Recipe for ODEs in Matlab

Simple Second Order 1D Example

Step 1: Start with the problem expressed as a typical second order ODE, e.g.:

$$\frac{d^2u}{dv^2} - 5\frac{du}{dv} + 4u = ve^{4v} \quad \text{or} \quad u''(v) - 5u'(v) + 4u(v) = ve^{4v}$$

where $u(0)=1$ and $u'(0)=0$.

Problem: Find $u(4.5)$.

Step 2: Add the subscript 1 to the dependent (upper) variable – in this case u becomes u_1:

$$\frac{d^2u_1}{dv^2} - 5\frac{du_1}{dv} + 4u_1 = ve^{4v}$$

Step 3: Define a new variable (subscript 2, i.e., u_2) to be the derivative of the first:

$$\frac{du_1}{dv} = u_2 \tag{1}$$

Step 4: Substitute new variable (u_2) into the ODE where possible:

$$\frac{du_2}{dv} - 5u_2 + 4u_1 = ve^{4v}$$

Step 5: Rearrange so that only the derivative term is on the left hand side:

$$\frac{du_2}{dv} = 5u_2 - 4u_1 + ve^{4v} \tag{2}$$

Step 6: Rewrite the two first order ODEs – Equations (1) and (2) – in vector form:

$$\frac{d}{dv}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ 5u_2 - 4u_1 + ve^{4v} \end{pmatrix} \tag{3}$$

Step 7: In MatLab, define a function that takes two variables (the independent variable, v in this case, should be first; the second is the vector **u**, and the subscripts above correspond to elements in the vector) and returns a column vector representing the right hand side of Equation (3):

```
>> f = @(v,u) [u(2);5*u(2)-4*u(1)+v*exp(4*v)];
```

Warning! Be very careful about spaces inside square brackets as these can confuse MatLab.

Step 8: Solve the ODE – for initial conditions, remember that u_1 is the original variable u and u_2 is the first derivative:

```
>> [V,U] = ode23 (f, [0,4.5], [1;0]);
```

Important! When specifying the time interval as start and stop values, make sure to use a <u>comma</u>. In MatLab, `[0:4.5]` would mean `[0,1,2,3,4]`, so the solution would stop at 4, not 4.5!

Finally: The answer to the problem ("Find $u(4.5)$.") is the last element in the first column of the matrix U (which is the solution for u_1; the second column is for u_2, and so on for larger ODE problems). The vector **V** contains the corresponding values of v.

Second Order 3D Example

A particle with charge q moving with velocity $\mathbf{v}$ in an electromagnetic field experiences the force $\mathbf{F}$:

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields respectively; both may be functions of time t and position $\mathbf{x}$. If the particle has mass m then the acceleration $\mathbf{a}$ is:

$$\mathbf{a}(t) = \frac{\mathbf{F}(t)}{m} = \frac{q}{m}\big(\mathbf{E}(t,\mathbf{x}(t)) + \mathbf{v}(t) \times \mathbf{B}(t,\mathbf{x}(t))\big)$$

where the dependence on t and $\mathbf{x}(t)$ is shown. The acceleration at any given time is therefore dependent on its current position and velocity. All three quantities are of course three-dimensional and can be written as column vectors with three components:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{q}{m}\left[\begin{pmatrix} E_1(t,\mathbf{x}) \\ E_2(t,\mathbf{x}) \\ E_3(t,\mathbf{x}) \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} B_1(t,\mathbf{x}) \\ B_2(t,\mathbf{x}) \\ B_3(t,\mathbf{x}) \end{pmatrix}\right]$$

$$= \frac{q}{m}\left[\begin{pmatrix} E_1(t,x_1,x_2,x_3) \\ E_2(t,x_1,x_2,x_3) \\ E_3(t,x_1,x_2,x_3) \end{pmatrix} + \begin{pmatrix} v_2 B_3(t,x_1,x_2,x_3) - v_3 B_2(t,x_1,x_2,x_3) \\ v_3 B_1(t,x_1,x_2,x_3) - v_1 B_3(t,x_1,x_2,x_3) \\ v_1 B_2(t,x_1,x_2,x_3) - v_2 B_1(t,x_1,x_2,x_3) \end{pmatrix}\right]$$

This can be thought of as a single three-dimensional vector ODE, or equivalently as a system of three inter-dependent ODEs. Written as above it is a second order ODE, since the acceleration and position variables are both present. Remembering that:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

we can replace the acceleration term by the time-derivative of velocity, but we also have to add in the three ODEs corresponding to velocity as the time-derivative of position, so that now there are six equations with six variables:

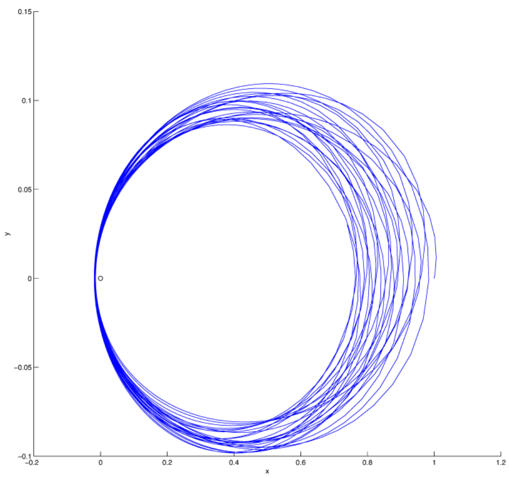
$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

