DEPARTMENT OF ENGINEERING MATHEMATICS

UNIVERSITY OF NEWCASTLE UPON TYNE

DIAGNOSYS

BACK-UP MATERIAL FOR BASIC MATHEMATICS

October, 1995

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

Introduction

The ***DIAGNOSYS*** test that you have just done is intended to assist you in identifying areas of mathematics you need to pay attention to, in order to succeed with your course. For some of them you may feel you need only a little practice; for others you may need to do rather more. Some of the conclusions of the Test are based on its 'expert-system' approach. For example, if you can expand two brackets multiplied together, we assume you can also expand a single pair of brackets. Conversely, if you have trouble with one pair of brackets, you presumably also have trouble with two pairs! In some cases, you may disagree with our conclusions.

The sheets in this booklet are for you to study to make up any weaknesses we have identified, or any you know of (even if you got the question right!). Each covers one or more skills, and gives a very brief explanation, some examples and some exercises. It is important that you try to understand the examples. No solutions are given to the exercises, and you should look for a similar example to help if you need to, or ask your maths tutor or a friend.

Some general issues about mathematics may also need attention. For example, you *must* be reliable at simple arithmetic, without your calculator, if only to estimate answers and so check that you pressed the right buttons. Also you must realise that: *algebra is not a set of rules to be remembered for moving* xs *and* ys *about.* An x or a y is actually a *symbol*, that is, it stands for something (usually a number). The rules for algebra are therefore only rules about how numbers behave, and what wouldn't be allowed with numbers (because it's *false*, not because we have said so!) isn't with symbols either. For example,

$$(5-3)(9-4) = 5\times 9 - 5\times 4 - 3\times 9 + 3\times 4 = 10$$

which is the same as 2x5. Here the operation — × — = + ('minus times minus equals plus') is used *because it works*, and gives the right answer, though this 'rule' can be justified in several ways (see the sheet for Skills 101,102,103). The same 'rule' applies to symbols, each of which represents a number.

The list below shows which sheets correspond to each skill in the Test. Most have a single sheet, some have two. Many sheets describe the material for more than one skill.

<u>Some skills are not covered in these materials</u>. There are no sheets for topics in Statistics, and only Geometric Progressions and Principles of Differentiation (not a numbered skill) are covered from the Calculus group of skills.

This booklet is an extended first edition, to accompany the diagnostic testing system *DIAGNOSYS* in Version 2.21, released in October 1995. Comments on the usefulness of these materials and on their format and content are welcomed.

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 101,102,103 OPERATIONS WITH MINUS SIGNS

Explanation

A minus sign is used in three basic ways, which are then combined in other operations:

Taking away part or all of what is there (subtraction)

$14-8=6,\ 5-5=0,\ 10-\pi=6.8584,\ 5x-3x=2x$

A negative quantity

If velocities are measured in the x-direction, then a velocity of -5.3m/s means movement in the $-x$ direction

Difference between quantities

The difference between a temperature of 10^0 above freezing and 5^0 below freezing is 10 - (-5) which is 15^0.

Examples

1. Two accounts, one in credit +£5 and one in debit -£3 have a total balance of (5) + (-3) = £2.
2. The time between 5 seconds into the future, and 3 seconds in the past is (5) - (-3) = 8 secs.
3. A car accelerates at 2m/s every second ($2ms^{-2}$). Five seconds ago, its velocity was $(-5)\times(2)=-10$m/s *greater, i.e.* 10 m/s/ less, if we count forward motion as positive.
4. A tank fills with water or empties. If we measure *increases* in the water level, and there is a rate of filling of -10 cm per minute (*i.e.* a decrease), then 5 minutes ago, the level was (-5) × (-10) = + 50 cm higher.

Brief examples

	9 - 6 = 3	4 - 11 = -7	2 + (-3) = -1	-2 + (-3) = -5
	3 - (-1) = 4	-4 - (-3) = -1	2 × (-3) = -6	(-4) × 5 = -20
	-3 × 2 = -6	(-4) × (-5) = 20	(-30) ÷ 6 = -5	45 ÷ (-9) = -5
	(-49) ÷ (-7) = 7	2·1×(-3·5)= -7·35	(-1·4)×(-2·6)=3·64	(-1·43)÷(-1·1) = 1·3
Also	$(-3)^2=9$	$(-4)^3=-64$		

Exercises (Use the values x = -3·5 and y = -0·5)

1. 5 + (-3) = ? 2. 4 + (-11) = ? 3. (-5) + 18 = ? 4. (-4·5) + (-3·6) = ?

5. 11 - (-5) = ? 6. 3·5 - (-5(-26) =? 7. -4·4 - (-5·6) = ? 8. 11·356 - (-13·444) = ?

9. $x + y$ = ? 10. $x - y$ = ? 11. $-x + y$ = ? 12. $-x - y$ = ?

13. $2x - 3y$ = ? 14. $-2(-x - 3y)$ = ? 15. 3×(-4·3) = ? 16. (-4) × (-3·3) = ?

17. $x \times y$ = ? 18. $x \times (-y)$ = ? 19. $(-3x) \times (-2y)$ = ? 20. $(-2.5)^2$ = ?

21. $(-1)^3$ = ? 22. 12 ÷ (-1·5) = ? 23. (-20) ÷ (-8) = ? 24. 1 ÷ (-1·5) = ?

SKILLS 104,105,201 DECIMALS, < > SIGNS

Explanation - decimals

An *integer*, or whole number, is usually written in *base 10* and needs no decimal point. The position of each digit indicates whether it represents *units, tens, hundreds* etc. Thus 307 means 3 hundreds + 0 tens + 7 units. We extend this, so that digits after the decimal point represent *tenths, hundredths, thousandths* etc. Thus $0 \cdot 14$ means 1 tenth + 4 hundredths, or $\frac{1}{10} + \frac{4}{100}$.

To compare numbers with decimal places, add 0s to make them of the same length, if you find that helpful.

Examples

1. $1 \cdot 52$ compared with $1 \cdot 6$, becomes $1 \cdot 52$ compared with $1 \cdot 60$. Overall, 160 hundredths is greater than 152 hundredths, so $1 \cdot 6 > 1 \cdot 52$.
2. 3 is greater than $2 \cdot 9948$.
3. $0 \cdot 09956$ is less than $0 \cdot 103$.

Explanation - < > signs

When we talk about the size of a number, we may mean "how big is it?", or we may mean "where is it, relative to other numbers?" If we think of negative numbers as being in the direction (left, say) opposite to positive numbers (right), then '<' or 'less than' means 'to the left of', and '>' or 'greater than' means 'to the right of'.

-3 -2 -1 0 +1 +2 +3

Examples

1. $2 \cdot 5 < 3 \cdot 5$
2. $1 \cdot 09 < 1 \cdot 1$
3. $-1 \cdot 5 < +2 \cdot 3$
4. $-3 < +2$
5. $4 \cdot 27 > -165$
6. $-5 > -6$
7. $-\pi < 2$
8. "If I am £600 overdrawn, and you have 45p credit, you have more in your account than I do", or, $-600 < 0 \cdot 45$.
9. $-(-4 \cdot 6)^2 < -4 \cdot 6$
10. $(0 \cdot 2)^2 < 0 \cdot 2$
11. $(-0 \cdot 2)^2 > -0 \cdot 2$

Exercises

Which is greater in each case - write your answer using a > or < sign:

1. $2 \cdot 763$ and $4 \cdot 0001$
2. $0 \cdot 0007$ and $0 \cdot 00103$
3. -2 and +4
4. $-3 \cdot 5$ and $+2 \cdot 1$
5. $-7 \cdot 003$ and $-4 \cdot 1$
6. $-5 \cdot 013$ and $-5 \cdot 031$
7. $(3 \cdot 6)^2$ and $3 \cdot 6$
8. $(0 \cdot 1)^2$ and $0 \cdot 1$
9. $(-1 \cdot 5)^2$ and $-1 \cdot 5$

DIAGNOSYS- BASIC MATHEMATICS FOLLOW-UP

SKILLS 105,202 DECIMAL PLACES & SIGNIFICANT FIGURES - ACCURACY

Explanation

Most practical calculations are concerned with results that are *sufficiently accurate*, rather than exact. There ae different levels of accuracy, appropriate to various needs.

Examples

1. 743 *to the nearest hundred* is 700
2. 3·05762 m. *to the nearest mm* is 3·058 m.
3. π (= 3·14159) *to three decimal places* is 3·142.
4. There are 39·4 inches in a metre, *to three significant figures* (39·3700...is more accurate).

Explanation

To express overall (relative) accuracy, we think of significant figures. e.g. 3·09 million miles is expressed as accurately as π = 3·14 (both to "3 sig. figs."). But to compare or combine two values, they must have the same (*absolute*) accuracy. e.g. adding 0·13 to -420 gives -420 to 2 sig.figs. Similarly, if our data is available to only 3 sig.figs, say, there is little point in calculating and recording answers to 10 sig. figs. (as given by most calculators).

To express a number to a required accuracy, we *round* it (as in the examples above). The only difficulty is how to deal with e.g. 650 to the nearest hundred. The most common convention is to round '5' up, so here our result is 700,

Examples

To 3 sig. figs , we get

5.	3·14159265...→ 3·14	6.	- 0·032486 → - 0·0325
7.	1728·31 → 1730	8.	1·0009 → 1·00

9. 3·0445 → 3·045 to 4 sig. figs.
10. -10513 →-11000 to 2 sig. figs.
11. 45 (exactly), expressed to 3 sig. figs., must be written 45·0

Exercises

Express each number to the number of significant figures given after it in ( ).

1.	42·458	(4)	2.	42·458	(2)
3.	-17051·4	(3)	4.	0·0002049	(1)
5.	- 4·0316	(3)	6.	741·6	(1)

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 106,204 POSITIVE POWERS

Explanation

x^2 ("x squared") is just a shorthand way of writing $x \times x$, and 4^3 ("4 cubed") means $4 \times 4 \times 4$. Other *powers* are interpreted the same way: y^4 ("y to the power 4", or just "y to the 4") means $y \times y \times y \times y$.

If we look at the sequence: $5^4, 5^3, 5^2$ we see that each is the previous one divided by 5. So we extend the idea and say 5^1 means 5, and 5^0 means 1. In the same way, x^1 means simply x, and x^0 is $x \div x$, which is 1. Any number to the power 0 is 1 for the same reason.

The 'rules' for powers come from the basic definition, and are best remembered this way.

Rule 1 $\quad x^3x^2 = (x.x.x.)(x.x.) = x^5$ hence: $\boxed{x^a x^b = x^{a+b}}$.

Remember this as "x times itself a times and b times more, so $a + b$ times altogether".

Also $u^5 \div u^2$ means $\dfrac{(u.u.u.u.u.)}{(u.u.)} = u^3$, hence

Rule 1b $\quad \boxed{x^a/x^b = x^{a-b}}$.

Rule 2 $\quad$ In the same way $(z^3)^2$ means $(z.z.z.)(z.z.z) = z^6$ hence

$\boxed{(x^a)^b = x^{a.b}}$.

Remember this as "x times itself a times, repeated b times, so $a.b$ times altogether."

