Pricing Solutions to the Bilateral Monopoly Problem under Uncertainty*

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I. Introduction

That (uniform) pricing solutions to the bilateral monopoly problem are sub-optimal is well known; see, for example, Scherer [19], and Waterson, [22]. Over the years, there has been considerable debate in the literature as to what constitutes a correct solution to the static bilateral monopoly problem under uncertainty. This is usefully reviewed in Machlup and Taber [10] and Blair, Kaserman and Romano [3]. As the latter paper points out, the principal conclusion of this debate is that the solution must involve

(i) an agreement to trade the joint-profit maximizing quantity (JPMQ), and
(ii) an agreement over the division of the associated joint profits.

However, the idea that pricing will not work in the above problem clearly rests with the assumption that prices must be uniform. With just a single seller, single buyer, and a single period model, there is scope for neither secondary trading nor storage; thus non-uniform pricing is feasible and it can be used to support the optimal solution. Appropriately designed, the offered non-uniform pricing schedule (NUPS) will be acceptable to the buying firm so long as the appropriate profit transfer is effected and the joint profit maximizing quantity traded. At the same time, the agreement need not involve requiring the buying firm to purchase the JPMQ; the NUPS can be designed to give it the marginal incentive to freely choose this. The NUPS which solves the bilateral monopoly problem is by no means unique, there being a continuum of such solutions.

This idea that nonuniform pricing solutions can often be used to solve the bilateral mo-

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1. It is a (possibly semantic) curiosity that almost all those who agree on (i) and (ii) conclude that (iii) both price and quantity will be determined through bilateral bargaining [3, 835]. despite the recognition that the price here does not serve as a rationing device but merely as a means of dividing the joint profits in some agreed proportions. In our view, it is probably better to expunge all mention of (uniform) price; that a given quantity is to be traded (JPMQ) and a certain sum of money (M) is to be transferred allows an implicit (uniform) price to be computed (p = M/JPMQ) but there is no need to mention price at all; introducing the term can at best merely serve to mislead.
The monopoly problem follows naturally from the literature on bilateral bargaining under asymmetric information—see for example Maskin and Riley [11; 12], Matthews [14], Rochet [18]—although a reasonably accessible presentation is difficult to find. Certainly, there is a whole literature on the bilateral monopoly problem which ignores the possibility of two part and other types of tariff and argues that price-quantity contracts are necessary. In view of this, it seems useful to discuss the nature of these pricing solutions in a simple and reasonably accessible model of the bilateral monopoly problem, and this is one purpose of the present paper.

Under certainty, the NUPS solution and the contract solution are fully equivalent and there is no reason to prefer one to the other. Introducing uncertainty on the demand (or supply) side entails that the JPMQ varies from period to period. A contract solution in this case would need to be continually renegotiated (which would entail significant negotiation costs) or, to avoid constant renegotiation, a "state contingent contract" might be provided which specifies quantity and money transfer for each possible state of the world. Naturally, such a contract would need to be incentive compatible. Alternatively, a single NUPS could be offered; under risk neutrality, the solution typically features marginal cost pricing over the range of uncertainty. However, non-uniform price schedules can yield the optimal solution to the problem only under certain conditions: that is, there are circumstances under which a NUPS will not support the joint profit maximizing outcome. In section III, the conditions under which the NUPS fails are explored and it is shown how a type of two part tariff, "take-or-pay" or "minimum bill" contract can be used to solve the problem in this particular case. It turns out that these pricing solutions involve marginal cost pricing over the range of uncertainty and an associated license fee or a take-or-pay quantity which is not more than the lowest possible state contingent joint profit maximizing quantity.

II. The Certainty Case

It is assumed that both parties know the buyer’s net marginal value (marginal revenue product) function, \( \nu(q, \theta) \), and the seller’s marginal cost of supply function, \( c(q) \), where \( q \) denotes the quantity traded. In this section, \( \theta \) is a fixed constant; in section III, the case of demand uncertainty will be examined in which \( \theta \) will feature as a random variable. It is assumed that \( \nu(q, \theta) \) and \( c(q) \) are differentiable and that, for any given \( \theta \), \( \nu(q, \theta) - c(q) \) is a strictly decreasing function of \( q \) such that \( \nu_q(q, \theta) - c'(q) < 0 \). It is also assumed that for any \( \theta \), \( \nu(q, \theta) - c(q) < 0 \) for sufficiently large \( q \) and that \( \nu(0, \theta) - c(0) > 0 \). That is, some positive trade is value enhancing.

