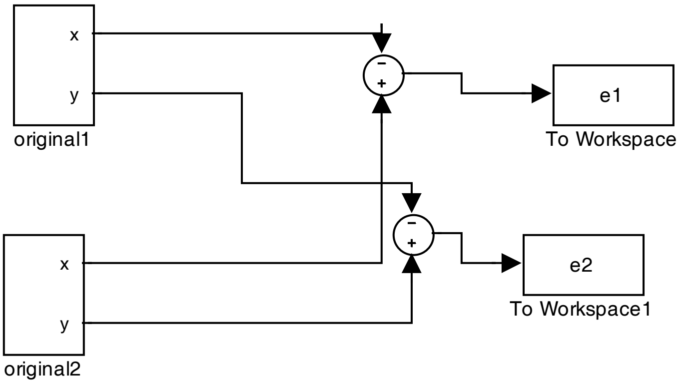
```
>> [DPA' DPB']
ans =
  -0.00762199518287  -0.90929742682579
   0.01665436331215  -0.41614683654700
```

So let's add a small perturbation $\Delta\phi(t_0, t_0, x_0) = \begin{bmatrix} 0.001 \\ 0.002 \end{bmatrix}$ and I expct the perturbation after 2s to be:

```
>> [DPA' DPB']*[0.001; 0.002]

ans =

  -0.00182621684883
  -0.00081563930978
```




```
>> [e1; e2]

ans =

  -0.00182360306785
  -0.00081662773015
```

Summary

So to summarise:

- To find how the perturbations evolve we use:
  $\Delta\phi(t,t_0,x_0)=\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0)$
- Thus we need $\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}$ which is the STM.
- Hence to find the STM:
  - For LTI systems $\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}=e^{A(t-t_0)}$
  - For LTV systems $\frac{d}{dt}\Phi_{STM}(t,t_0)=A(t)\Phi_{STM}(t,t_0),\ \Phi_{STM}(t_0,t_0)=I_{n\times n}$
  - For NL systems again solve the previous equation but now $A(t)=\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_0)}$.

7. Stability Analysis

Now, let's assume that we have a periodic orbit then we can still apply the previous method and we can try to find the perturbations using:

$$\Delta\phi(t,t_0,x_0)=\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0) \text{ or } \Delta\phi(t,t_0,x_0)=\Phi(t,t_0,x_0)\Delta\phi(t_0,t_0,x_0)$$

Notice that we used $\Phi(t,t_0,x_0)$ instead of $\Phi(t,t_0)$ as in the general nonlinear case $\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}$ may still be a function of x_0.

This matrix now can be found by solving

$$\frac{d}{dt}\Phi_{STM}(t,t_0)=\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_0)}\Phi_{STM}(t,t_0),\ \Phi_{STM}(t_0,t_0)=I_{n\times n}$$

Let's evaluate the above perturbations at $t=t_0+T$:

$$\Delta\phi(t_0+T,t_0,x_0)=\Phi(T+t_0,t_0,x_0)\Delta\phi(t_0,t_0,x_0)$$

Then we call the matrix $\Phi(T+t_0,t_0,x_0)$ the Monodromy Matrix (MM) of the limit cycle.

$$\Delta\phi(2T+t_0,t_0,x_0)=\Phi(T+t_0,t_0,x_0)\Delta\phi(T+t_0,t_0,x_0)\Leftrightarrow$$

Now, $\Delta\phi(2T+t_0,t_0,x_0)=\Phi(T+t_0,t_0,x_0)\Phi(T+t_0,t_0,x_0)\Delta\phi(t_0,t_0,x_0)\Leftrightarrow$

$$\Delta\phi(2T+t_0,t_0,x_0)=\Phi^2(T+t_0,t_0,x_0)\Delta\phi(t_0,t_0,x_0)$$

And also, $\Delta\phi(2T+t_0,t_0,x_0)=\Phi(2T+t_0,t_0,x_0)\Delta\phi(t_0,t_0,x_0)$

Thus we immediately see that $\Phi(2T+t_0,t_0,x_0)=\Phi^2(T+t_0,t_0,x_0)$ which implies that: $\Phi(kT+t_0,t_0,x_0)=\Phi^k(T+t_0,t_0,x_0)$ or that

$$\Delta\phi(kT+t_0,t_0,x_0)=\Phi^k(T+t_0,t_0,x_0)\Delta\phi(t_0,t_0,x_0)$$

Using eigenvalue decomposition: $\Delta\phi(kT+t_0,t_0,x_0)=U^{-1}\Lambda^k U\Delta\phi(t_0,t_0,x_0)$

Which implies that if the all eigenvalues of the monodromy matrix have absolute value less than 1 the orbit is stable.

Summary:

So to summarise:

- $\Delta\phi(t,t_0,x_0)=\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0)$
- Thus we need $\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}$ which is the STM.
- Hence to find the STM:
    - For LTI systems $\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}=e^{A(t-t_0)}$
    - For LTV systems $\dfrac{d}{dt}\Phi_{STM}(t,t_0)=A(t)\Phi_{STM}(t,t_0)$, $\Phi_{STM}(t_0,t_0)=I_{n\times n}$
    - For NL systems again solve the previous equation but now $A(t)=\left.\dfrac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_0)}$.
- Evaluate the STM at $t=t_0+T$

8. Floquet Theory

Another way to see the above is to linearise around the limit cycle:

$$\dot{x}_p(t) = f\left(x_p(t), t\right)$$

$$\dot{x}(t) = f\left(x, (t)t\right) = f\left(x_p(t) + \Delta x(t), t\right) = f\left(x_p(t), t\right) + \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = x_p} \Delta x(t) \Leftrightarrow$$

$$\dot{x}(t) - \dot{x}_p(t) = \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = x_p} \Delta x(t) \Leftrightarrow$$

$$\frac{d}{dt}\Delta x(t) = \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = x_p} \Delta x(t)$$

Which in the previous notation this can be written as:

$\frac{d}{dt}\Delta\phi(t, t_0, x_0) = \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = \phi(t, t_0, x_0)} \Delta\phi(t, t_0, x_0)$ and thus the solution is:

$\Delta\phi(t, t_0, x_0) = \Phi_{STM}(t, t_0, x_0) \times \Delta\phi(t_0, t_0, x_0)$ which again implies that

$\frac{d}{dt}\Phi_{STM}(t, t_0, x_0) = \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = \phi(t, t_0, x_0)} \Phi_{STM}(t, t_0, x_0)$ and that $\Phi_{STM}(t, t_0, x_0) = I$.