Examples

1.	$3^2 \times 3 = 3^2 \times 3^1 = 3^{2+1} = 3^3$	2.	$9 \times 9 = 9^{1+1} = 9^2$
3.	$x^4x^6 = x^{10}$	4.	$(2x)^3 = 2x.2x.2x. = 2^3x^3 = 8x^3$ (not really another rule)
5.	$8^6 \div 8^4 = 8^{6-4} = 8^2$	6.	$y^5 \div y = y^{5-1} = y^4$
7.	$x \div x = x^{1-1} = x^0 = 1$	8.	$(3^4)^2 = 3^{4\times2} = 3^8$
9.	$(w^3)^3 = w^9$	10.	$(x^2y)^3 = x^6y^3$
11.	$(x/y^2)^4 = x^4/y^8$	12.	$u^3/u^3 = u^{3-3} = u^0 = 1$

Exercises

1.	$7^3 \times 7^2 = ?$	2.	$6^2 \times 6 = ?$	3.	$x^4x^5 = ?$
4.	$y^3y^7 = ?$	5.	$(2x)^4 = ?$	6.	$p^1.p^1 = ?$
7.	$12^6 \div 12^3 = ?$	8.	$x^4 \div x^2 = ?$	9.	$z^7 \div z = ?$
10.	$a^5/a^5 = ?$	11.	$(5^2)^3 = ?$	12.	$(5^3)^2 = ?$
13.	$(z^4)^3 = ?$	14.	$(2u^3)^2 = ?$		

SKILL 107 RATIOS

Explanation

A ratio is a comparison of two numbers, lengths, or any other quantities. "Three times as big" or "twice as much" are easy to understand. But if two bars are of lengths 3 m and 2 m, we say their lengths are in the ratio "three to two", or 3:2. If their lengths were 6 m and 4 m, the ratio would be the same, namely 3:2. To see this, imagine measuring 2 m lengths of each; notice also that $3 \div 2$ is the same as $6 \div 4$.

Proportion is a related idea. The area of a rectangle is length $\times$breadth, or $A = LB$. We can say $A \propto L$, or "A is proportional to L". This means that, if L increases by the factor r, A increases by the same factor, i.e. $L \to rL$ and $A \to rA$ (if B stays fixed). This can also be stated: for a fixed value of B, $A{:}L$ is a constant ratio. Note that also, if L is fixed, $A \propto B$.

Examples

1. 3:2, 6:4, 12:8, 9:6, $1{:}\frac{2}{3}$, -3: -2, $\frac{1}{2}{:}\frac{1}{3}$ are all the same ratio.
2. 2:1, 26:13, $11\cdot5{:}5\cdot75$, $-\frac{1}{2}{:}-\frac{1}{4}$, $0\cdot1{:}0\cdot05$ are all the same ratio.
3. 3:5, 27:45, - 12: - 20, $1{:}1\cdot6667$, 20:30 are all the same ratio except the last one!
4. If $Q = \dfrac{\pi a^4 p}{8\mu L}$,then we can say e.g. $Q \propto p$, also $Q \propto a^4, Q \propto 1/L$ if the other variables are fixed.

Exercises

Give five other ratios the same as each of

1. 3:1, 2. 1:2

3. 4:3, 4. 2:5

- include some negative, fractional and decimal examples.

5. If $y = \dfrac{3x^2 z}{1+w}$, give two quantities that y is proportional to.

SKILL 108 FACTORISATION OF INTEGERS

Explanation

A factor of a positive integer (whole number) is another positive integer that divides exactly into it.

Examples

(i) The factors of 6 are 6, 3, 2 and 1.

(ii) The factors of 49 are 1, 7 and 49.

(iii) 8 is *not* a factor of 12, 20 or 100.

(iv) The only factors of 83 are 1 and 83 (it is a *prime number*).

(v) The factors of 24 are 1, 2, 3, 4, 6, 8, 12, 24.

Explanation

A positive integer can also be the *product* of two negative integers, whilst a negative integer can be the product of one positive and one negative integer.

Examples

(vi) -6 = -6 x 1 or -3 x 2 or -2 x 3 or -1 x 6
(vii) 16 = 1 x 16 or 2 x 8 or 4 x 4 or (-1) x (-16) or (-2) x (-8) or (-4) x (-4)

Exercises

Write down all the positive factors of :

(1) 8 (2) 20 (3) 35 (4) 100 (5) 41

Write down all possible pairs of factors, positive and negative, for the following:

(6) 6 (7) 12 (8) -12 (9) -3 (10) 8

What three numbers (give all possibilities) could multiply to give each of:

(11) +12 (12) -8 ?

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 112,215 PRECEDENCE OF OPERATIONS

Explanation

"Double and add 3" is not the same as "Add 3, then double" - the order in which we do things matters, and must be clearly understood in any calculation or piece of mathematics. *Precedence* means *order* or *priority*. Some rules can be handled by your calculator, e.g. examples 1 - 4, 6 and 7, but not 8. The rules are:

1. *Brackets* - the contents of these are worked out first.
2. *Powers* or other *functions* come next.
3. *Multiply* and *Divide* are next. 4. *Add* and *Subtract* are last.

Examples

1. $7 - 3 \times 4$ is $7 - 12$ which is -5 *not* 4×4.
2. $(7 - 3) \times 4$ *is* 4×4.
3. $2x - 1$ means $2 \times x$, then subtract 1.
4. $3x^2$ means x^2, then multiply by 3.
5. $x \sin x$: find $\sin(x)$, then multiply by x.
6. $3 - \frac{1}{2}$: divide 1 by 2 then subtract from 3.
7. $4 + 2/x$: divide 2 by x then add to 4.
8. $\dfrac{1}{2+3}$ indicates 1 divided by the *whole* of $2 + 3$, i.e. it means the same as $\dfrac{1}{(2+3)}$. It is *not the same as* $\dfrac{1}{2} + \dfrac{1}{3}$. This is one of the commonest mistakes in basic mathematics!
9. $\dfrac{x-2}{x+1} = \dfrac{x}{x+1} - \dfrac{2}{x+1}$ but $\neq \dfrac{x-2}{x} + \dfrac{x-2}{1}$
10. The calculator sequence for $\dfrac{1}{2+3}$ is $1 \div (2 + 3)$.
11. The sequence for $\dfrac{1}{2 \times 3}$ is $1 \div (2 \times 3)$ or, if you like, $1 \div 2 \div 3$ but *not* $1 \div 2 \times 3$.
12. What does $x - y - z$ mean? By convention work left - to - right, so $15 - 8 - 3$ is 4 not 10.

Exercises (Use the values $x = 3$, $y = -2$)

1. $11 + 2 \times 5 =$?
2. $-14 + 3 \times 7 =$?
3. $4 \times 5 - 3 =$?
4. $-(2 + 7) \times 5 =$?
5. $2 + 3^2 =$?
6. $(2 + 3)^2 =$?
7. $4 \times 5^2 =$?
8. $(4 \times 5)^2 =$?
9. $2x - y =$?
10. $x(x + 3y) =$?
11. $\dfrac{21}{4+x} =$?
12. $\dfrac{13+22}{7} =$?

Give the sequence of keys for:

13. $2 - 3x^2$
14. $3x^2 \sin x - 1$

SKILLS 203,303,304 NEGATIVE POWERS

Explanation

The positive powers 5^2, 5^3, 5^4etc. are extended to powers 1 and 0 by extending the sequence backwards (Skills 106/204). Thus reducing the power by 1 ,means dividing by 5 (or whatever it is). We can continue, giving a meaning to negative powers:

$5^3 = 125,\ 5^2 = 25,\ 5^1 = 5,\ 5^0 = 1,\ 5^{-1} = \frac{1}{5},\ 5^{-2} = \frac{1}{25}$ etc.

Clearly, for any numbers x, n, we can say $x^{-n} = 1/(x^n)$.

The reason this is useful is that it's easy to show that the rules of powers (Skill 204) $(x^a x^b = x^{a+b},\ (x^a)^b = x^{ab})$ work for negative powers as well. In fact, think of $x^a \div x^b = x^{a-b} = x^{a+(-b)} = x^a x^{-b}$ - another way of getting at negative powers.

Examples

1. $2^{-1} = 1/2^1 = \frac{1}{2}$
2. $3^{-3} = 1/3^3 = \frac{1}{27}$
3. $2^4 \times 2^{-3} = 2^{4-3} = 2^1$
4. $(x^{-2})^3 = x^{-6}$
5. $y^3 \div y^{-2} = y^{3-(-2)} = y^5$
6. $\left(x^2/y\right)^{-3} = \left(y/x^2\right)^3$ or y^3/x^6 or y^3x^{-6} etc.

Exercises

Rewrite without negative powers:

1. 5^{-2}, 2. $\left(\frac{1}{2}\right)^{-1}$, 3. $(xy^{-1})^{-4}$,

Rewrite using negative powers:

4. $\frac{1}{3^2}$, 5. $\frac{1}{x^3}$, 6. $\frac{x^2}{y}$,

Find each of:

7. $5^2 \times 5^{-4}$, 8. $3^{-1} \times 3^{-2}$, 9. $x^{-2}/\ x^{-3}$,

10. $(p^2/q^3)^{-3}$.

SKILL 205 INVERSE PROPORTION

Explanation

The area of a rectangle is length × breadth, or LB. Another rectangle with twice the length and half the breadth has the same area. If L increases by a factor r, and B decreases by the same factor, i.e. $L \to rL$ and $B \to B / r$, then LB remains the same. We can say $LB = A$, so $L = \frac{A}{B}$ or $L \propto \frac{1}{B}$, which means,

"L is proportional to $\frac{1}{B}$", or "L is *inversely proportional* to B."

For a constant speed, we can say distance = speed × time, or d = st. So in the same way we can write $s \propto 1/t$ or $t \propto 1/s$; t and s are inversely proportional to each other.

Examples

1. For a capacitor, charge Q, voltage V and capacitance C are related by $Q = VC$. We have $C \propto 1/V$ (or $V \propto 1/C$).
2. If $Q = \frac{\pi a^4 p}{8 \mu L}$, then Q is inversely proportional to L. We can also say that $p \propto 1/a^4$ if Q and L are fixed (and π, μ are constants), so p is inversely proportional to a^4.

Exercises

1. If $V = IR$ for a resistance, which quantities are proportional to each other (Skill 107) or inversely proportional?
2. If a car travels for 3 hours at 50 mph, how long would it travel at 15 mph to return the same distance?

SKILL 206 - CANCELLING FRACTIONS

Explanation

A fraction is a division - one number divided by another. Algebraic fractions behave in just the same way as numerical fractions. *Cancelling* means dividing the top and bottom (numerator and denominator) by the same number or variable. This does not change the value of the fraction, for example:

$$^{30}/_{48} = {}^{10}/_{16} \text{ (dividing both by 3)}$$
$$= {}^{5}/_{8} \text{ (dividing both by 2)}$$

It is usually a good idea to cancel as many factors as possible.

Examples

1. $^{12}/_{18} = {}^{2}/_{3}$ (divide by, or cancel, the common factor 6).
2. $^{36}/_{49}$ - no cancellation possible.
3. $\dfrac{4x^3y}{2xy^2} = \dfrac{2x^2}{y}$ - cancel factors 2, x, y.
4. $\dfrac{3p(p-1)}{p^3-p^2} = \dfrac{3(p-1)}{p^2-p} = \dfrac{3(p-1)}{p(p-1)} = \dfrac{3}{p}$ cancelling the factor $(p - 1)$ at the end.
5. *You can't do this:* $\dfrac{2x^2-x}{x^2} = \dfrac{\cancel{2x^2}-x}{\cancel{x^2}}$ This is wrong because, to keep the same value, you must divide the *whole* top and bottom.
6. Also remember that a cancellation is a division, so e.g. x cancelled from x gives 1 and not 0, e.g. $\dfrac{x+x^2}{x} = \dfrac{1+x}{1} = 1+x$.