Denote the joint profit maximizing output as \( q^* \), with associated profits \( \pi^* \). These are determined as the solution to the problem:

\[
\max_{q} \pi = \int_{0}^{q} (\nu(\tau, \theta) - c(\tau))d\tau
\]

for which first and second order necessary conditions are:

\[
\nu(q^*, \theta) - c(q^*) = 0
\]

\[
\nu_q(q^*, \theta) - c'(q^*) \leq 0.
\]

\( \pi^*, q^* \) would of course be realized by a vertically integrated firm (and there is an incentive to vertically integrate to the extent that the bilateral monopoly result falls short of this). The bargaining solution to this problem is not analyzed here (various approaches have been taken; for example, Zeuthen [23], Nash [16], Harsanyi [6], Foldes [4]). We simply note that it always in-
volves trading \( q^* \) at a price which will entail a particular division of the joint profit between the two parties (conditions (i) and (ii) identified in section I). The division of profits in proportion \( \alpha \) to the buyer, \( 1 - \alpha \) to the seller usually requires \( \alpha = 0.5 \) although \( \alpha \) is in general contingent upon the specific assumptions made in the bargaining game. In this paper, it is assumed merely that the bargaining process would result in some \( \alpha \in (0, 1) \). Since, with full information, this outcome is known, there is no need to actually go through the bargaining process; \( q^* \) and \( \alpha \) can be agreed on immediately. The question then becomes one of whether a pricing solution can support the equilibrium.

Let \( p(q) \) denote the (possibly only piecewise continuous) price schedule. The selling firm offers the price schedule; once accepted, the buying firm is free to purchase any quantity it desires. The buyer will be concerned to maximize profits, so will select a quantity \( Q \) so as to:

\[
\text{Maximize } \int_0^Q \{\nu(q, \theta) - p(q)\}dq
\]

for which a first order necessary condition is that, if \( Q > 0 \),

\[
\nu(Q, \theta) - p(Q) = 0.
\]

In addition, the buying firm will accept this price schedule as a basis for purchases only if it obtains at least the share \( \alpha \pi^* \) of profits. That is, if

\[
\int_0^Q \{\nu(q, \theta) - p(q)\}dq \geq \alpha \pi^*.
\]

The selling firm in choosing a price schedule will recognize that the schedule must satisfy constraints (5) and (6). It thus chooses \( p(q) \) so as to

\[
\text{Maximize } \int_0^Q \{p(q) - c(q)\}dq
\]

subject to (5) and (6). The solution to this variational problem is fairly trivial (see Figure 1) and the derivation is omitted. The necessary conditions include:

\[
P(Q) = \nu(Q, \theta) = c(Q)
\]

and the constraint (6) holding with equality. Equations (2) and (8) entail that the schedule is chosen by the seller so that the buyer then chooses to purchase the joint profit maximizing quantity \( Q = q^* \). That is, if \( p(q) \) is continuous it must pass through the intersection of the \( c(q) \) and \( \nu(q, \theta) \) schedules in Figure 1 (and cut \( \nu(q, \theta) \) from below).\(^3\) The function \( p(q) \) can take on almost any path to the left of \( q^* \) so long as it divides the joint profits (area \( abc \)) in the proportions \( \alpha \pi^* \) and \( (1 - \alpha)\pi^* \) (indeed it is possible to construct an optimal price path \( p(q) \) which lies everywhere outside of the region \( abc \)).\(^4\)

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2. Alternatively the buyer could offer the schedule and the seller would then be free to choose.
3. The price schedule could, of course, feature an upward discontinuity at the output level \( Q = q^* \).
4. Technically, the price schedule must also satisfy the global rationality condition that \( q^* \) be at least as good as any other \( q \); that is

\[
\int_0^{q^*} \{\nu(\tau, \theta) - p(\tau)\}d\tau \geq \int_0^q \{\nu(\tau, \theta) - p(\tau)\}d\tau \text{ for all } q \geq 0.
\]