Hence we have a similar equation to the one before and hence we get the same result. Thus to find the stability properties we have to find the monodromy matrix and each eigenvalues.

Summary:

- Linearise the system around the periodic orbit and hence express the perturbations through a LTV differential equation
- To solve that we need the STM which can be found by solving a matrix differential equation:
- $\frac{d}{dt}\Phi_{STM}(t, t_0, x_0) = \left.\frac{\partial f\left(x(t), t\right)}{\partial x(t)}\right|_{x = \phi(t, t_0, x_0)} \Phi_{STM}(t, t_0, x_0)$, $\Phi_{STM}(t, t_0, x_0) = I$
- The stability is found by evaluating $\Phi_{STM}(t, t_0, x_0)$ after $t = T + t_0$.

9. Poincaré Map

Going back to the general solution of a nonlinear (it can also be linear) system:

$$\phi(t,t_0,x_0)=x_0+\int_0^t f(\phi(\tau,t_0,x_0),\tau)d\tau$$

Now, the stroboscopic Poincare map is nothing more than:

$$\phi(T+t_0,t_0,x_0)=x_0+\int_0^{T+t_0} f(\phi(\tau,t_0,x_0),\tau)d\tau \Leftrightarrow x_{n+1}=P(x_n)=x_n+\int_0^{T+t_0} f(\phi(\tau,t_0,x_n),\tau)d\tau$$

The fixed point is $x_{FP}+\int_0^{T+t_0} f(\phi(\tau,t_0,x_{FP}),\tau)d\tau=x_{FP}\Leftrightarrow \int_0^{T+t_0} f(\phi(\tau,t_0,x_{FP}),\tau)d\tau=0$

The stability analysis is similar to the previous case:

$$\frac{\partial P(x_{FP})}{\partial x_{FP}}=I+\int_0^{T+t_0}\frac{\partial f(\phi(\tau,t_0,x_0),\tau)}{\partial\phi(\tau,t_0,x_0)}\frac{\partial\phi(\tau,t_0,x_0)}{\partial x_0}d\tau$$

Hence the monodromy matrix is nothing more than the Jacobian of the Poincaré map!

Unfortunately again in the general case we have to numerically solve the matrix differential equation:

$$\frac{d}{dt}\left(\phi_{x_{FP}}(t,t_0,x_{FP})\right)=\left.\frac{\partial f(x,t)}{\partial x}\right|_{x=\phi(t,t_0,x_{FP})}\phi_{x_0}(t,t_0,x_{FP}) \text{ for } t\in[0,T]$$

Summary:

- Evaluate the solution after T time: $\phi(T+t_0,t_0,x_0)$.
- The stability can be found from $\frac{\partial\phi(t,t_0,x_0)}{\partial x_0}$.
- To calculate this, again we have to solve a matrix differential equation.

Overall Summary:

1. In all cases we start from assuming that we have the solution $\phi(t,t_0,x_0)$.
2. Express the perturbations as: $\Delta\phi(t,t_0,x_0)=\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0)$
3. To find $\Phi_{STM}(t,t_0,x_0)=\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}$ solve the matrix DE: $\dfrac{d}{dt}\Phi_{STM}(t,t_0,x_0)=A(t)\Phi_{STM}(t,t_0,x_0)$, $\Phi_{STM}(t_0,t_0,x_0)=I_{n\times n}$
4. The solution of the above equation will give $\Phi_{STM}(t,t_0,x_0)$ and hence we can get the MM: $\Phi_{STM}(T+t_0,t_0,x_0)$. The eigenvalues of this will determine the stability as $\Delta\phi(t_0+kT,t_0,x_0)=\left(\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}\right)^k\Delta\phi(t_0,t_0,x_0)$
5. For the Floquet Theory we must first derive $\dfrac{d}{dt}\Delta\phi(t,t_0,x_0)=\left.\dfrac{\partial f(x(t),t)}{\partial x(t)}\right|_{x=\phi(t,t_0,x_0)}\Delta\phi(t,t_0,x_0)$ which is LTV and hence we need the STM to express $\Delta\phi(t,t_0,x_0)=\dfrac{\partial\phi(t,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0)$.
6. For the Poincare map it is exactly the same as before but now we write directly: $\Delta\phi(t_0+T,t_0,x_0)=\dfrac{\partial\phi(t_0+T,t_0,x_0)}{\partial x_0}\Delta\phi(t_0,t_0,x_0)$.

Thus

- The Floquet Theory needs an extra step than the Trajectory Sensitivity
- The Poincaré map is just a special case of the Trajectory Sensitivity

Example: Poincaré Map vs Floquet Theory

1. Initial numerical analysis

A system is given by

$$\left.\begin{aligned} \frac{dx}{dt} &= -y + x\left(1 - x^2 - y^2\right) \\ \frac{dy}{dt} &= x + y\left(1 - x^2 - y^2\right) \end{aligned}\right\}$$

The EPs are:

$$\left.\begin{aligned} 0 &= -y + x\left(1 - x^2 - y^2\right) \\ 0 &= x + y\left(1 - x^2 - y^2\right) \end{aligned}\right\}$$

From the 1st eqn: $-y + x\left(1 - x^2 - y^2\right) = 0 \Leftrightarrow y = x\left(1 - x^2 - y^2\right)$

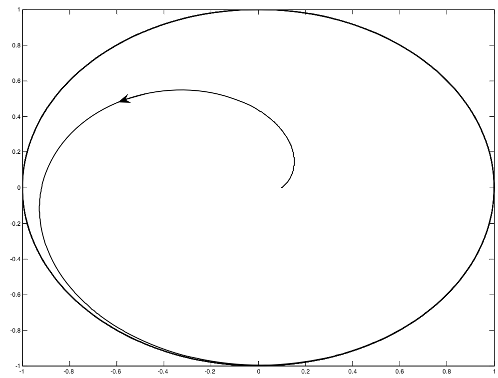
And by replacing that at the 2nd eqn:

$$0 = x + x\left(1 - x^2 - y^2\right)\left(1 - x^2 - y^2\right)$$

$$0 = x + x\left(1 - x^2 - y^2\right)^2 \Rightarrow \begin{cases} x = 0 \Rightarrow y = 0 \\ 1 + \left(1 - x^2 - y^2\right)^2 = 0 \text{ which is inconsistent} \end{cases}$$

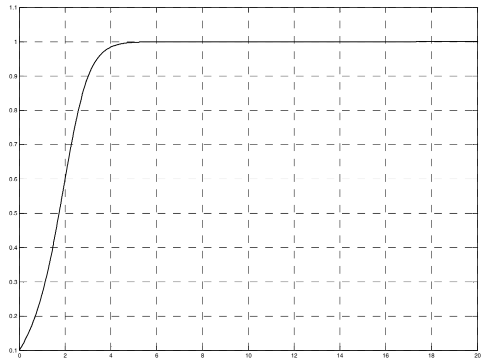
Hence there is only one equilibrium point at the origin.