Exercises

Cancel as far as possible (you can work in stages):

1. $^{14}/_{35}$
2. $^{90}/_{36}$
3. $\dfrac{2x^2yz}{3xz^3}$
4. $\dfrac{4(a+b)}{6(a^2-b^2)}$ (factorise the denominator)
5. Explain what this student has done (wrongly!):

   Given $y = \dfrac{x}{x-1}$, let $x = t+1$, then

   $$y = \frac{t+1}{t+1-1} = -1$$

SKILL 207,307 MULTIPLYING AND DIVIDING FRACTIONS

Explanation

Numerical and algebraic fractions behave in exactly the same way, so the rules for, e.g. multiplying them are the same.

Consider $\frac{2}{5}\times\frac{2}{3}$. Think of it as $\frac{2}{5}\times 2 \div 3$ and it is clear that we get $\frac{4}{5}\div 3$ and then $\frac{4}{15}$. In short, this is $\frac{2\times 2}{5\times 3}$. We may be able to cancel a common factor afterwards.

Now look at $1\div\frac{1}{3}$ which is obviously 3. Division by $\frac{1}{3}$ is equivalent to multiplication by 3. So division by, e.g. $\frac{2}{3}$, is equivalent to multiplication by 3 and division by 2. Thus $\frac{4}{15}\div\frac{2}{3}=\frac{4}{15}\times 3\div 2$ or $\frac{4}{15}\times\frac{3}{2}$. The answer is $\frac{12}{30}=\frac{2}{5}$ after cancelling the factor 6.

Examples

1. $\frac{1}{4}\times\frac{2}{3}=\frac{1\times 2}{4\times 3}=\frac{2}{12}=\frac{1}{6}$
2. $\frac{1}{x}\times\frac{x^2}{y}=\frac{x^2}{xy}=\frac{x}{y}$
3. $\frac{p-1}{p}\times\frac{p^2}{p^2-1}=\frac{p^2(p-1)}{p(p^2-1)}=\frac{p(p-1)}{(p^2-1)}=\frac{p(p-1)}{(p+1)(p-1)}=\frac{p}{p+1}$
4. $\frac{3}{2}\div\frac{1}{2}=\frac{3}{2}\times\frac{2}{1}=\frac{6}{2}=3$
5. $\frac{4}{3}\div\frac{3}{7}=\frac{4}{3}\times\frac{7}{3}=\frac{28}{9}$
6. $\left(\frac{2x^2}{x+1}\right)\Big/\left(\frac{x}{2}\right)=\frac{4x^2}{x(x+1)}=\frac{4x}{x+1}$

Exercises (cancel afterwards if possible)

1. $\frac{3}{10}\times\frac{5}{6}=?$
2. $\frac{x}{2}\times\frac{3(x+1)}{x}=?$
3. $\frac{2p^2q}{q+p}\times\frac{3(q+1)}{q^2p}=?$
4. $\frac{2}{5}\div\frac{3}{20}=?$
5. $\frac{2}{7}\div\frac{5}{7}=?$
6, $\frac{u^2v}{2w}\div\frac{4u}{vw}=?$

SKILL 208 FACTORS OF ALGEBRAIC PRODUCTS

Explanation

To factorise expressions such as $3x-6x^2 \to 3x(1-2x)$ (Skill 209) we need to identify possible factors of each term. For example, $6x^2$ has factors 1, 2, 3, 6, x, x^2, $2x, 3x, 6x, 2x^2, 3x^2, 6x^2$ - a lot of possibilities! We don't actually write them down each time, but we need to recognise that they exist. In fact, $6x^2$ can also have the same factors again, but negative, giving 24 possible factors!

When we see more than one variable (x and y, say), there are still more possibilities. We also need to recognise what *isn't* a factor. For example, $12x^3y$ has factors $12x^2y, 2x, 6y, x^3$ and many others, but *not* $3xy^2$, because there is no y^2 factor in the original. ($12x^3y$ has actually got 48 possible factors, excluding negative ones).

A negative term will also have positive and negative factors, although one of each will be needed to produce the original. For example $-3x$ has factors $-1, +3, +x, -x$ etc.

Examples

1. $4x$ has factors $1, 2, 4, x, 2x, 4x$
2. xy has factors $1, x, y, xy$ (also $-x$, -1 etc.)
3. $2u^3$ has factors $1, 2, u, 2u, u^2, u^3, 2u^3$
4. θ^4 has factors $1, \theta, \theta^2, \theta^3, \theta^4$ (also $-\theta, -\theta^2$ etc.)
5. $-6z^2$ has factors $+1, -1, +2, -3z, -6z, 2z^2$ etc.
6. $15u^3v^2$ has factors $3, u^2, uv, 5uv^2, -15u^3$ etc. (96 altogether including negative ones).

Note When we say here 'a positive factor' or a 'negative factor', we mean simply, 'with a + sign' or 'with a - sign'. Since x, y, u, v etc. may be negative numbers, the actual value of the factor is not yet known.

Exercises

Give *all* the positive factors of (total given in ( )):

1.	$2x$ (4)	2.	$8u$ (8)	3.	$6xy$ (16)
4.	$5z^3$ (8)	5.	abc (8)	6.	$2xy^2$ (12)

Give at least five varied factors (+ or -) of:

7.	$-6y^3$	8.	$4x^2y^2$	9.	$18w^2z^3$
10.	$-9abd^4$				

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 209 SIMPLE FACTORISATION

Explanation

Factorisation is one of a range of methods for writing something in an alternative way. Our objective may be to make use, calculation or subsequent algebra simpler. *There is no point* in trying to factorise until you are expert at expanding brackets, collecting terms and identifying factors (Skills 210,211,208)

Aim to 'take out' the highest possible factor, but it's often easiest to work in stages e.g.

$$18x - 24x^2 = x(18 - 24x) = 6x(3 - 4x)$$

Examples

1. $3 + 6x = 3(1 + 2x)$
2. $4u - 12u^2 = 4(u - 3u^2) = 4u(1 - 3u)$
3. $-3y + 6 = 3(-y + 2)$ or $-3(y - 2)$ (check your — signs!)
4. $x + x^2 - 2x^3 = x(1 + x - 2x^2)$
5. $8abc - 12b^2c + 6ac^2 = 2c(4ab - 6b^2 + 3ac)$
6. $4y - 3z + 9z^2 = 4y - 3z(1 - 3z)$ (there is no factor common to all terms)

Exercises

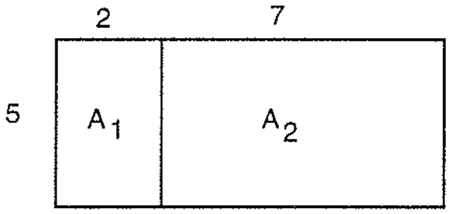
Factorise as far as possible (work in stages if you like)

1.	$4 + 8x = ?$	2.	$y + y^2 = ?$
3.	$uv + 2v = ?$	4.	$3x^4 - 6x^2 = ?$
5.	$8p + 12p^3 = ?$	6.	$-q - q^2 = ?$
7.	$-3 + 9y = ?$	8.	$10x^2z + 15xz^2 = ?$
9.	$4r^2 - 8r + 10r^3 = ?$	10.	$48t^3s - 30t^2s^2 = ?$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 210 EXPANDING A BRACKET

Explanation
Brackets are always evaluated first (Skill 215) *in principle* , e.g. 5(2 + 7) is 5×9 or 45. However, it is possible to expand the bracket as e.g. $5(2 + 7) = (5 \times 2) + (5 \times 7)$. (This is called the *distributive law*). To see that it's true, consider the areas shown:



also

$$Area = 5(2+7)$$
$$Area = A_1 + A_2 = 5 \times 2 + 5 \times 7$$

The same rule applies to subtraction and division, e.g.

$$(3 - 5)/2 = (3/2) - (5/2)$$

After expanding, collect terms if possible (Skill 211)

Note This basic rule does not apply to functions such as $(2+5)^2$ or $\sin(x+y)$. Rewriting these without brackets is more complex. Expanding multiple brackets is an extension of the rule - see Skill 312.

Examples

1. $12(3 + 4) = 36 + 48$
2. $5(5 - 2) = 25 - 10$
3. $(8 + 20)/4 = 8/4 + 20/4 = 2 + 5$
4. $-3(-2 - 7) = 6 + 21$
5. $3(x+y) = 3x + 3y$
6. $2x(3 - x^2) = 6x - 2x^3$
7. $9x + 1 - 4x(1 + 2x) = 5x + 1 - 8x^2$
8. $3u(u - \frac{1}{u}) = 3u^2 - 3$
9. $2y + 1 - 3y(y + 2 - \frac{1}{y}) = 4 - 4y - 3y^2$

Exercises (for the numerical exercises, check by evaluating the bracket first)

1. $6(2 + 4) = ?$
2. $5(-2 + 6) = ?$
3. $(12 - 8)/2 = ?$
4. $4(1 - x) = ?$
5. $2x(x + 3) = ?$
6. $3 - 2(u - 1) = ?$
7. $4v - 2v^2(1 + 2/v) = ?$
8. $5x(x \times y + 2z) = ?$
9. $2x(1 - 3x) - 3x(1 - 2x) = ?$
10. $2xy(2x - y + 2) = ?$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 110,211 COLLECTING TERMS

Explanation

At every stage of a piece of mathematical work, it is helpful to 'tidy up' or simplify what you have. This makes the results easier to understand, check and use, and may reduce work in subsequent stages.

Collecting terms means putting together similar parts, such as $-3+5 \to +2$ or $2x+7x \to 9x$, often after expanding brackets, such as

$$(x+1)(x-1) = x^2 + x - x - 1 = x^2 - 1$$

A *term* is part of a mathematical expression added or subtracted to other parts (whereas *factors* multiply together etc.). Be careful to recognise that e.g. $3x^2y$ and $-2yx^2$ can be combined, but $4x^2y$ and $3xy^2$ can't (except by factorising - see Skill 209). Also be careful of any fractions, e.g. $\frac{2}{x} - \frac{3y}{x} \to \frac{2-3y}{x}$ but $\frac{2x^2}{y} + \frac{4x^2}{y^2}$ must be left separate.

Examples

1. $3+2x-1+4x \to 2+6x$
2. $2y-4+7-3y^2+6y \to 3+8y-3y^2$
3. $2u-3a+4au+3u^2$ - no similar terms
4. $5x^3y^2+7xy-11x^2y^3+8xy \to 5x^3y^2+15xy-11x^2y^3$
5. $\frac{3}{x}+\frac{2y}{x}-\frac{4}{x^2} \to \frac{3+2y}{x}-\frac{4}{x^2}$
6. $4x^2y - \pi x^2y \to 0\cdot8584x^2y$

Also note

7. $5\sin x - 2\cos x + 3\sin x \to 8\sin x - 2\cos x$
8. $3x(x+1)^2 - 2x(x+1)^2 \to x(x+1)^2$
9. $\frac{2u}{v} - \frac{u}{2v} = \frac{4u-u}{2v} = \frac{3u}{2v}$

Exercises

1. $2+4x+6x-3+x = ?$
2. $-u^2+4u-6+2u+3u^2-u = ?$
3. $3x^2yz+4x^2yz^2-2xyz+6x^2yz = ?$
4. $-\frac{x}{2}+3x^2+\frac{1}{2}+2x = ?$
5. $2\cdot1w-4+3w^2-1\cdot7+3\cdot6w = ?$
6. $\frac{4y^2}{x}+\frac{2y}{x}-\frac{y^2}{x}+\frac{y}{2x} = ?$
7. $4x\sin x - 2\sin x + x\sin x - 4 = ?$
8. $xy+2xy(x+1)^2-3xy = ?$
9. $(x+1)^3 = (x+1)(x^2+2x+1)$
   $= x^3+2x^2+x+x^2+2x+1 = ?$
10. $(x-1)(x^2+x+1) = x^3+x^2+x-x^2-x-1 = ?$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 111,212,315,412 SOLUTION OF LINEAR EQUATIONS

Explanation

An *equation* is a relationship *satisfied* by a number or several numbers (or functions). This number (or numbers) is called the *solution* of the equation; we *solve* the equation to obtain such solutions. For example, two objects, with different initial velocities and constant accelerations have velocities at time t seconds given by

$$V_1 = 5 + 10t, \quad V_2 = 20 + 7t \qquad \text{(units } ms^{-1}\text{)}$$

At what time are their velocities equal? Clearly, when $5 + 10t = 20 + 7t$

We now *solve* this equation for the value of the *unknown t*, by doing things to both sides of the equation, *so that they remain equal.* Subtract $7t$ from both sides to get (after collecting terms)

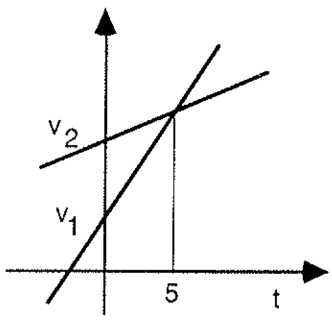
$$5 + 3t = 20$$

Subtract 5 from both sides, then divide both sides by 3, to get

$$3t = 15, \text{ then, } t = 5 \text{ seconds}$$

(At this time $V_1 = V_2 = 55ms^{-1}$)

We could also draw two straight line graphs (hence the term 'linear') giving V_1 and V_2; where the graphs cross V_1 and V_2 are equal for the same value of t.