Clearly, this condition would be satisfied by a price schedule which lies in the area \( abc \) in Figure 1 on \([0, q^*]\) although this is by no means required.
Hence with full information and certainty, a pricing solution is just as viable as a contract. This is not too surprising since a contract may be thought of as a particular kind of non-linear pricing schedule (one in which marginal price becomes 'infinite' at a particular quantity). As mentioned earlier, despite its simplicity, this solution does not seem to have been explicitly recognized in the literature on bilateral monopoly under certainty; the only paper to do so to our knowledge is a note by Blair and Kaiserman [2] in which they discuss a particular solution which in terms of Figure 1 amounts to a linear price path passing through the point $b$. The above analysis emphasizes the fact that this is merely one of many optimal NUPS which solve the problem—in the next section we show that, in the presence of uncertainty, a linear price path will no longer do as a solution to the problem.

III. Pricing Solutions under Uncertainty

The question of the relative performance of the pricing solution vis-à-vis the contract solution becomes more interesting in the presence of uncertainty. We shall deal exclusively with demand uncertainty (the results obtained carry over to the symmetric case where there is supply uncertainty although not to the case where there is both demand and supply uncertainty) and where, for simplicity, both parties are risk neutral.\(^5\)

The information environment assumed is as follows; both parties know the supplier's cost function, the structure of the demand (marginal revenue product) function $\nu(q, \theta)$, and the associated density function $f(\theta)$.\(^5\) However, only the buyer knows the ex post value of $\theta$. As before, $\nu_q(q, \theta) - c'(q) < 0$ is assumed and in addition $\nu_\theta(q, \theta) > 0$. The latter inequality entails that $\nu(q, \theta)$ schedules are ordered by the parameter $\theta$ which is scaled, without loss of generality, to have range $[0, 1]$. As in section II, it is also assumed that $\nu(q, \theta) - c(q) < 0$ for sufficiently large

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5. The nature of the bargaining solution in the case where the agents are risk averse and there may be both supply and demand side uncertainty is discussed in Rochet [18].
$q$ and that $\nu(0,0) > c(0)$. The latter is the reasonable assumption that, even when demand is at its minimum ($\theta = 0$), bilateral monopoly is a situation where trade is value enhancing.

From an ex ante viewpoint, the maximum joint profit is now a random variable defined by:

$$\pi(\theta) = \int_0^{Q(\theta)} [\nu(q, \theta) - c(q)] dq$$

(9)

where $Q(\theta)$ is given by:

$$\nu(Q(\theta), \theta) - c(Q(\theta)) = 0.$$  

(10)

Note that $Q(\theta) > 0$ for all $\theta \geq 0$ given our assumptions. Furthermore, from (10), $dQ(\theta)/d\theta > 0$ since $\nu_q(q, \theta) - c'(q) < 0$ and $\nu_\theta(q, \theta) > 0$. This in turn means that $d\pi(\theta)/d\theta > 0$; that is, increases in demand increase joint profits. Let $\pi^*$ denote the maximum expected joint profits:

$$\pi^* = \int_0^1 \pi(\theta)f(\theta)d\theta$$

(11)

Clearly, vertical integration would attain the maximum expected joint profit level $\pi^*$. The question is whether a NUPS can support the same solution and agreed split of joint profit, $\alpha \pi^*$ to the buyer, $(1 - \alpha)\pi^*$ to the seller. The basic result is:

**Proposition 1.** A non-uniform pricing schedule exists which solves the bilateral monopoly problem if and only if

$$(1 - \alpha)\pi^* \leq \pi_m = \int_0^{q_1} [\nu(q, \theta) - c(q)] dq$$

where $q_1$ is defined by $\nu(q_1, 0) = c(q_1)$. The NUPS involves marginal cost pricing, $p(q) = c(q)$, on the interval of uncertain demand $[q_1, q_2]$ where $q_2$ is given by $\nu(q_2, 1) = c(q_2)$.