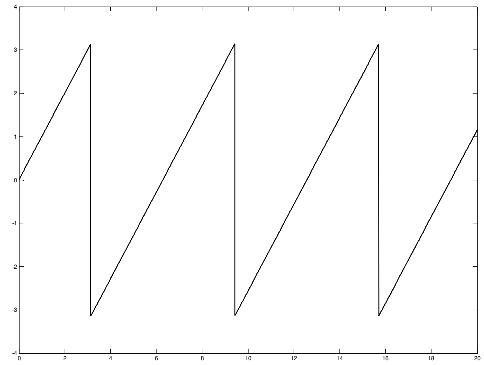
The system has a stable limit cycle as it can be seen by its numerical simulation.



To study the system we can use polar coordinates:

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \end{aligned} \quad \text{which will give: } r^2 = x^2 + y^2 \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$


1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0 2 4 6 8 10 12 14 16 18 20


4
3
2
1
0
-1
-2
-3
-4
0 2 4 6 8 10 12 14 16 18 20

2. Polar Coordinates

Lets calculate the model in polar coordinates:

$$\frac{dr^2}{dt} = \frac{d}{dt}\left(x^2 + y^2\right)$$

$$2r\frac{dr}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dr}{dt} = \frac{1}{r}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$\frac{dr}{dt} = \frac{1}{r}\left(x\left(-y + x\left(1 - x^2 - y^2\right)\right) + y\left(x + y\left(1 - x^2 - y^2\right)\right)\right)$$

Also, $\begin{matrix} x = r\cos(\theta) \\ y = r\sin(\theta) \end{matrix}$, so $x^2 + y^2 = r^2$, Hence $\frac{dr}{dt} = \frac{1}{r}\left(x\left(-y + x\left(1 - r^2\right)\right) + y\left(x + y\left(1 - r^2\right)\right)\right)$

$$\frac{dr}{dt} = \frac{1}{r}\left(-xy + x^2\left(1 - r^2\right) + yx + y^2\left(1 - r^2\right)\right)$$

$$\frac{dr}{dt} = \frac{1}{r}\left(-xy + x^2\left(1 - r^2\right) + yx + y^2\left(1 - r^2\right)\right)$$

$$\frac{dr}{dt} = \frac{1}{r}\left(x^2\left(1 - r^2\right) + y^2\left(1 - r^2\right)\right)$$

$$\frac{dr}{dt} = \frac{1}{r}\left(\left(x^2 + y^2\right)\left(1 - r^2\right)\right)$$

$$\frac{dr}{dt} = \frac{1}{r}\left(r^2\left(1 - r^2\right)\right)$$

$$\frac{dr}{dt} = r\left(1 - r^2\right)$$

And

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{d\theta}{dt} = \frac{d}{dt}\left(\frac{y}{x}\right)\frac{1}{1 + \left(\frac{y}{x}\right)^2} = \frac{\frac{dy}{dt}x - \frac{dx}{dt}y}{x^2}\frac{x^2}{x^2 + y^2} =$$

$$\left(\frac{dy}{dt}x - \frac{dx}{dt}y\right)\frac{1}{x^2 + y^2} =$$

$$\left(\left(x + y\left(1 - x^2 - y^2\right)\right)x - \left(-y + x\left(1 - x^2 - y^2\right)\right)y\right)\frac{1}{x^2 + y^2} =$$

$$\left(\left(x + y\left(1 - r^2\right)\right)x - \left(-y + x\left(1 - r^2\right)\right)y\right)\frac{1}{r^2} =$$

$$\left(\left(x^2 + yx\left(1 - r^2\right)\right) - \left(-y^2 + xy\left(1 - r^2\right)\right)\right)\frac{1}{r^2} =$$

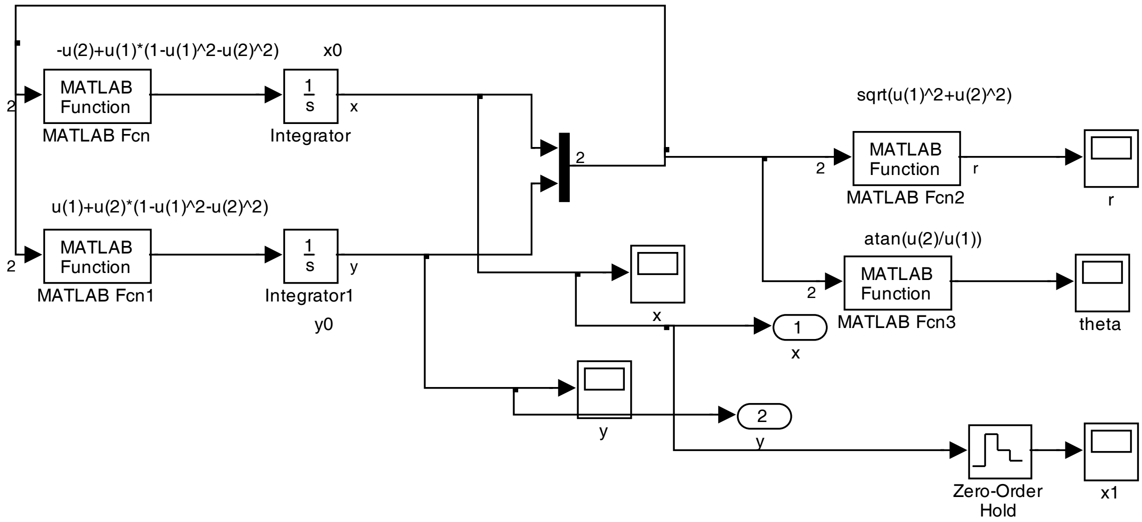
$$\left(x^2 + yx\left(1 - r^2\right) + y^2 - xy\left(1 - r^2\right)\right)\frac{1}{r^2} =$$

$$\left(x^2 + y^2\right)\frac{1}{r^2} = r^2\frac{1}{r^2} = 1$$

So the system in polar coordinates is:

$$\boxed{\begin{aligned}\frac{dr}{dt} &= r\left(1-r^2\right)\\ \frac{d\theta}{dt} &= 1\end{aligned}}$$

, Simulink validated this as well.