Examples

In all cases change both sides in the same way to get $x =$..... or $t =$ .... etc.

1. $4x = -8$, divide by 4 to get, $x = -2$.
2. $3t - 2 = 12$, add 2, divide by 3, $t = \frac{14}{3} = 4 \cdot 667$.
3. $2y - 3 = 7 - 5y, \quad 7y = 10, \quad y = \frac{10}{7}$.
4. $-(2 + z) = 3(z + 4), \quad -14 = 4z, \quad z = -3 \cdot 5$.
5. Solve $2x = 1 - ax$ for x in terms of a. $2x + ax = 1, x = \dfrac{1}{2 + a}$.

The following would not have straight-line graphs, but are solvable by the same means, because more complicated terms cancel out:

6. $x^2 + 3x - 1 = (x + 1)^2, \quad 3x - 1 = 2x + 1, \quad x = 2$.
7. $\dfrac{1}{x} = \dfrac{2}{x + 1}$. Multiply both sides by x, then by $(x+1)$ to get $x+1=2x, x = 1$.
8. $\dfrac{x - 3}{2x + 3} = \dfrac{4 - x}{1 - 2x}$. Multiply by $(2x + 3)$ and by $(1 - 2x)$ to get $(x - 3)(1 - 2x) = (4 - x)(2x + 3)$ and finally $x = \frac{15}{2} = 7 \cdot 5$.

Exercises

Solve each for the unknown:

1. $3t + 1 = -8$
2. $2 + 2x = -3 + 4x$
3. $2p + 5 = -(3 - p)$
4. $2(1 - \theta) = -4(\theta + 7)$

Solve for the variable given in ( ) in terms of the other symbols:

5. $w = 3 + 4z$ (z)
6. $v = u + at$ (a)

Solve for the unknown:

7. $\dfrac{1}{u+1} = \dfrac{1}{5-u}$
8. $\dfrac{q-4}{3-2q} = \dfrac{7-q}{2q+1}$
9. Solve 2. above by drawing a straight line graph of both sides of the equation, i.e.
   $y = 2 + 2x$ and $y = -3 + 4x$

SKILL 213 TRANSPOSITION OF FORMULAE

Explanation

A *formula* is a rule for calculating something, starting with other quantities which may take different values. For example, for a circle A= πr^2, and we can calulate A using different values of *r*. An *equation* is similar to a formula, but is not primarily for calculating a value; it expresses a *relationship* between physical or mathematical quantities. Both have an = sign which must be well understood.

The sign = means 'the same as'. That is, what is on each side, or before and after, must be the same number (or same function). For example, 12 = 9 + 3, A = πr^2, $x(x + 2) = x^2 + 2x$. Therefore, *if we change the value of one side* in any way, the two sides will remain equal only if *the other side is changed in the same way* (or *"do the same to both sides"*). We may add, subtract, multiply, divide, square, differentiate etc. This idea is crucially important for solving equations (Skill 212 etc.)

A formula is transposed to change its purpose, e.g. to find the radius of a circle when we know its volume. Our objective is, therefore, to obtain a formula with *r* =... ("make *r* the subject of the formula").

Examples

1. Make *r* the subject of A = πr^2. Divide both sides by π: $A/\pi = r^2$
   Take square root: $\sqrt{A/\pi} = r$ or $r = \sqrt{A/\pi}$
2. To convert temperature in ^{o}F to ^{o}C, we use $T_c = \frac{5}{9}(T_F - 32)$.
   Multiply both sides by 9 and divide by 5: $\frac{9}{5}T_c = T_F - 32$
   Add 32 to both sides: $T_F = \frac{9}{5}T_c + 32$
3. 1 metre = 39·4 inches, so $L_I = 39 \cdot 4 L_M$. Hence, dividing by 39·4,
   $L_M = L_I / 39 \cdot 4$
4. For a pendulum, $T = 2\pi\sqrt{L/g}$. Hence $L = g(T/2\pi)^2$
5. $x = \dfrac{3}{2+4y}$, $x(2+4y) = 3$, $2 + 4y = \dfrac{3}{x}$, $y = \dfrac{(3/x - 2)}{4}$
6. $r = 2s/(s + 1)$, $s + 1 = 2s/r$, $s(1 - 2/r) = -1$, $s = -1/(1-2/r)$
   (uses factorisation, Skill 209)

Exercises

Make the ( ) variable the subject of each formula:

1.	$a = 6b$,	(b)	2.	$x = 3 - 4y$,	(y)
3.	$p = 2q/5r$	(q)	4.	$m = 2n/3c$,	(c)
5.	$x = 2/(y+1)$	(y)	6.	$1/z = 2z/w$	(z)
7.	$Q = \dfrac{\pi a^4}{8\mu L}$	(L)	8.	As 7., but (a)	

SKILLS 113,214 EVALUATION OF A FORMULA

Explanation

To evaluate a formula, you replace all the symbols by their values and follow the usual rules of arithmetic, recognising the following points:

1. Multiply signs are not usually shown in the formula.
2. The order, or *precedence* of calculations matters (Skill 215)
3. Learn how to use the memory on your calculator, to avoid writing down intermediate results. If you do write them down, think how accurately you need to do so. Planning the sequence of calculations can also help.
4. *Simplify* if you can first. Also go on to *estimate* the result as a check (it's easy to press the wrong buttons!).
5. If sin, cos etc. are being calculated, make sure whether degree or radians are to be used.
6. If units (dimensions) are given, put them in your answer too.

Examples

1. $y = \frac{3x^3}{2a}$ with $x = 2,\ a = 4$ gives $y = (3 \times 8) \div (8) = 3$ - no need for the calculator.
2. $T = 2\pi\sqrt{L/g}$ with $L = 0 \cdot 5m$, (time period for a pendulum in seconds), gives
   $T = 2 \times 3 \cdot 1416 \times \sqrt{0 \cdot 5 / 9 \cdot 81}$
   $\approx 2 \times 3 / \sqrt{20} \approx 1 \cdot 5$ secs estimate, and $T = 1 \cdot 42$ secs to 3 sig. figs.

Exercises

1. If $a = \frac{2b^2(1-c)}{2+d}$, and $b = 3,\ d = 7,\ c = -8$ find a without using your calculator.
2. Simplify, then estimate and calculate $z = \frac{\sqrt{4-7y}}{p+1}$ if $y = -3,\ p = 2 \cdot 7$.
3. Find the circumference and area of circles of radius 0·2m and 0·4m (Skill 262). Find the ratios of the circumferences and of the areas. What do you notice?
4. Evaluate $(\cos x)^2$ - $(\sin x)^2$ and also $\cos(2x)$ for the two different values of x (in radians or degrees). Comment.

SKILL 301 SIMPLE INEQUALITIES

Explanation

When we solve an *equation,* we usually look for one or two values of the variable that satisfy the equation exactly, e.g. $3x - 5 = 4$ gives $x = 3$. An *inequality* is, in real problems, a requirement such as: "the capital spent on the two projects must be no more than $ 10,000". We could write this as: $c_1 + c_2 \leq 10,000$.

We sometimes obtain an inequality in the form $3x > 12$, or $2x - 1 \leq 3x + 4$, and we need to find *all values* of x, or the *range of values* of x, that satisfy this requirement. The first gives $x > 4$; the second is true for all values of $x \geq -5$ (try some values to check, such as -6, -5, -4, 0 etc.).

To solve an equation we can do *anything* to *both* sides and they remain equal. To solve an inequality we can add or subtract as usual, but we can multiply or divide only by positive numbers. (To see this, try multiplying 4 > 3 by -2 on both sides).

Examples

1. $2u \leq 6$ gives $u \leq 3$.
2. $4p - 1 \geq 7$ gives $4p \geq 8$ and $p \geq 2$.
3. $y + 1 < 2 - y$ gives $2y + 1 < 2$, $2y < 1$, $y < \frac{1}{2}$.
4. $1 - 3x > 3 - 2x$ gives $-2 > x$ or $x < -2$.
   Note, if you get $-x > 2$ here, you must go to $0 > 2 + x$, then $-2 > x$.

Exercises Solve for the range of values in each case. Check afterwards with two or three values.

1.	$5v < 15$.	2.	$4 \geq 2 + 3p$.
3.	$3 < 2 - x$.	4.	$5 + 2z > 3 - 3z$.

SKILLS 302,401 SCIENTIFIC NOTATION

Explanation

In scientific and engineering work we usually want answers to e.g. 3 or 4 significant figures (Skill 202). We also often deal with very large and very small numbers, e.g. the speed of light $c \approx 186{,}000$ miles per second. It is very convenient to write our numbers in a standard form: a number (+ or -) between 1·0 and 9·999 multiplied by a power of 10. For example, $c \approx 1.86 \times 10^5$ miles/sec.

Small numbers require negative powers of 10 (Skill 203). For example, $0{\cdot}002304 = 2{\cdot}304 \times 10^{-3}$.

Notes

1. Some people choose to have a number between 0·1 and 0·9999 instead.
2. To enter a number like this on your calculator, use the E or EE (or EXP) key, thus: 1·86E5. The calculator also displays numbers like this (except it might show 1·86E + 05). Your calculator can probably handle numbers up to 10^{99} or so. Computer programs usually accept numbers using 'E' or 'e' also.
3. To add and subtract numbers by hand they must have the same power of 10. For example:

   $3{\cdot}561 \times 10^5 - 2{\cdot}483 \times 10^4 = 3{\cdot}561 \times 10^5 - 0{\cdot}248 \times 10^5 = 3{\cdot}313 \times 10^5$

Examples

1. $-356{\cdot}014 = -3{\cdot}560 \times 10^2$ (to 4 sig. figs.)
2. $0{\cdot}000031467 = 3{\cdot}15 \times 10^{-5}$ (to 3 sig. figs.)
3. $\dfrac{(3 \times 10^5)(2 \times 10^{-2})}{0{\cdot}9 \times 10^{-4}} = \dfrac{6}{0{\cdot}9} \times 10^{5-2+4} = \dfrac{6}{0{\cdot}9} \times 10^7 = 6{\cdot}667 \times 10^7$
4. $-2{\cdot}103 \times 10^{-2} + 1{\cdot}512 \times 10^{-1} = -0{\cdot}210 \times 10^{-1} + 1.512 \times 10^{-1}$

   $= 1{\cdot}302 \times 10^{-1}$

Exercises

Write in scientific notation (keeping all significant figures):

1. 4137·5 2. - 0·0426

Write in ordinary decimal notation:

3. $-7{\cdot}61 \times 10^2$ 4. $8{\cdot}3 \times 10^{-3}$
5. Which is the largest of

   $256{\cdot}31 \times 10^{-3}$, $1{\cdot}047 \times 10^{-1}$, $0{\cdot}00813 \times 10^2$?