Proof is provided in the Appendix.

Figure 2 illustrates the solution described in Proposition 1. Marginal cost pricing over the range of uncertainty guarantees that whatever the value of $\theta$, the joint profit maximizing quantity will be purchased; ex post profits, and hence ex ante expected profits are maximized. The price schedule to the left of $q_1$ can be used to ensure the seller receives the required share of this joint profit, $(1 - \alpha)\pi^*$. Figure 2 also makes clear why the solution will not always work; if $(1 - \alpha)\pi^* > \pi_m$ (i.e., area $abc$), then $(1 - \alpha)\pi^* = \int_0^{q_1} [p(q) - c(q)] dq > f_0^{q_1} [\nu(q, 0) - c(q)] dq$ which implies that $f_0^{q_1} [\nu(q, 0) - p(\theta)] dq < 0$. It is easy to see that the buyer would choose zero purchases on some interval of low $\theta$ i.e., on $[0, \varepsilon)$, for some $\varepsilon > 0$. Clearly this entails that joint expected profits are no longer maximised and the solution no longer optimal. The seller can only move towards the target level $(1 - \alpha)\pi^*$ by either excluding purchases for some $\theta$ or through raising price above marginal cost in the range of uncertain demand. If there is exclusion or if prices are above marginal cost then joint profit maximising quantities will not be traded and joint expected profits will be less than $\pi^*$.

6. An electricity utility selling to a large industrial firm provides a good example where the seller's cost structure is likely to be known but where the demander's willingness to pay is uncertain. It is not unreasonable to assume knowledge of the density function since, under any given price schedule, information on the density function is being continually provided by the firm's consumption choices.

7. We skip over some of the mathematical (measure theoretic) intricacies in presenting the following analysis—thus here the result $p(q) = c(q)$ need only hold "almost everywhere" on $(q_1, q_2)$ etc.
Given that a NUPS may not solve the problem, we now consider the effect of introducing a license fee, minimum bill or take-or-pay element into the pricing contract.

**Proposition 2.** A take-or-pay pricing contract exists which solves the bilateral monopoly problem under uncertainty. The pricing contract involves marginal cost pricing, \( p(q) = c(q) \), on the interval of uncertain demand, \([q_1, q_2]\), and, if \((1 - \alpha)\pi^* > \pi_m\), a take-or-pay quantity \(q_{tp}(0 \leq q_{tp} \leq q_1)\) and minimum payment, \(M\), of at least \((1 - \alpha)\pi^* - \pi_m\) (i.e. \((1 - \alpha)\pi^* - \pi_m \leq M \leq (1 - \alpha)\pi^*\)).

Proof is provided in the Appendix.

The take-or-pay pricing contract takes the form that the total payment, \(R(q)\) by the buyer for purchasing \(q\) units of the good is now

\[
R(q) = \begin{cases} 
M & \text{for } 0 \leq q \leq q_{tp} \\
M + \int_{q_{tp}}^{q} p(\tau) d\tau & \text{for } q > q_{tp}.
\end{cases}
\]

That is, there is a positive marginal price for \(q > q_{tp}\), but a zero marginal price for \(q < q_{tp}\). The minimum bill for any quantity purchased (including zero consumption) is \(M\). With non-uniform pricing, the buyer can always choose to buy nothing and spend nothing. With the take-or-pay contract by contrast, this is no longer the case. The money transfer \(M\) is a sunk cost to the buying firm and hence has no effect upon its ex post purchasing decision; this take-or-pay element in the pricing contract can be used to obviate the potential for low \(\theta\) exclusion of consumption associated with a pure NUPS.
The optimal solution is by no means unique and, where the take-or-pay element is required, any \( q_{tp} \in [0, q_1] \) can be chosen so long as an appropriate \( M \) and pricing schedule \( p(q) \) for \( q \geq q_{tp} \) is defined. The case where \( q_{tp} = 0 \) differs from a two part tariff in that in this case the minimum bill, \( M \), continues to be payable; with a two part tariff, non-consumption allows the consumer to evade the fixed charge. To illustrate the point about non-uniqueness, consider the following two optimal solutions:

(a) Set \( q_{tp} = 0, M = (1 - \alpha)\pi^* \) and \( p(q) = c(q) \) for \( q \geq q_{tp} \).