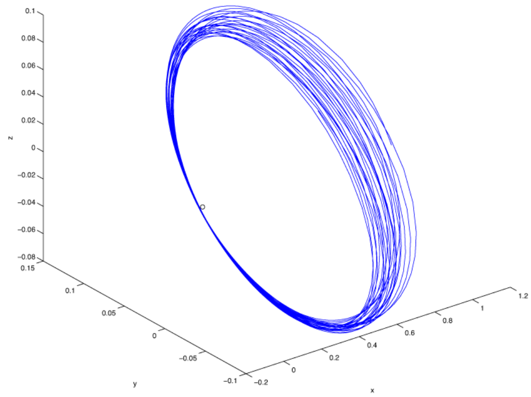
$$\frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \frac{q}{m}\left[\begin{pmatrix} E_1(t,x_1,x_2,x_3) \\ E_2(t,x_1,x_2,x_3) \\ E_3(t,x_1,x_2,x_3) \end{pmatrix} + \begin{pmatrix} v_2 B_3(t,x_1,x_2,x_3) - v_3 B_2(t,x_1,x_2,x_3) \\ v_3 B_1(t,x_1,x_2,x_3) - v_1 B_3(t,x_1,x_2,x_3) \\ v_1 B_2(t,x_1,x_2,x_3) - v_2 B_1(t,x_1,x_2,x_3) \end{pmatrix}\right]$$

In MatLab this needs to be expressed as a single six-dimensional vector ODE, so we replace v_1, v_2, v_3 by x_4, x_5, x_6:

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_5 \\ x_6 \\ (q/m)\left[E_1(t,x_1,x_2,x_3) + x_5 B_3(t,x_1,x_2,x_3) - x_6 B_2(t,x_1,x_2,x_3)\right] \\ (q/m)\left[E_2(t,x_1,x_2,x_3) + x_6 B_1(t,x_1,x_2,x_3) - x_4 B_3(t,x_1,x_2,x_3)\right] \\ (q/m)\left[E_3(t,x_1,x_2,x_3) + x_4 B_2(t,x_1,x_2,x_3) - x_5 B_1(t,x_1,x_2,x_3)\right] \end{pmatrix}$$

To solve this in Matlab we first need to define functions for the EM field and various constants. Here I have chosen an oscillating EM field with a second charged particle fixed at the origin, and the resulting path of the particle is shown below.





Note: It would be impossible to read the code if the whole 6D vector ODE was defined at once, so here I have broken it and the EM functions down into individual component functions, and grouped them at the end. (For this example it would be better to create a function ***m***-file, but this is an example of how to use multiple @ functions together.)

Note: The 6D vector is really two 3D vectors. These can be separated easily: `x(1:3)` creates a 3-element vector from the first three elements of `x`, i.e., the position vector; `x(4:6)` creates the 3D velocity vector from the last three elements of `x`.

```
k = 7 * 2 * pi; % fields oscillate with frequency 7Hz

a = 1.3E-3; % amplitude of magnetic field oscillation

m = 1E-3; % mass of particle in motion

q = 1;  % charge of particle in motion
Q = -2; % charge of a heavy mass at the centre of the field

% Electric field:
%   each function takes time and 3D coordinate vector
%   each returns a single value
E1 = @(t,x) Q*x(1)/norm(x)^3 - x(2)*k*a;
E2 = @(t,x) Q*x(2)/norm(x)^3 + x(1)*k*a*sin(k*t);
E3 = @(t,x) Q*x(3)/norm(x)^3;

%   returns 3D column vector representing electric field
E = @(t,x) [E1(t,x); E2(t,x); E3(t,x)];

% Magnetic field:
%   each function takes time and 3D coordinate vector
%   each returns a single value
B1 = @(t,x) 0;
B2 = @(t,x) 0;
B3 = @(t,x) a*cos(k*t);

%   returns 3D column vector representing magnetic field
B = @(t,x) [B1(t,x); B2(t,x); B3(t,x)];

% ODE functions
%   each function takes time and 6D vector coordinate/velocity
%   each returns a single value
f1 = @(t,x) x(4);
f2 = @(t,x) x(5);
f3 = @(t,x) x(6);
f4 = @(t,x) (q/m)*(E1(t,x(1:3)) + x(5)*B3(t,x(1:3)) - x(6)*B2(t,x(1:3)));
f5 = @(t,x) (q/m)*(E2(t,x(1:3)) + x(6)*B1(t,x(1:3)) - x(4)*B3(t,x(1:3)));
f6 = @(t,x) (q/m)*(E3(t,x(1:3)) + x(4)*B2(t,x(1:3)) - x(5)*B1(t,x(1:3)));

%   returns 6D column vector representing the ODE
f = @(t,x) [f1(t,x); f2(t,x); f3(t,x); f4(t,x); f5(t,x); f6(t,x)];

% Alternatively, we could do:
F123 = @(t,x) x(4:6);
F456 = @(t,x) (q/m) * (E(t,x(1:3)) + cross (x(4:6), B(t,x(1:3))));
F = @(t,x) [F123(t,x); F456(t,x)];

% Set initial coordinate (1,0,0); initial velocity (1,1,1)
x0 = [1;0;0;5;5;5];

% Solve from t = 0 until t = 1
[T,X] = ode23 (f, [0,1], x0);

plot3 (0,0,0,'ko')
hold on
plot3 (X(:,1),X(:,2),X(:,3))
xlabel ('x')
ylabel ('y')
zlabel ('z')
```