6. Simplify $\dfrac{(4{\cdot}5 \times 10^{-3})(0{\cdot}2 \times 10^4)}{-1{\cdot}5 \times 10^2}$

Calculate

7. $2{\cdot}514 \times 10^2 + 3{\cdot}421 \times 10^3$
8. $1{\cdot}561 \times 10^{-2} - 1{\cdot}603 \times 10^{-1}$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 305 USING FRACTIONAL POWERS

Explanation

The idea of fractional powers and the way one rule applied to them was given as Skill 306. Here you will be applying the usual rules of powers (Skill 204) to fractional values. The same rules also apply for decimal values.

Rules

$$x^a x^b = x^{a+b}$$

$$x^a / x^b = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a$$

Examples

1. $x^{\frac{1}{2}} x^{\frac{3}{2}} = x^{\frac{1}{2}+\frac{3}{2}} = x^2$
2. $y^2 y^{-\frac{5}{2}} = y^{-\frac{1}{2}}$
3. $\left(x^{\frac{2}{3}}\right)^3 = x^{\frac{2}{3}\times 3} = x^2$
4. $q^{-\frac{1}{2}} / q^{-\frac{3}{2}} = q^{-\frac{1}{2}-(-\frac{3}{2})} = q^1 = q$
5. $h^{\frac{1}{2}} h^{\frac{1}{3}} = h^{\frac{1}{2}+\frac{1}{3}} = h^{\frac{5}{6}}$
6. $(x^2 y)^{\frac{1}{3}} = x^{\frac{2}{3}} y^{\frac{1}{3}}$

Exercises

1. $x^2 x^{\frac{1}{2}} = ?$
2. $a^{\frac{1}{3}} a^{\frac{4}{3}} = ?$
3. $b^{\frac{1}{3}} / b = ?$
4. $(ab)^{\frac{1}{2}} = ?$
5. $(g^2)^{\frac{1}{4}} = ?$
6. $\left(p^{-\frac{1}{2}}\right)^4 = ?$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 306 FRACTIONAL POWERS

Expanation

If we use the rules which work for positive and negative powers for fractions also, we find we can give a useful meaning to e.g. $4^{\frac{1}{2}}$ and even $x^{-0.301}$.

We know that $(x^a)^b = x^{ab}$, so $(x^a)^2 = x^{2a}$. Suppose $a = \frac{1}{2}$, then we have $(x^{\frac{1}{2}})^2 = x^1 = x$. So the square of $x^{\frac{1}{2}}$ is x, so we can interpret $x^{\frac{1}{2}}$ as $\sqrt{x}$. (It is not at all the same as $\frac{1}{x^2}$ or x^{-2}!)

In the same way we say $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$ etc. Once we can interpret fractions like $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ etc., we can go on to other fractions and even decimals. For example, $8^{\frac{2}{3}} = 8^{\frac{1}{3}\times 2} = (8^{\frac{1}{3}})^2 = 2^2 = 4$.

Examples

1. $25^{\frac{1}{2}} = \sqrt{25} = 5.$
2. $9^{-\frac{1}{2}} = 1\big/9^{\frac{1}{2}} = \frac{1}{3}.$
3. $64^{\frac{2}{3}} = (64^{\frac{1}{3}})^2 = 4^2 = 16.$
4. $9^{\frac{3}{2}} = (9^{\frac{1}{2}})^3 = 3^3 = 27.$

Exercises

Without your calculator, find

1.	$4^{\frac{1}{2}}$	2.	$27^{\frac{1}{3}}$	3.	$1000^{\frac{1}{3}}$
4.	$4^{\frac{5}{2}}$	5.	$25^{\frac{3}{2}}$	6.	$32^{\frac{1}{5}}$

7. Find out how to use the x^y key (or y^x) on your calculator. Check that it works for the examples above.
8. Find $10^{0.3010}$ and $10^{0.4771}$, rounded to 4 sig. figs.

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 109,216,308,404 ADDING AND SUBTRACTING FRACTIONS

Explanation

A fraction is a number divided by another number, e.g. $\frac{3}{4}$ or $\frac{x}{y}$. To add or subtract, we need both (or there might be three or more) to have the same (common) denominator (bottom). To achieve this, we multiply the top and bottom of each fraction by a chosen number, which doesn't change the value of the fraction.

Examples

1. $3\frac{1}{2}$ is really $\frac{3}{1}+\frac{1}{2}$. Multiply the first, top and bottom, by 2 to get $\frac{6}{2}+\frac{1}{2}=\frac{7}{2}$
2. $\frac{1}{2}+\frac{1}{3}=\frac{1}{2}\times\frac{3}{3}+\frac{1}{3}\times\frac{2}{2}=\frac{3}{6}+\frac{2}{6}=\frac{5}{6}$
3. $\frac{3}{4}-\frac{1}{6}=\frac{3}{4}\times\frac{6}{6}-\frac{1}{6}\times\frac{4}{4}=\frac{18}{24}-\frac{4}{24}=\frac{14}{24}=\frac{7}{12}$
   But it's neater to spot that 12 is a suitable *lowest common denominator* and so get
   $$\frac{3}{4}-\frac{1}{6}=\frac{3}{4}\times\frac{3}{3}-\frac{1}{6}\times\frac{2}{2}=\frac{9}{12}-\frac{2}{12}=\frac{7}{12}$$
4. $x+\frac{1}{x}=\frac{x}{1}+\frac{1}{x}=\frac{x}{1}\times\frac{x}{x}+\frac{1}{x}=\frac{x^2}{x}+\frac{1}{x}=\frac{x^2+1}{x}$
5. $\frac{3}{2}-\frac{1}{y}=\frac{3y}{2y}-\frac{2}{2y}=(3y-2)/(2y)$
6. $\frac{x-1}{2xy}+\frac{y+1}{4y^2}=\left(\frac{x-1}{2xy}\right)\left(\frac{2y}{2y}\right)+\left(\frac{y+1}{4y^2}\right)\left(\frac{x}{x}\right)$
   $=\frac{(x-1)(2y)+(y+1)(x)}{4xy^2}=\frac{3xy+x-2y}{4xy^2}$
7. $\frac{1}{2}+\frac{1}{3}+\frac{1}{5}=\frac{15+10+6}{30}=\frac{31}{30}$

If you use a higher denominator than necessary, it is possible to cancel later, but it makes more work and the cancellation may not be obvious.

Exercises (Cancel to simplify your answer when possible)

1. $2\frac{5}{7}=?/7$
2. $1+\frac{3}{8}=?$
3. $\frac{5}{2}-\frac{1}{3}=?$
4. $\frac{5}{6}+\frac{11}{12}=?$
5. $\frac{1}{x}+\frac{1}{y}=?$
6. $\frac{1}{2x}+\frac{1}{2y}=?$
7. $u+1+1/u=?$
8. $V^2-1/V^2=?$
9. $\frac{1}{2}+\frac{1}{x}-\frac{1}{4xy}=?$

SKILL 309 SOLUTION OF A SIMPLE QUADRATIC

Explanation

If a quadratic equation can be re-written using factors of the quadratic function, it can be solved easily. If the term in x is missing, solution is immediate; if the constant is missing, factorisation is easy.

But remember, for a quadratic equation you are looking for two solutions, though there may turn out to be only one (roots equal) or none (unless complex numbers are used). Beware of cancelling out a factor and so 'losing' a root altogether.

Examples

1. $x^2 - 9 = 0$ (no x term)

   $x^2 = 9$

   $x = \pm\sqrt{9} = \pm 3$

2. $2p^2 = 24$

   $p = \pm\sqrt{12} = \pm 2\sqrt{3}$ (since $12 = 4 \times 3$)

3. $3y^2 + 17 = 0$

   $y^2 = \pm\sqrt{-17/3}$ - no real roots

4. $x^2 = -4x,$

   $x^2 + 4x = 0,$

   $x(x+4) = 0$

   $x = 0$ or $x + 4 = 0$, so $x = 0$ or -4

   But not $x^2 + 4x = 0$ so $x + 4 = 0$ so $x = -4$, losing one root.

5. $2p^2 - p = 2p$

   $2p^2 - 3p = 0,\ p(2p-3) = 0,\ p = 0$ or $\dfrac{3}{2}$

Exercises

Solve to obtain two, one or no roots:

1.	$2x^2 - 18 = 0$	2.	$y^2 + 3 = 3$
3.	$4 = -p^2$	4.	$x^2 - 2x = 0$
5.	$3q^2 = 21q$	6.	$h^2 + 2h = 4h$
7.	$2u^2 - 3u = -3u$		

SKILL 310 SOLN OF A QUADRATIC EQUN BY FACTORS

Explanation

We know that $0a = 0$ for any number a. Therefore, if $ab = 0$ we know that either $a = 0$, or $b = 0$, or possibly both.

If we have a quadratic equation, and we can factorise the quadratic function (Skill 311), we can then solve it. For example, given $x^2 - 3x + 2 = 0$, we factorise to get $(x-2)(x-1) = 0$ (Check this!). Now we know that either $(x-2) = 0$, giving $x = 2$, or $(x-1) = 0$, giving $x = 1$.

Similarly, $u^2 - 3u - 28 = 0$ becomes $(u+4)(u-7) = 0$ (check!), and hence $u = -4$ or $+7$ (be careful about the signs here!).

We can also work the other way. We can write down a quadratic equation with known roots. For example, if the roots are $z = -5$ and $z = 2$, we know that $z + 5 = 0$ or $z - 2 = 0$, so we get $(z + 5)(z - 2) = 0$, or $z^2 + 3z - 10 = 0$.

Examples

1. From $(x + 3)(x - 4) = 0$ we get $x = -3$ or $+ 4$
2. From $(u - 2)(u - \frac{1}{2}) = 0$ we get $u = 2$ or $\frac{1}{2}$.
3. From $(2p - 1)(p + 3) = 0$ we get $p = \frac{1}{2}$ if $2p - 1 = 0$ or $p = -3$.
4. From $3x(x + 3) = 0$ we get $x = 0$ or $x = -3$.
5. The equation with roots $s = 2$ and $s = -3$ is $(s - 2)(s + 3) = 0$, or $s^2 + s - 6 = 0$.
6. The equation with roots $v = \frac{1}{3}$ and $v = 1$ is $(v - \frac{1}{3})(v - 1) = 0$, or $v^2 - \frac{4}{3}v + \frac{1}{3} = 0$, or $3v^2 - 4v + 1 = 0$.
7. From $(x+2)^2 = 0$, we get $x = -2$ *only* (a *repeated root*).

Exercises

Write down the roots from the equation in its factorised form:

1. $(x-2)(x-3) = 0$,
2. $(v+4)(v+2) = 0$,
3. $(3y-1)(y-2) = 0$,
4. $(2z-3)(z+1) = 0$,
5. $4(x+1)(3x+2) = 0$,

Write down the equation with the given roots:

6. $x = 1$ and $x = -4$,
7. $y = -2$ and $y = -5$,
8. $u = \frac{1}{2}$ and $u = \frac{1}{3}$,
9. $w = 0$ and $w = 2$,
10. $x = 3$ *only* (think of it as $x = 3$ and $x = 3$ again)

SKILL 311 FACTORS OF A QUADRATIC FUNCTION

Explanation

Our aim is to rewrite a quadratic function, such as x^2+3x+2 as the product of two brackets, in this case $(x+1)(x+2)$. (Each bracket is a *linear* function of x). This is often a route to solving a quadratic equation (Skill 310). This skill depends on being able to expand such brackets quickly and reliably (Skill 312). Do more practice at this *now* if you need to!