(b) Set \( q_{tp} = q_1, M = (1 - \alpha)\pi^* \) and \( p(q) = c(q) \) for \( q \geq q_{tp} \).

Both these solutions feature \( M = (1 - \alpha)\pi^* \). However, the minimum bill does not have to equal \( (1 - \alpha)\pi^* \); in the case where \( q_{tp} = 0 \) for example, a smaller value for \( M \) could be chosen along with \( p(q) > c(q) \) on a subinterval of \([0, q_1]\) to raise the difference. Since the maximum that can be raised by the latter in this case is \( \pi_m \), the minimum value for \( M \) is \( (1 - \alpha)\pi^* - \pi_m \) (this minimum holds more generally—see appendix).

Although the chosen quantities always exceed \( q_{tp} \) in optimal solutions, it may be worth emphasising that the take-or-pay constraint will in fact be active in some \( \theta \)-states if \( (1 - \alpha)\pi^* > \pi_m \). That is, in sufficiently low \( \theta \)-states, the buyer would prefer to choose zero consumption if this would avoid the minimum charge. It is the fact that the buyer is committed to paying at least \( M \) whatever the consumption level that induces a choice of \( q > q_{tp} \) rather than \( q = 0 \).

The take-or-pay pricing contract described above is of a form particularly appropriate for long term contracts where the two parties are involved in period by period trading under demand uncertainty.

As mentioned in the introduction, an incentive compatible state-contingent contract also exists which will solve the bilateral monopoly problem under uncertainty. Here, we briefly point out the characteristics of this contract as a prelude to discussing the practical merits of the schemes in section IV. With both the seller (principal) and buyer (agent) risk neutral, we may expect the state contingent-contract solution to involve the buyer bearing the whole of the risk involved—see for example Myerson [15] and Matthews [14]—and this in fact proves to be the case.

The state contingent contract offered by the seller specifies, for all \( \theta \in [0, 1] \), a state contingent quantity to be traded, \( \bar{q}(\theta) \) and a price \( \bar{p}(\theta) \). So long as \( \bar{q}(\theta) \) satisfies the equation

\[
\nu(\bar{q}(\theta), \theta) = c(\bar{q}(\theta)),
\]

joint expected profits will be maximized. Ignoring the incentive compatibility problem, \( \bar{p}(\theta) \) will yield the requisite profit shares so long as it satisfies the condition

\[
\int_0^1 \bar{p}(\theta)\bar{q}(\theta) \, d\theta - \int_0^1 \bar{q}(\theta) \, d\theta = (1 - \alpha)\pi^*.
\]

However, with the resolution of uncertainty, the ex post value of \( \theta \) is known to the buyer, but not to the seller. From an operational viewpoint, the value of \( \theta \) must be imparted to the seller in order that the contract can be implemented. Depending upon the type of contract, it may be that the buyer has an incentive to misreport the true value of \( \theta \). Let \( \hat{\theta} \) denote the true value and \( \tilde{\theta} \) the reported value. Naturally, if the contract is not incentive compatible, the solution is inferior to that available through the use of a take-or-pay contract or NUPS. This is so since, if there is an incentive to misreport (\( \theta \neq \hat{\theta} \) for some \( \theta \)), the joint profit maximizing quantity will not be traded, so joint profits will be reduced. Misreporting confers a gain to the buyer, but a gain which will be
of lower magnitude than the loss incurred by the seller as a result of the misreporting strategy. The nature of the incentive compatible contract is given in the following easily verified proposition;

**Proposition 3.** An optimal and incentive compatible state contingent price and quantity contract solution to the bilateral Monopoly problem involves trading a quantity $\bar{q}(\theta)$ at a price $\bar{p}(\theta)$ when the buyer announces $\theta$ where $\bar{q}(\theta)$ and $\bar{p}(\theta)$ are defined by the equations

$$
\nu(\bar{q}(\theta), \theta) = c(\bar{q}(\theta))
$$

$$
\bar{p}(\theta)\bar{q}(\theta) - \int_{0}^{\bar{q}(\theta)} c(\tau)d\tau = (1 - \alpha)\pi^*
$$

for $\theta \in [0, 1]$.