2.1 Equilibria and solution of system in polar coordinates

The 2nd equation has no EP but the first has at:

$$r\left(1-r^2\right)=0 \Rightarrow \begin{cases} r=0 \\ r=1 \end{cases}$$

The Jacobian is $\frac{\partial}{\partial r}\left(r-r^3\right)=1-3r^2 \Rightarrow \begin{cases} r=0:1 \\ r=1:-2 \end{cases}$. So the origin is unstable and r=1 is stable.

The ODE in polar coordinates can be solved and we have:

$$r(t,r_0)=\left[1+\left(\frac{1}{r_0^2}-1\right)e^{-2t}\right]^{-1/2}$$

$$\theta(t,\theta_0)=\theta_0+t$$

So we can see that the angle θ will continuously increase while the amplitude r will converge to 1.

2.2 Poincare map and Jacobian from polar coordinates

But regarding the original system the cosine/sine of the angle will be periodic of period 2π and as the amplitude will converge to a steady state value then this implies that the system will have a limit cycle which will be a perfect circle of radius 1, centre at the origin and a period of 2π .

So if we start at r_0 then the next point after one period (Poincare map) is:

$r(2\pi, r_0) = P(r_0) = \left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-1/2}$, which can be written as:

$$r_{n+1} = P(r_n) = \left[1 + \left(\frac{1}{r_n^2} - 1\right)e^{-4\pi}\right]^{-1/2}$$

To find the fixed point:

$$r_n = \left[1 + \left(\frac{1}{r_n^2} - 1\right)e^{-4\pi}\right]^{-1/2} \Leftrightarrow r_n\left[1 + \left(\frac{1}{r_n^2} - 1\right)e^{-4\pi}\right]^{1/2} = 1 \Leftrightarrow r_n^2\left[1 + \left(\frac{1}{r_n^2} - 1\right)e^{-4\pi}\right] = 1 \Leftrightarrow$$

$$r_n^2 + \left(1 - r_n^2\right)e^{-4\pi} = 1 \Leftrightarrow r_n^2 + e^{-4\pi} - r_n^2 e^{-4\pi} = 1 \Leftrightarrow r_n^2\left(1 - e^{-4\pi}\right) = 1 - e^{-4\pi} \Leftrightarrow r_n^2 = 1 \Rightarrow r_n = 1$$

As the FP is 1 then the Jacobian at that point is:

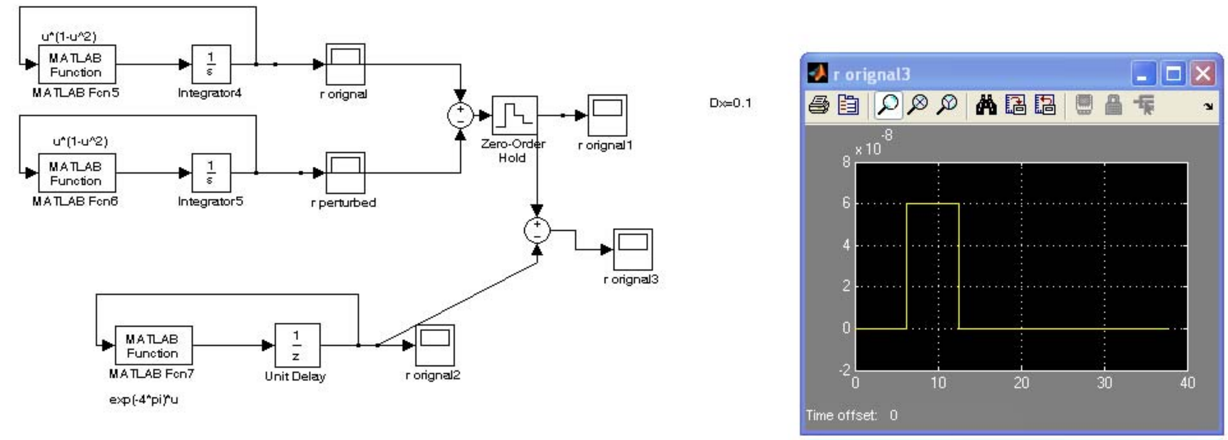
$$\frac{\partial P(r_n)}{\partial r_n^2} = e^{-4\pi} r_n^3\left[1 + \left(\frac{1}{r_n^2} - 1\right)e^{-4\pi}\right]^{-3/2} \Rightarrow \left.\frac{\partial P(r_n)}{\partial r_n^2}\right|_{r_n = r_0} = e^{-4\pi} r_0^3\left[1 + \left(\frac{1}{r_0^2} - 1\right)e^{-4\pi}\right]^{-3/2}$$

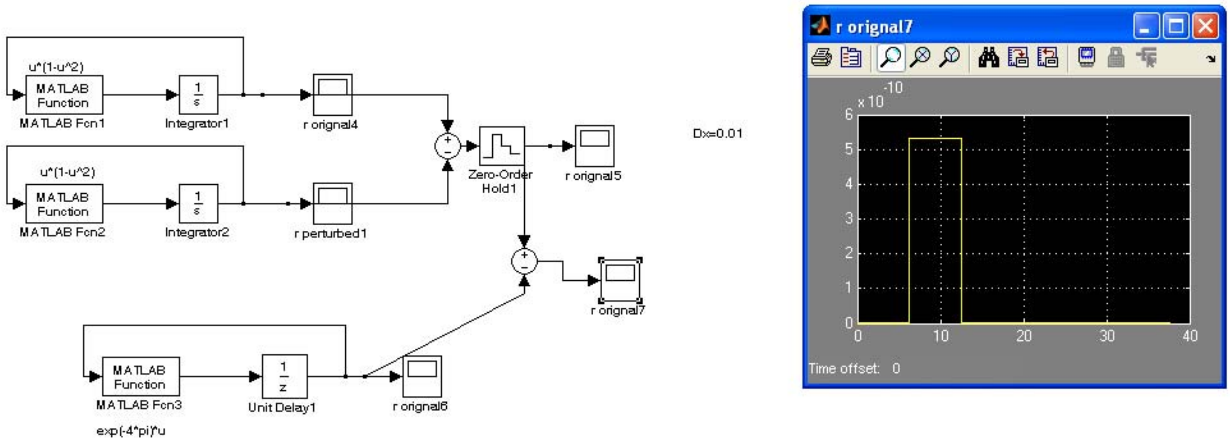
And hence: $\left.\frac{\partial P(r_n)}{\partial r_n^2}\right|_{r_n = 1} = e^{-4\pi} = 3.48 \times 10^{-6} < 1$ and hence it is stable.