If we expand $(x+a)(x+b)$ we get $x^2+(a+b)x+ab$. So we need to find numbers a and b that multiply to give the 'units' and add to give the right number of x's. Minus signs make this more difficult.

Examples

1. $x^2+2x+1=(x+a)(x+b)$. We need $ab=1$ and $a+b=2$. Obviously $a=b=1$, and we get $x^2+2x+1=(x+1)(x+1)=(x+1)^2$.
2. $y^2+4y+3=(y+a)(y+b)$. $ab=3$, so we try $a=3$ and $b=1$ to get (since $a+b=4$) $y^2+4y+3=(y+3)(y+1)$
3. $u^2+5u+6=(u+a)(u+b)$. We could take $a=6, b=1$ or $a=3, b=2$. The second choice works.
4. $x^2-3x+2=(x+a)(x+b)$. If a and b are both positive, we'll get $+3x$. Take $a=-2$, $b=-1$.
5. $w^2-w-6=(w+a)(w+b)$. We have *four* choices for a,b, namely (+6, -1) or (-6, +1) or (+3, -2) or (-3, +2). The last one works.

Note: With a little experience we go through the possibilities for a and b in our heads, but we always check by expanding afterwards!

6. $2x^2+5x+2=(2x+1)(x+2)$ - this is more difficult and best left until you are good at the others!

Exercises

Factorise into two brackets. *Check every time* by expanding!

1.	x^2+5x+4.	2.	u^2+6u+8.
3.	u^2+9u+8 (compare with 2.)	4.	y^2-2y+1.
5.	p^2+4p+4.	6.	$q^2+8q+16$.
7.	$q^2-8q+16$ (compare with 6.)	8.	x^2-4x+3.
9.	u^2-6u+8 (compare with 2.)	10.	y^2+2y-3.
11.	y^2-2y-3 (compare with 10.)	12.	$r^2+2r-15$.

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 312 EXPANDING MULTIPLE BRACKETS

Explanation

The basic rule for expanding one bracket (see Skill 210) is $a(b+c)=ab+ac$. This can be extended to two brackets:

To expand $(a+b)(c+d)$, let $a+b=e$

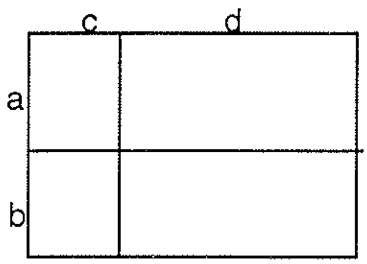
Now $e(c+d)=ec+ed$

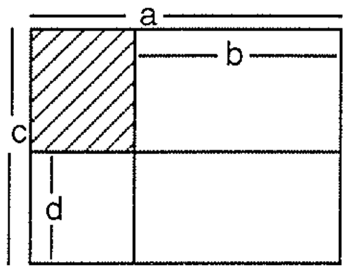
Put back $a+b$: $(a+b)c+(a+b)d=ac+bc+ad+bd$.

Thus we multiply all terms in the first bracket by all terms in the second. The same idea works for more terms, more brackets, and for subtraction instead of addition.

Also consider the areas shown below. For the first case, we see that

Area = $(a+b)(c+d)= ac+ad+bc+bd$.





If we consider the second case, we see that the shaded area is

$(a-b)(c-d)=ac-ad-bc+bd$

because one part has been subtracted twice - this is another way of explaining why "$-\times-=+$"

Examples

1. $(2+4)(3+5)=2\times3+2\times5+4\times3+4\times5=6+10+12+20$
2. $(11-6)(12-7)=132-77-72+42$
3. $(3+x)(2+x)=6+3x+2x+x^2=6+5x+x^2$
4. $(u+{}^1\!/_u)(2u-{}^1\!/_u)=2u^2-1+2-{}^1\!/_{u^2}=2u^2+1-{}^1\!/_{u^2}$
5. $(x+w)^2=x^2+2xw+w^2$
6. $(a+b)(a-b)=a^2-ab+ba-b^2=a^2-b^2$
7. $(x+y-1)(x+y)=x^2+xy+yx+y^2-x-y=x^2+2xy+y^2-x-y$
8. $(z+1)(z+2)(z+3)=(z+1)(z^2+5z+6)=z^3+6z^2+11z+6$
9. $(a+b)^3=a^3+3a^2b+3ab^2+b^3$ (check by trying some values for a and b)

Exercises Where the exercise is purely numerical, check by evaluating the brackets first. For the others, check by trying a value for each variable.

1. $(6+4)(3+5)=?$
2. $(6-4)(8-6)=?$
3. $(x+2)(x+1)=?$
4. $(y+2z)(y-z)=?$
5. $(2x-1)(3-x)=?$
6. $-(u+v)(u+3v)=?$
7. $(y+3)^2=?$
8. $(2x-1)^2=?$
9. $(a+1)(a+1)(a+3)=?$
10. $(c+1)^3=?$
11. $(x-y)^2=?$
12. $(x+\frac{1}{w})^2=?$

SKILL 313 DIFFERENCE OF TWO SQUARES

Explanation

Compare 7 x 7 = 49 with 6 x 8 = 48, or 6 x 6 = 36 with 5 x 7 = 35. Notice that, for any number n, $n^2 = (n - 1)(n + 1) + 1$ or $(n - 1)(n + 1) = n^2 - 1$. We can also obtain this by expanding the brackets. In fact, $(a + b)(a - b) = a^2 + ba - ab - b^2 = a^2 - b^2$ for *any* values of a and b, because two terms always cancel. Whenever you see $(\ )^2 - (\ )^2$ you can always factorise it in this way, even when it's *not* obvious there is a 'square' (example 4).

Note $(x - y)^2$ is *not* $x^2 - y^2$! Always check by expanding the brackets carefully.

Examples

1. $x^2 - 1 = (x+1)(x-1)$ or $(x-1)(x+1)$ or $-(1-x)(x+1)$ if you like
2. $s^2 - t^2 = (s+t)(s-t)$
3. $4w^2 - 1 = (2w)^2 - (1)^2 = (2w+1)(2w-1)$
4. $x^2 - 2 = x^2 - (\sqrt{2})^2 = (x+\sqrt{2})(x-\sqrt{2})$
5. $2p^2 - q^2 = (\sqrt{2}p + q)(\sqrt{2}p - q)$
6. $x - y = (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})$ (unusual!)
7. $4 - m^2 = (2+m)(2-m)$
8. $x^2 + 1$ or $x^2 + y^2$ *can't be factorised* (unless we use complex numbers)
9. $x^4 - y^4 = (x^2)^2 - (y^2)^2 = (x^2 + y^2)(x^2 - y^2)$

Exercises

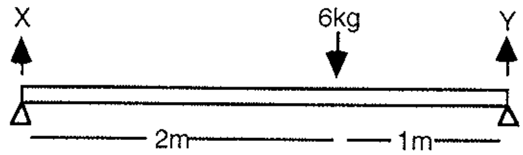
Factorise into two brackets, or state that it's impossible.

1. $b^2 - 4$
2. $9 - x^2$
3. $4x^2 - y^2$
4. $9x^2 - 16y^2$
5. $1 - 2x^2$ (remember $2 = (\sqrt{2})^2$)
6. $3 - p^2$
7. $4 + 9x^2$
8. $4x^2y^2 - 1$
9. $p^4 - 1$

SKILL 314 SIMULTANEOUS EQUATIONS

Explanation

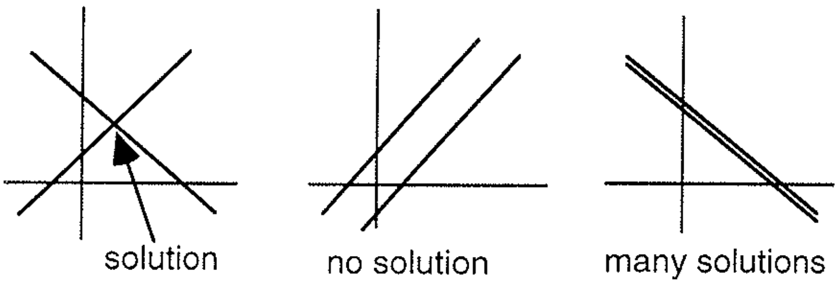
For example, a weight of 6 kg supported as shown gives rise to forces X,Y at the supports:



By equilibrium of forces and moments we get two equations, $X + Y = 6$, $2X - Y = 0$. We aim to find values of X and Y that satisfy *both* equations ' simultaneously'. There are three methods:

Method 1

Draw two graphs. Where the lines intersect, the X and Y values must satisfy both equations, and so give the solution. If the lines are parallel, there is *no* solution possible. If the lines coincide, there are *many* possible solutions.



Method 2

From one equation, get a formula for one variable in terms of the other. *Substitute* into the other equation.

This method is unsuitable for large numbers of equations and unknowns.

Method 3

Multiply one or both equations by a chosen number to make the coefficient of one unknown equal in each, or equal and opposite. Then subtract (or add) the *whole* of one equation (both sides) from the other to *eliminate* one unknown.

Equivalently, add or subtract a multiple of one equation to or from the other. This method is efficient for large 'systems' of equations.

Examples

1. $X + Y = 6$ *and* $2X - Y = 0$ - by substitution.
   2nd equation gives $Y = 2X$. Substitute in 1st to get $X + (2X) = 6$, $3X = 6$, $X = 2$. Now use $Y = 2X$ to get $Y = 4$. (units are kg force)
2. $X + Y = 6$ *and* $2X - Y = 0$ - by elimination
   Coefficients of Y are already equal and opposite.
   Add equations to get $3X = 6$ and so on.
3. $2u + 3v = -4$ *and* $3u + 5v = -7$ - by substitution
   1st equation gives:

   $u = (-4 - 3v) / 2$, put in 2nd,

   $3(-4 - 3v) / 2 + 5v = -7, \quad -6 - \frac{9}{2}v + 5v = -7,$

   $\frac{1}{2}v = -1, \quad v = -2$ and finally $u = 1$
4. Same as 3., but by elimination.
   Multiply 1st by 3 and 2nd by 2 to get
   $6u + 9v = -12$

   $6u + 10v = -14$
   Subtract 1st from 2nd to eliminate u:
   $v = -2$, and then $u = 1$ again by substituting the value of v into one equation.

Exercises

Solve at least one by each method:

1. $2x + y = 8$
   $3x + 2y = 14$
2. $-3x + 3y = 12$
   $4x + 5y = 11$
3. $2p - \frac{1}{2}q = 5$
   $-4p + q = -10$
   use substitution or elimination, *then* draw the graph
4. $2s - 5t = 4$
   $-3s - 4t = 11$

SKILLS 316,411 SOLN OF QUADRATIC EQUNS BY THE FORMULA

Explanation

The formula for solving a quadratic equation in the form $ax^2+bx+c=0$ is

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Provided you are careful with '–' signs, it is easy to use, and will give 2 roots if b^2-4ac is positive, 1 root if b^2-4ac = 0, and no (real) roots if b^2-4ac is negative. The formula is derived by the method of *completing the square* (Skills 406,407) which you should also learn.

We start with the equation $ax^2+bx+c=0$ and complete the square as follows:

$$x^2+\frac{b}{a}x+\frac{c}{a}=0,\quad \left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0,$$

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}=\frac{\pm\sqrt{b^2-4ac}}{2a}$$

which finally gives the formula above.