This contract motivates the buyer to reveal the true $\theta$; that is, the buyer chooses $\hat{\theta} = \theta$ for all $\theta \in [0, 1]$. The return to the seller is a constant, so this individual or firm is unconcerned about which value $\hat{\theta}$ is reported. The expected return to the buyer is thus the joint expected profits minus this constant amount, so it is in his interests to report the true $\theta$, so maximizing expected joint profits and hence his own expected return. Naturally enough, the contract parallels the non-uniform pricing result; again the seller is profit-neutral with respect to the buyer’s decision. Thus, for any $\theta \in [0, 1]$, the ex post profit gained by the seller under the incentive compatible contract will be exactly the same as under the non-uniform pricing solution (since they are both constant and raise the same total $(1 - \alpha)\pi^*$, when integrated over $[0,1]$). Similarly, ex post profits to the buyer are the same for the two cases.

**IV. Concluding Comments**

As an alternative to a state contingent (incentive compatible) price-quantity contract, non-uniform pricing (possibly in conjunction with a take-or-pay or license fee provision) will solve the bilateral monopoly problem (i.e., support maximum joint expected profits and an agreed division of this between the two parties). In practice there is no doubt that the costs of writing and administering the state contingent solution would exceed the costs associated with implementing the NUPS (with take-or-pay provisions). Indeed, it is hard to point to any examples of state contingent contracts while take-or-pay (minimum bill) contracts are quite common in markets such as natural gas, coal and bauxite, etc. [8; 7; 21; 17] and have recently received attention as a practical means of getting pricing to approximate marginal costs where demand is uncertain and marginal costs are rising [13].

It is perhaps worth mentioning that the above analysis also applies to the problem of successive monopoly (discussed by Machlup and Taber [10] and Spengler [20]) so long as the downstream firms are unable to set up secondary trading (electricity companies selling to firms for example). Furthermore, the bilateral monopoly situation arises not only where there are distinct firms, but also in the multi-divisional and multi-national firm context where transfer pricing and arms length trading occur. The minimum bill/non-uniform pricing contract discussed above can often be used in such cases and indeed in the case where there are several buying divisions, so long as there are barriers to secondary trading.

Well documented contracts are rarely available in the public domain. Arrow [1, 136] discusses weapons procurement by the US government; agreements typically involved reimbursement of all costs plus a fixed agree profit, a contract which approximates that discussed above.
(the information environment becomes critical in any bilateral agreement of course; with asymmetric information there may be little incentive for suppliers to minimize costs). A good recent example of a minimum bill contract discussed in Kay, Manning and Szymanski [9] is the agreement between EuroTunnel and BR, SNCF (the British and French Railways). This involves a fixed fee and uniform prices both for passengers and rail freight, along with a minimum payment of (1987) £105.5M per annum (indexed by RPI) for the first twelve years of operation. There is some evidence that the unit price may be slightly above marginal cost, but otherwise, it represents an excellent example of the contract in operation.

Appendix

Proof for Proposition 1

The problem is to find a price schedule which satisfies

\[
\pi_s = \int_0^1 \int_0^{Q(q)} [p(q) - c(q)] f(\theta) dq d\theta \geq (1 - \alpha) \pi^*, \quad (i)
\]

\[
\pi_b = \int_0^1 \int_0^{Q(q)} \nu(q, \theta) - p(q) f(\theta) dq d\theta \geq \alpha \pi^*, \quad (ii)
\]

\[
\nu(Q(\theta), \theta) - p(Q(\theta)) = 0, \quad (iii)
\]

\[
\int_0^{Q(q)} [\nu(\tau, \theta) - p(\tau)] d\tau \geq \int_0^q [\nu(\tau, \theta) - p(\tau)] d\tau \quad (iv)
\]

for all \( q \geq 0 \).