Or that simply the perturbations around the fixed point are given by:

$$\Delta r(n+1) = e^{-4\pi}\Delta r(n)$$

which has been crosschecked using Simulink. To do that I simulated the original system at r_0=1 and the same system with r_0=0.1 and I sampled their difference with $T = 2\pi$. Then I simulated the $\Delta r(n+1) = e^{-4\pi}\Delta r(n)$ with initial condition 1-0.9=0.1 and I compared the previous difference and that output. The smaller the IC the smaller their difference:





3. Cartesian coordinates

3.1 Solution in Cartesian coordinates

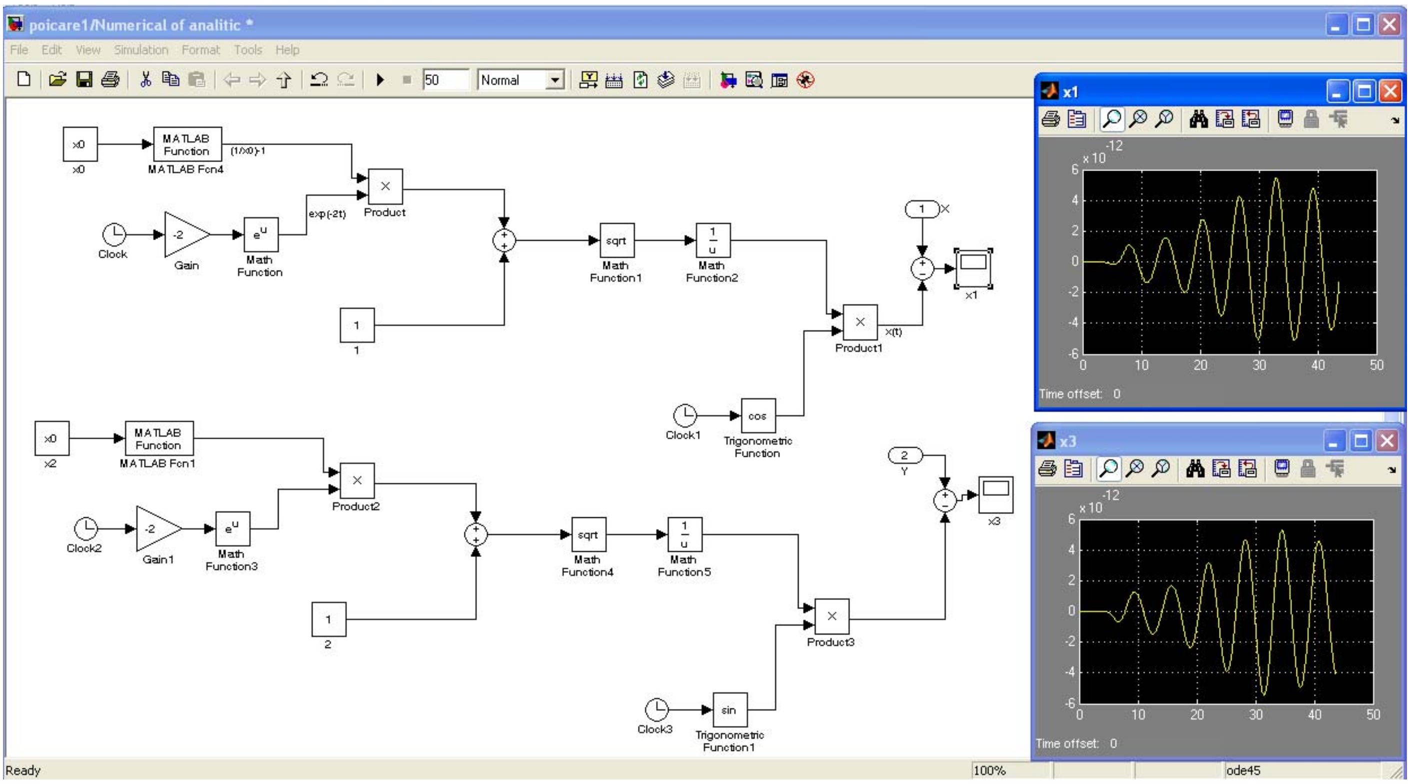
Now let's find the solution in the x-y plane:

$$\left.\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\} \overset{\theta_0=0}{\Rightarrow} \left.\begin{array}{l} x = \left[1+\left(\dfrac{1}{r_0^2}-1\right)e^{-2t}\right]^{-1/2}\cos(t) \\ y = \left[1+\left(\dfrac{1}{r_0^2}-1\right)e^{-2t}\right]^{-1/2}\sin(t) \end{array}\right\}$$

$$\left.\begin{array}{l} x(0) = r(0)\cos(0) \\ y(0) = r(0)\sin(0) \end{array}\right\} \Rightarrow \left.\begin{array}{l} x(0) = r(0) \\ y(0) = 0 \end{array}\right\}$$

$$\Rightarrow \left.\begin{array}{l} x = \left[1+\left(\dfrac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\cos(t) \\ y = \left[1+\left(\dfrac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\sin(t) \end{array}\right\}$$

This has been validated by Simulink. I.e. I simulated the previous set of equations that give analytical solution and compare that with the numerical that I get from Simulink. The error was 10^{-14}.