Examples

1. $x^2+3x+2=0$. Here a =1, b = 3, c = 2 and we get
   $$x=\frac{-3\pm\sqrt{9-8}}{2}=\frac{-3\pm1}{2}=-1 \text{ or } -2$$
2. $t^2-3t-10=0$. Here a = 1, b = -3, c = - 10, so
   $$t=\frac{+3\pm\sqrt{9+40}}{2}=\frac{3\pm7}{2}=5 \text{ or } -2.$$
3. $2u^2+4u-1=0$. Here a =2, b = 4, c = - 1, so
   $$u=\frac{-4\pm\sqrt{16+8}}{4}=\frac{-4\pm\sqrt{24}}{4}.$$
   $$=0\cdot2247 \text{ or } -2\cdot2247$$
4. $p^2+3p+2\cdot25=0$. We get $p=\dfrac{-3\pm\sqrt{0}}{2}=-1\cdot5$ - only one root.
5. $x^2-2x+3=0$. We find no real roots.

Exercises

Solve to get 2, 1 or 0 roots:

1. $x^2+5x+6=0$
2. $y^2-7y+12=0$
3. $2u^2-u-1=0$
4. $-3v^2+2v+5=0$
5. $-w^2+2w-3=0$
6. $h^2=-2h+4$ (get 0 on one side first)

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILLS 317, 408 COMPLEX NUMBERS

Explanation

Complex numbers can be thought of as a new kind of number which allows us to e.g. solve *all* quadratic equations, or as a sort of trick which offers a powerful method of dealing with real problems like oscillations. In this sheet we'll look only at how to manipulate them.

If we write $2+3i$ or $-4+i$ or $a-ib$, we say that these are *complex numbers* with a *real part* and an *imaginary part.*

$2 - 5i$ has real part 2 and imaginary part -5
7 is purely real (imaginary part 0)
$-3i$ is purely imaginary (imaginary part - 3)

When adding and subtracting complex numbers, treat i like any other symbol in collecting terms.

To make complex numbers interesting and useful, we give the symbol i a special property, namely $i^2=-1$. When multiplying complex numbers, replace i^2 by -1 whenever it appears.

NB The symbol j is often used in place of i, especially by electrical engineers, for whom i could mean 'current'.

Examples

1. $(2+4i)+(-3+2i)=-1+6i$
2. $(-3+6i)-(-4-i)=1+7i$
3. $\left(-2+\sqrt{3}i\right)+\left(-2-\sqrt{3}i\right)=-4$
4. $(1+2i)(3+4i)=3+4i+6i+8i^2=3+10i+8(-1)=-5+10i$
5. $(-2+i)(6-3i)=-12+12i+3=-9+12i$

Exercises Add, subtract and multiply as indicated.

1. $(3+2i)+(4+5i)$
2. $(-4-i)+(-6+7i)$
3. $(2-i)-(4+2i)$
4. $(2+5i)(1+2i)$
5. $(-3+6i)(2-4i)$
6. $(4+2i)(4-2i)$
7. $(-3+i)(-3-i)$

SKILL 405 COMMON ERRORS

Explanation and Examples

Some of the commonest errors in mathematics are to do with expanding brackets, simplifying fractions, and minus signs. The examples in the *DIAGNOSYS* test are:

1. $(x-1)^2 = x^2+1$? Wrong! It should be x^2-2x+1. (Skill 312)

2. $x^2+2x-8=(x-4)(x+2)$? Wrong! The right-hand side should be $(x+4)(x-2)$.

3. $\dfrac{1}{x^2-x} = \dfrac{1}{x^2} - \dfrac{1}{x}$? Wrong! The division 'bar' takes precedence over the subtraction (Skill 215), so $\dfrac{1}{x^2-x}$ means $\dfrac{1}{(x^2-x)}$ and we can't divide then subtract. To rewrite this fraction, use partial fractions or complete the square (Skill 407).

4. $(-3x)^2 = -9x^2$? Wrong! It should be $+9x^2$.

5. $\left(x+\dfrac{1}{x}\right)^2 = 2+x^2+\dfrac{1}{x^2}$? Correct!

Exercise?

This 'skill' concerns care and attention, not knowledge. Work at a steady speed, and check results as you go along, not at the end, preferably by an independent method. For example, put in some values of x to check for correct algebra.

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 409 DIVISION BY ZERO

Explanation
When it is said " you can't divide by zero", we should ask "why should we want to?" It arises *either* because we are trying to solve a problem in a 'general' form, and we divide by (or cancel) something that later turns out to be zero, *or* because we need the ratio of two vanishingly small quantities, as in differentiation. If we go back a few steps, we can usually resolve the difficulty.

The examples below are for single equations, but often the problems are 'hidden' in e.g. a pair of simultaneous equationa (see Skill 314 and Exercise 3).

Examples

1. The equations $3x = 5$ and $4x = 0$ can both be solved, giving $x = \frac{5}{3}$ and $x = 0$ respectively.
2. The equation $0x = -7$ can't be solved, that is, there is *no* value of x for which this is true. So in this case, "dividing by zero" to get x is rightly "impossible".
3. The equation $0x = 0$ is satisfied by *any* value for x. Therefore, we *can* solve it (meaning, we can find a solution), but *not uniquely*. This equation actually gives us *no information* about x at all!
4. $ax = b$ can be solved for a unique value of x in terms of a and b provided $a \neq 0$. If $a = 0$ but $b \neq 0$ there is *no* solution. If $a = b = 0$ then there are *many* solutions.
5. Solving $x^2 - 4x = 0$ we should not cancel a factor x to give $x - 4 = 0$, $x = 4$ because we overlook the possibility that $x = 0$. We should factorize: $x(x-4) = 0$, and conclude $x = 0$ or $x = 4$.

Exercises

1. Comment on the solutions:
   (i) $2x = 0$, (ii) $0x = 5$, (iii) $0y = 0$

2. Solve
   (i) $2u^2 = u$ (ii) $v - 3v^2 = 0$

3. Solve, or comment:

   (i) $\begin{cases} 2x - y = 3 \\ -4x + 2y = -6 \end{cases}$, (ii) $\begin{cases} -\frac{1}{2}u + 3v = -2 \\ u - 6v = -4 \end{cases}$

SKILL 402 LOGARITHMS

Explanation

Logarithms and powers are really the same thing. For example:

$2^3 = 8$ is the same as $\log_2 8 = 3$ ("the log base 2 of 8 is 3")
$10^2 = 100$ is the same as $\log_{10} 100 = 2$
$e^{-1} = 0 \cdot 368$ is the same as $\log_e 0 \cdot 368$ or $\ln 0 \cdot 368 = -1$

The rules of logs are just the rules of powers rewritten. For example:

$2 \times 3 = 10^{0 \cdot 3010} 10^{0 \cdot 4771} = 10^{0 \cdot 7781} = 6$ or
$\log_{10}(2 \times 3) = \log_{10}(2) + \log_{10}(3)$

and

$3^2 = \left(10^{0 \cdot 4771}\right)^2 = 10^{0 \cdot 9542} = 9$ or
$\log_{10}\left(3^2\right) = 2 \log_{10} 3$

Note because 10^x or e^x or 2^x is positive for all real x, we cannot find a log of a negative number.

Examples

1. $\log_{10} \frac{1}{1000} = \log_{10}\left(10^{-3}\right) = -3$
2. $\log_2 32 = 5$, since $32 = 2^5$
3. $\log_{10} 5 + \log_{10} 7 = \log_{10} 35$
4. $\log_{10} 6 - \log_{10} 2 = \log_{10}(6 \times 2^{-1}) = \log_{10} 3$
5. $6 \log_2 4 - 3 \log_2 8 + \log_2 \frac{1}{2} = \log_2\left(4^6 8^{-3} 2^{-1}\right) = \log_2\left(2^{12-9-1}\right) = 2$

Exercises

1. Check Example 3 by finding the numbers using your calculator (the "log" key does $\log_{10}$)
2. What is (without your calculator)

   (a) $\log_{10} 100$, (b) $\log_2 \frac{1}{8}$, (c) $\ln \frac{1}{e}$, (d) $\log_2 \sqrt{2}$
3. Simplify $2\log_{10} 9 - 4\log_{10} 27 + \log_{10} \frac{1}{3}$

SKILL 403 ARBITRARY FACTORS

Explanation

The idea of a *factor* is something that divides into something else, so 12 is a factor of 60 (but not of 30), and x^2 is a factor of $6x^3y$. However, we can extend the idea usefully. Since $(x^2)\left(\frac{y}{x}\right) = xy$, we can think of x^2 and $\frac{y}{x}$ as possible factors of xy. In fact, we can take *any factor* outside a bracket.

Examples

1. $1 + x$ can be written as $x\left(\frac{1}{x}+1\right)$ or $x\left(1+\frac{1}{x}\right)$.
2. $3 - 2x$ can be written as $3x\left(\frac{1}{x}-\frac{2}{3}\right)$.
3. $4 - 2x + x^2 = x\left(\frac{4}{x}-2+x\right) = x^2\left(\frac{4}{x^2}-\frac{2}{x}+1\right)$.
4. $x + y = xy\left(y^{-1} + x^{-1}\right)$.
5. $\dfrac{x}{x+y} = \dfrac{x}{x\left(1+y/x\right)} = \dfrac{1}{1+y/x}$.
6. $\sqrt{u} + u = \sqrt{u}\left(1+\sqrt{u}\right)$.

Exercises

1. $3 - 2y = y(.....)$?
2. Take the factor x out of $2x^2 - 3$.
3. $x+\frac{1}{x} = \frac{1}{x}(.....)$?
4. $\sqrt{x} + 2x = \sqrt{x}(.....)$?

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 406 SOLVING QUADRATICS BY COMPLETING THE SQUARE

Explanation

A quadratic equation like $x^2+6x+7=0$ can be solved by rewriting the quadratic function by completing the square (skill 407)

Examples

1. To solve $x^2+6x+7=0$, rewrite as $(x+3)^2-2=0$.
   Therefore $(x+3)^2=2$
   $$x+3=+\sqrt{2} \text{ or } -\sqrt{2}$$
   $$x=\sqrt{2}-3 \text{ or } x=-\sqrt{2}-3$$

2. To solve $u^2-3u+1=0$, rewrite as
   $$\left(u-\frac{3}{2}\right)^2-\frac{5}{4}=0$$
   $$u-\frac{3}{2}=\pm\sqrt{5/4}$$
   $$u=\frac{3}{2}+\sqrt{5/4} \text{ or } \frac{3}{2}-\sqrt{5/4}$$

3. To solve the <u>general quadratic equation</u> $ax^2+bx+c=0$, we divide by a then follow the same pattern:
   $$x^2+\frac{b}{a}x+\frac{c}{a}=0$$
   $$\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}+\frac{c}{a}=0$$
   $$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}=\pm\frac{1}{2a}\sqrt{b^2-4ac}$$
   $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Exercises solve each by completing the square, dividing through first if necessary

1. $x^2+3x+2=0$
2. $p^2-4p-1=0$
3. $2y^2-y-3=0$

SKILL 407 COMPLETING THE SQUARE

Explanation

If we expand and collect terms for the examples

$$(x+1)^2+2=x^2+2x+1+2=x^2+2x+3,$$
$$(x-3)^2+5=x^2-6x+9+5=x^2-6x+14$$

we see that, because of squaring the bracket, we obtain a quadratic function with an x^2 term, and an x term with coefficient twice the number in the bracket. Completing the square means reversing this process. Our objective is to have a bracket squared, with only numbers outside it.

This is useful for solving quadratics (skill 406), interpreting graphs (skills 451, 452) etc.