Equations (i) and (ii) are the joint profit maximum share constraints associated with seller, buyer respectively whilst (iii) is the marginal condition associated with a free buyer's choice of \( Q(\theta) \) and (iv) the global condition for \( Q(\theta) \) to be optimal. Summing (i) and (ii) entails that joint profits must be maximized, so the constraints will hold with equality if a feasible solution to (i)--(iv) exists. Proposition 1 states that the optimal solution involves

\[
p(q) = c(q) \text{ for } q \in [q_1, q_2] \quad (v)
\]

(\( q_1, q_2 \) are defined by \( \nu(q_1, 0) = c(q_1) \) and \( \nu(q_2, 1) = c(q_2) \) respectively (see Figure 1) and that

\[
(1 - \alpha) \pi^* \leq \int_0^{q_1} [\nu(q, 0) - c(q)] dq. \quad (vi)
\]

To show these are necessary conditions (sufficiency is trivial), first suppose that \( p(Q(\theta)) \neq c(Q(\theta)) \) on some sub-interval of \( \theta \). Then, from (iii), \( \nu(Q(\theta), \theta) \neq c(Q(\theta)) \) and this contradicts the necessary condition (10) for joint maximum profits \( \pi^* \) to be attained. Now, suppose that (v) holds but (vi) does not; that is,\n
\[
(1 - \alpha) \pi^* > \pi_m = \int_0^{q_1} [\nu(q, 0) - c(q)] dq. \quad (vii)
\]

In view of (i), (v) and (vii), we have

\[
\int_0^1 \int_0^{Q(q)} [p(q) - c(q)] f(\theta) dq d\theta = \int_0^{q_1} [p(q) - c(q)] dq \geq (1 - \alpha) \pi^*
\]

\[
> \int_0^{q_1} [\nu(q, 0) - c(q)] dq
\]

which implies

\[
\int_0^{q_1} [\nu(q, 0) - p(q)] dq < 0
\]
which contradicts (iv) (to see this, set $\theta = 0$, so $Q(\theta) = q_1$ and set $q = 0$ in (iv)).

**Proof for Proposition 2**

(a) Necessity of $p(q) = c(q)$ on $[q_1, q_2]$; as above.

(b) Necessity of $(1 - \alpha)\pi^* - \pi_m \leq M \leq (1 - \alpha)\pi^*$:

Clearly if $M > (1 - \alpha)\pi^*$, the buying firm cannot attain its profit share $\alpha\pi^*$. To demonstrate the other inequality, suppose it does not hold; that is suppose

$$M < (1 - \alpha)\pi^* - \pi_m.$$  \hspace{1cm} (ix)

The rationality condition (iv) becomes (noting $p(\tau) = 0$ for $\tau < q_{\pi}$)

$$\int_0^{Q(\theta)} \nu(\tau, \theta)d\tau - \int_{q_{\pi}}^{Q(\theta)} p(\tau)d\tau - M \geq \int_0^{q} \nu(\tau, \theta)d\tau - \int_{q_{\pi}}^{q} p(\tau)d\tau - M$$  \hspace{1cm} (x)

for all $q \geq 0$.

In particular, when $\theta = 0$, $Q(\theta) = q_1$ so, setting $q = 0$, (x) implies

$$\int_0^{q_1} \nu(\tau, 0)d\tau - \int_{q_{\pi}}^{q_1} p(\tau)d\tau \geq 0.$$  \hspace{1cm} (xi)

Since (v) holds, (i) becomes

$$\pi_x = \int_{q_{\pi}}^{q_1} p(\tau)d\tau - \int_0^{q_1} c(\tau)d\tau + M \geq (1 - \alpha)\pi^*.$$  \hspace{1cm} (xii)

Using (ix) and the definition of $\pi_m$ (see (vii)), (xii) reduces to

$$\int_0^{q_1} \nu(\tau, 0)d\tau - \int_{q_{\pi}}^{q_1} p(\tau)d\tau < 0.$$  \hspace{1cm} (xiii)

Equation (xiii) contradicts (xi), hence the inequality stated in Proposition 2 must hold.

**References**