If I do not assume that θ=0:

$$\left.\begin{array}{l} x = r\cos(\theta) \\ y = r\sin(\theta) \end{array}\right\} \overset{\theta_0=0}{\Rightarrow} \left.\begin{array}{l} x = \left[1+\left(\dfrac{1}{r_0^2}-1\right)e^{-2t}\right]^{-1/2}\cos(\theta_0+t) \\ y = \left[1+\left(\dfrac{1}{r_0^2}-1\right)e^{-2t}\right]^{-1/2}\sin(\theta_0+t) \end{array}\right\}$$

$$\left.\begin{array}{l} r(0) = \sqrt{x^2(0)+y^2(0)} \\ \theta(0) = \tan^{-1}\left(\dfrac{y(0)}{x(0)}\right) \end{array}\right\} \Rightarrow \begin{array}{l} x = \left[1+\left(\dfrac{1}{x_0^{\,2}+y_0^{\,2}}-1\right)e^{-2t}\right]^{-1/2}\cos\left(\tan^{-1}\left(\dfrac{y_0}{x_0}\right)+t\right. \\ y = \left[1+\left(\dfrac{1}{x_0^{\,2}+y_0^{\,2}}-1\right)e^{-2t}\right]^{-1/2}\sin\left(\tan^{-1}\left(\dfrac{y_0}{x_0}\right)+t\right. \end{array}$$

3.2 Jacobian in Cartesian coordinates

Now let’s find the Poincare map in the x-y domain:

$$\mathbf{P}(x_0, y_0)\Big|_{t=2\pi} = \begin{bmatrix} \left[1+\left(\dfrac{1}{x_0^{\ 2}+y_0^{\ 2}}-1\right)e^{-4\pi}\right]^{-1/2} \cos\left(\tan^{-1}\left(\dfrac{y_0}{x_0}\right)+2\pi\right) \\ \left[1+\left(\dfrac{1}{x_0^{\ 2}+y_0^{\ 2}}-1\right)e^{-4\pi}\right]^{-1/2} \sin\left(\tan^{-1}\left(\dfrac{y_0}{x_0}\right)+2\pi\right) \end{bmatrix}$$

The Jacobian of that is rather cumbersome so I will use Matlab

```
clc, clear
syms x0 y0 T

r2=x0^2+y0^2; ra=(1/r2)-1;
rb=ra*exp(2*T); rb=ra*exp(-2*T);
rc=1/sqrt(1+rb); rat=y0/x0;
th1=atan(rat)+T;

f1=rc*cos(th1); f2=rc*sin(th1);
Df11=diff(f1,x0); Df12=diff(f1,y0);
Df21=diff(f2,x0); Df22=diff(f2,y0);
Df11=subs(Df11,{T,x0,y0},{2*pi,1,0});
Df12=subs(Df12,{T,x0,y0},{2*pi,1,0});
Df21=subs(Df21,{T,x0,y0},{2*pi,1,0});
Df22=subs(Df22,{T,x0,y0},{2*pi,1,0});
DF=[Df11 Df12; Df21 Df22]
eig(DF)
```

DF =

```
   0.00000348734236   0.00000000000000
  -0.00000000000000   1.00000000000000
```

We get 2 eigenvalues:
  0.00000348734236
  1.00000000000000

The 1st one is the same as the one that we got from the polar coordinates and 1 is from the fact that the system is autonomous.

Notice that we could have calculated the Jacobian if we had solved

$$\frac{d}{dt}\left(\phi_{x_{FP}}(t,t_0,x_{FP})\right) = \frac{\partial f(x,t)}{\partial x}\bigg|_{x=\phi(t,t_0,x_{FP})} \phi_{x_0}(t,t_0,x_{FP}) \text{ for } t \in [0,T]$$

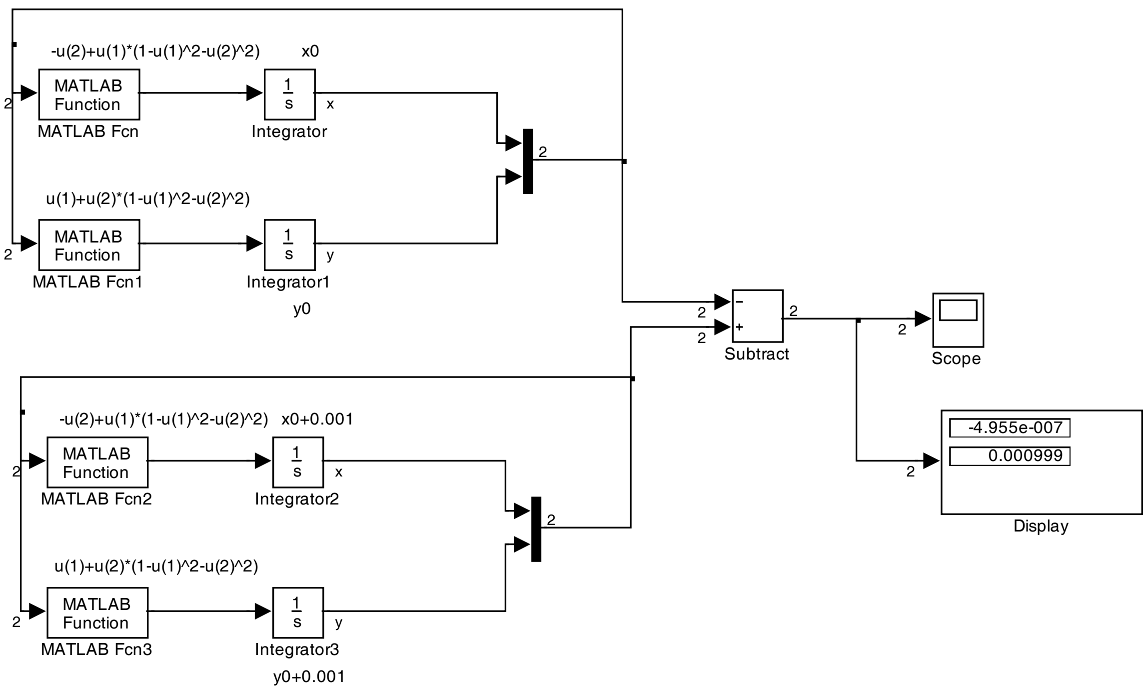
3.3 Perturbation calculation using the Jacobian

```
>> DF*[0.001;0.001]

ans =

  1.0e-003 *

   0.00000348734236
   1.00000000000000
```



Note: If I did not know the IC for the Poincare map, i.e. its fixed point I have to use a Newton Rapshon method.

4. Jacobian around the periodic orbit

So now I have a nonlinear system with a stable periodic orbit with T=6.28… and also I know its solution. Now I will try to use the Floquet Theory.

The nonlinear vector field is:

$$\mathbf{f}(x,y)=\begin{bmatrix}-y+x\left(1-x^2-y^2\right)\\ x+y\left(1-x^2-y^2\right)\end{bmatrix} \text{ and hence } \frac{d\mathbf{X}}{dt}=\mathbf{f}(\mathbf{X})=\mathbf{f}(x,y)$$

So the Jacobian around the orbit is:

$$\frac{\partial\mathbf{f}(\mathbf{X})}{\partial\mathbf{X}}=\begin{bmatrix}\frac{\partial\left(-y+x\left(1-x^2-y^2\right)\right)}{\partial x} & \frac{\partial\left(-y+x\left(1-x^2-y^2\right)\right)}{\partial y}\\ \frac{\partial\left(x+y\left(1-x^2-y^2\right)\right)}{\partial x} & \frac{\partial\left(x+y\left(1-x^2-y^2\right)\right)}{\partial y}\end{bmatrix}=$$

$$\frac{\partial\mathbf{f}(\mathbf{X})}{\partial\mathbf{X}}=\begin{bmatrix}\frac{\partial\left(-y+\left(x-x^3-xy^2\right)\right)}{\partial x} & \frac{\partial\left(-y+\left(x-x^3-xy^2\right)\right)}{\partial y}\\ \frac{\partial\left(x+\left(y-yx^2-y^3\right)\right)}{\partial x} & \frac{\partial\left(x+\left(y-yx^2-y^3\right)\right)}{\partial y}\end{bmatrix} \text{ which will give:}$$

$$\frac{\partial\mathbf{f}(\mathbf{X})}{\partial\mathbf{X}}=\begin{bmatrix}1-3x^2-y^2 & -1-2xy\\ 1-2xy & 1-x^2-3y^2\end{bmatrix}$$

Now this must be evaluated around the orbit:

$$\left.\begin{aligned} x&=\left[1+\left(\frac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\cos(t)\\ y&=\left[1+\left(\frac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\sin(t)\end{aligned}\right\} \text{ starting}$$

on the point $x_0=1$, $y_0=0$ as this is a point on the cycle.

$$\left.\begin{aligned} x_p(t)&=\cos(t)\\ y_p(t)&=\sin(t)\end{aligned}\right\} \text{ so:}$$

$$\left.\frac{\partial\mathbf{f}(\mathbf{X})}{\partial\mathbf{X}}\right|_{\mathbf{X}=\mathbf{X_P}}=\begin{bmatrix}1-3\cos(t)^2-\sin(t)^2 & -1-2\cos(t)\sin(t)\\ 1-2\cos(t)\sin(t) & 1-\cos(t)^2-3\sin(t)^2\end{bmatrix}$$

Or:

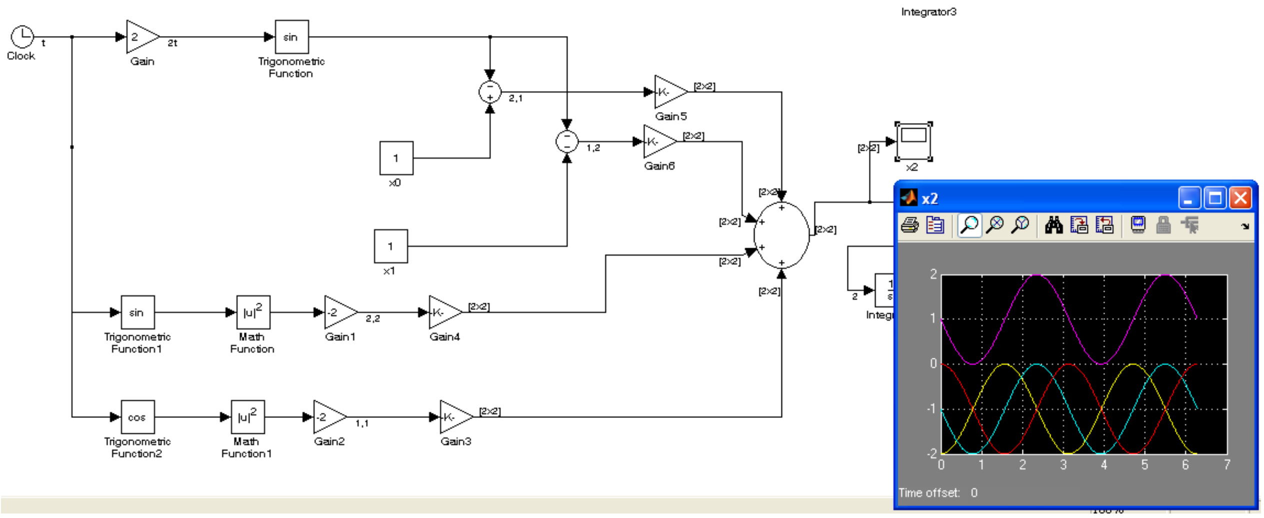
$$\left.\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X_P}} = \begin{bmatrix} 1-2\cos(t)^2-\cos(t)^2-\sin(t)^2 & -1-2\cos(t)\sin(t) \\ 1-2\cos(t)\sin(t) & 1-\cos(t)^2-2\sin(t)^2-\sin(t)^2 \end{bmatrix}$$

$$\left.\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X_P}} = \begin{bmatrix} 1-2\cos(t)^2-1 & -1-2\cos(t)\sin(t) \\ 1-2\cos(t)\sin(t) & 1-1-2\sin(t)^2 \end{bmatrix}$$

$$\left.\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X_P}} = \begin{bmatrix} -2\cos(t)^2 & -1-\sin(2t) \\ 1-\sin(2t) & -2\sin(t)^2 \end{bmatrix}$$

For t=0 to t=T=6.28…

The elements of the Jacobian matrix are:



5. Solution of Matrix Differential Equation

5.1 Calculation of Monodromy matrix

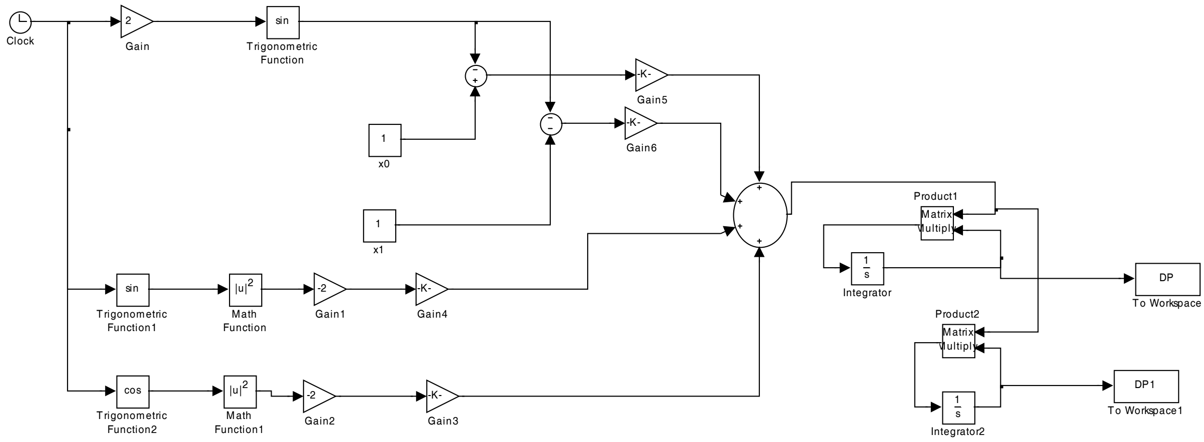
In order to find the modnoromy matrix I need to to solve the MDE:

$$\frac{d\mathbf{\Phi}(t,t_0)}{dt} = \left.\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}}\right|_{\mathbf{X}=\mathbf{X_P}} \mathbf{\Phi}(t,t_0)$$

$$\frac{d\mathbf{\Phi}(t,t_0)}{dt} = \mathbf{A}(t)\mathbf{\Phi}(t,t_0)$$

With the IC = I_2

The following model will calculate the matrix A from t=0 to t=T.




```
>> DP=[DP1' DP2']
DP =
   0.00000348734236  -0.00000000000004
   0.00000000000004   0.99999999999983
```

Notice that this result is very close to the one that we got from the Poincare Map.

Now, that the monodromy matrix is found we can find the Floquet multipliers:

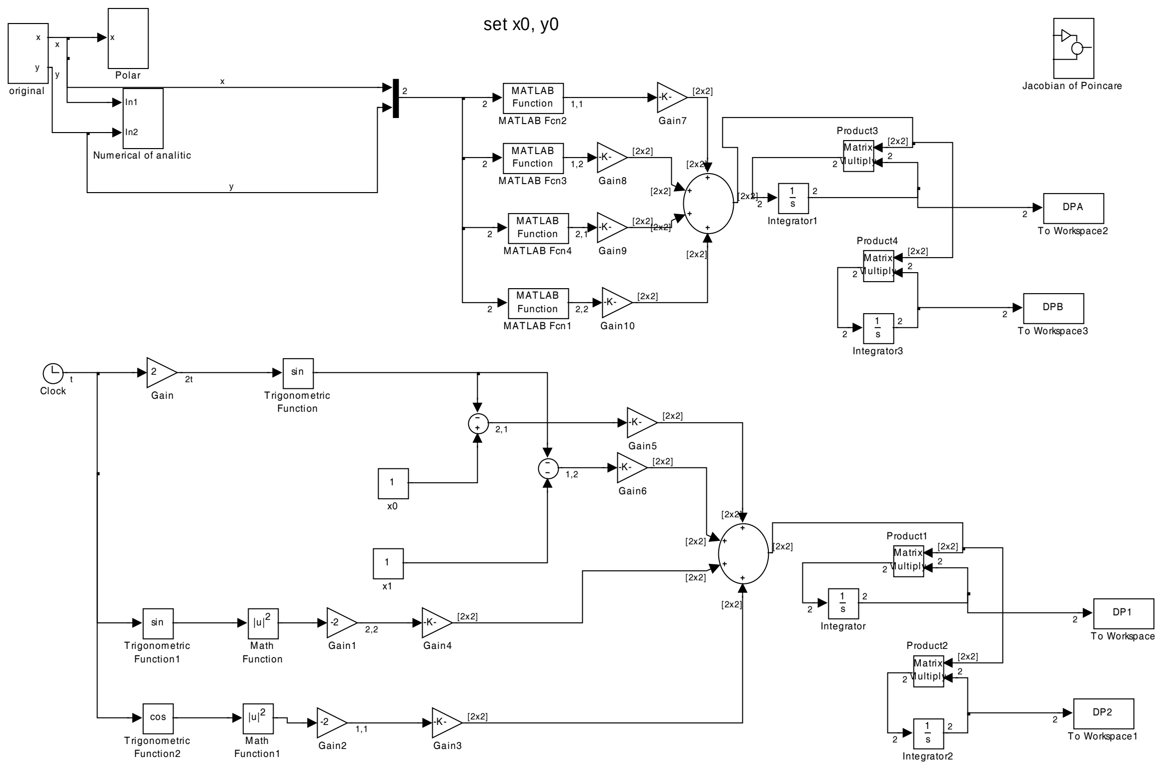
```
>> eig(DP)
ans =
   0.00000348734236
   0.99999999999983
```

Which are very close to the ones from the PM.

Another way to do that is to numerically solve the original system and then to use the numerical solution to $\frac{\partial \mathbf{f}(\mathbf{X})}{\partial \mathbf{X}} = \begin{bmatrix} 1-3x^2-y^2 & -1-2xy \\ 1-2xy & 1-x^2-3y^2 \end{bmatrix}$ with

$$\left.\begin{aligned} x &= \left[1+\left(\frac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\cos(t) \\ y &= \left[1+\left(\frac{1}{x_0^2}-1\right)e^{-2t}\right]^{-1/2}\sin(t) \end{aligned}\right\}.$$

It is important though to remember that the system must start from $x_0=1$, $y_0=0$.



>> eig(DPa)
ans =
   0.00000348734236
   1.00000000000004

Hence we get the same results.

5.2 Perturbation calculation using the Monodromy matrix

So I have found $\mathbf{\Phi}(t,0)$, now I will use this matrix to prove that $\Delta\mathbf{X}(t) = \mathbf{\Phi}(t,0)\Delta\mathbf{X}(0)$ by knowing that $\Delta\mathbf{X}(0) = [0.1 \quad 0]$. So for t=0.1T I have got from the NL system:
   0.02572402284312
   0.01868959658690
And from the product
>> E1=[DP1' DP2']*[0.1;0]
E1 =
   0.02302539573201
   0.01672892922346

These are in a very good agreement.