Examples

1. $x^2+4x+5=(x+?)^2+?$
   To obtain $4x$, we must have $(x+2)^2$, so we write
   $x^2+4x=(x+2)^2-4$. Now we get the 'numbers' right:
   $[x^2+4x]+5=[(x+2)^2-4]+5=(x+2)^2+1$
2. $x^2+6x-1=[(x+3)^2-9]-1=(x+3)^2-10$
3. $v^2-2v+3=[(v-1)^2-1]+3=(v-1)^2+2$
4. $w^2-3w+\frac{1}{2}=\left[\left(w-\frac{3}{2}\right)^2-\frac{9}{4}\right]+\frac{1}{2}=\left(w-\frac{3}{2}\right)^2-\frac{7}{4}$

Exercises

Complete the square in each case.

1. x^2+2x+2
2. x^2+8x+5
3. y^2-4y-2
4. $v^2-10v+24$
5. x^2-x-1

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 410 ALGEBRAIC SUBSTITUTION

Explanation

A *function* like $f(x) = 2x^2 - 3$ can be thought of as a set of instructions, taking one number as an 'input' and producing an 'output'. For example, if $x = 5$, we get $f(5) = 2 \times 25 - 3 = 47$.

The function we call f means that particular set of instructions, whatever we start with. So if our input is a, we get $f(a) = 2a^2 - 3$, whilst if our input is $2y$-1, we get

$$f(2y-1) = 2(2y-1)^2 - 3$$
$$= 8y^2 - 8y - 1 \qquad \text{(after expanding)}$$

Examples

1. If $f(x)$ = 1 - 3x, then
   $f(-2) = 1 - 3(-2) = 7$, $f(2x) = 1 - 3(2x) = 1 - 6x$,
   $f(1 - 2x) = 1 - 3(1-2x) = 6x - 2$.

2. If $g(u) = u^2 - 2u + 4$, then
   $g(x) = x^2 - 2x + 4$, $g(4u-3) = (4u-3)^2 - 2(4u-3) + 4$
   $= 16u^2 - 32u + 19$.
   $g(x+h) = (x+h)^2 - 2(x+h) + 4$
   $= x^2 + 2hx + h^2 - 2x - 2h + 4$

3. If $f(x) = x^2 + x$, then
   $$\frac{f(x+h) - f(x)}{h} = \frac{\left[(x+h)^2 + (x+h)\right] - \left[x^2 + x\right]}{h} = \frac{2hx + h^2 + h}{h}$$
   $$= 2x + 1 + h$$

Exercises

1. If $f(x) = 2x - 3$, what is
   (a) $f(-5)$, (b) $f(a)$, (c) $f(2x)$?

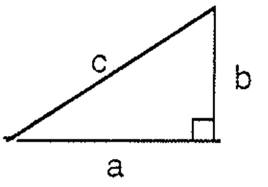
2. If $g(s) = 2 - 3s^2$, what is
   (a) $g(2s-1)$, (b) $g(-s)$, (c) $g(s+t)$?

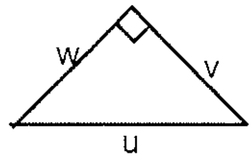
3. If $f(x) = 3 - x^2$, what is
   $\dfrac{f(x+h) - f(x)}{h}$?

SKILL 221 PYTHAGORAS' THEOREM

Explanation

A *right-angled triangle* is a triangle with one angle of 90^0 or $\frac{\pi}{2}$ radians. This angle is often marked with a small square. The slant side (opposite the right-angle) is called the *hypotenuse*.

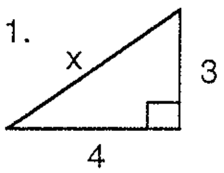




The hypotenuse is c in the first triangle and u in the second.
Pythagoras' theorem states that the square of the length of the hypotenuse equals the sum of the squares of the lengths of the other two sides, for *any* right-angled triangle. So, for the examples above: $c^2 = a^2 + b^2$ and $u^2 = v^2 + w^2$

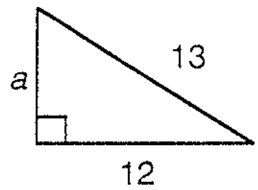
Examples

1.



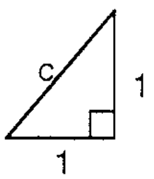
By Pythagoras' theorem,
$x^2 = 3^2 + 4^2$, so $x^2 = 9 + 16 = 25$,
so $x = 5$

2.



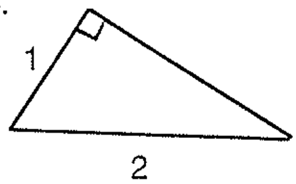
$a^2 = 13^2 - 12^2 = 169 - 144 = 25$
so $a = 5$

3.



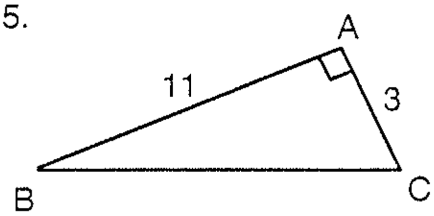
$c^2 = 1^2 + 1^2 = 2,\ c = \sqrt{2} = 1 \cdot 414.....$

4.



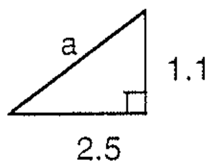
Let z be the third side, then
$z = \sqrt{2^2 - 1^2} = \sqrt{3}$

5.



$BC = \sqrt{AC^2 + AB^2} = \sqrt{130}$

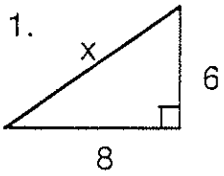
6.

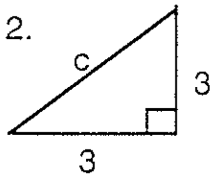


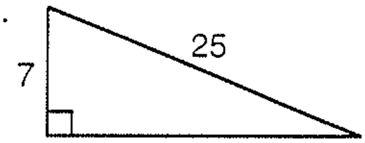
$a = \sqrt{1 \cdot 21 + 6 \cdot 25} = \sqrt{7 \cdot 46} = 2 \cdot 731$

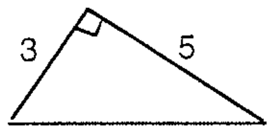
Note Sometimes a root can be simplified without writing it in decimal form, and this may be neater, e.g. $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4}\sqrt{2} = 2\sqrt{2}$ or $\sqrt{99} = \sqrt{9}\sqrt{11} = 3\sqrt{11}$ etc.

Exercises Find the side which is labelled, or choose a label and then find it:

1.


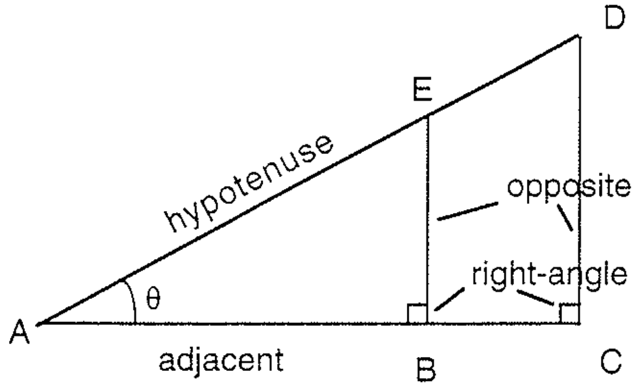
2.


3.


4.


SKILL 222 SINE AND COSINE DEFINITIONS

Explanation



The ratio DC : EB is the same as AC : AB and AD : AE. These are *similar triangles,* and what they have in common is their *angles.*

Therefore, the ratios BE : AE and CD : AD are also the same. We associate this ratio with the angle θ and call it the *sine* of θ, or sin θ for short, so

$$\sin\theta = BE/AE = \text{opposite/hypotenuse}$$

Similarly, the ratios AB : AE and AC : AD are the same; we call this the *cosine* of θ, or

$$\cos\theta = AB/AE = \text{adjacent/hypotenuse}$$

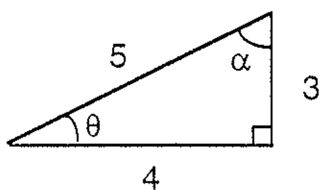
The ratio BE/AB is called the *tangent* of θ ; it is also *the gradient* of the sloping line (skill 251), so

$$\tan\theta = BE/AB = \text{opposite/adjacent}$$

The same idea works for angles bigger than 90^o. Also note that sin θ, cos θ exist even when no right-angled triangle is visible in a diagram; they are functions of the angle θ and can be calculated in other ways too.

Examples

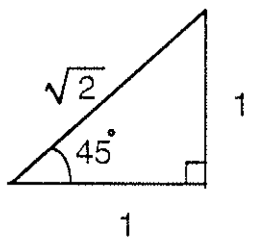
1.

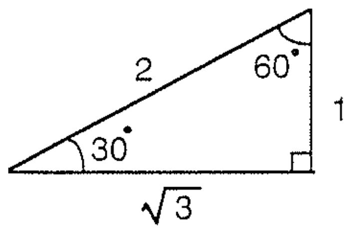


For this triangle we can write $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, but also,

$\sin\alpha = \frac{4}{5}$, $\cos\alpha = \frac{3}{5}$, $\tan\alpha = \frac{4}{3}$. $(\alpha + \theta = 90^0)$

2. The second triangle is half of an *equilateral* triangle (equal sides); these two give us some useful results.





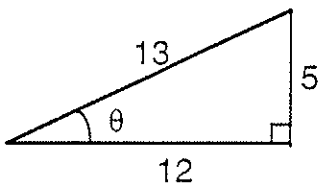
$\sin 45^0 = \dfrac{1}{\sqrt{2}} = \cos 45^0$, $\sin 30^0 = \dfrac{1}{2}$ $\cos 30^0 = \dfrac{\sqrt{3}}{2}$

$\tan 45^0 = 1$ $\tan 30^0 = \dfrac{1}{\sqrt{3}}$

3. $\sin 0^0 = \tan 0^0 = \cos 90^0 = 0$
$\sin 90^0 = \cos 0^0 = 1$, $\tan 90^0$ is infinite

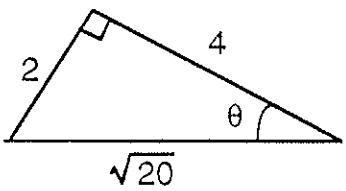
Exercises

1.



Write down $\sin\theta$, $\cos\theta$, $\tan\theta$

2.



Write down $\sin\theta$, $\cos\theta$, $\tan\theta$

3. From the triangle in example 2, write down $\sin 60^0$, $\cos 60^0$, $\tan 60^0$

SKILLS 223,324 PERCENTAGES

Explanation

Per cent means *for each hundred.* Therefore, 60 out of 200 is, 30 for each hundred, or 30 per cent, written 30%. Similarly, 20 out of 50 is, 40 for each hundred, or 40%. The calculation is, logically, find how many hundreds, then divide by that. Thus,

(i) 200 is 2 hundreds, $60 \div 2 = 30\%$.

(ii) 50 is $\frac{50}{100} = 0{\cdot}5$ hundreds, $20 \div 0{\cdot}5 = 40\%$.

However, the common way of doing this is, find the answer as a fraction, then multiply by 100 to get a fraction of 100. Thus:

(i) $\frac{60}{200} = 0\cdot 3, \quad 0\cdot 3 \times 100 = 30\%$

(ii) $\frac{20}{50} = 0\cdot 4, \quad 0\cdot 4 \times 100 = 40\%$

Reversing the process, we can find e.g. 25% of 84. This means simply $\frac{25}{100}$ of 84, which is 21.

Extending this, we can say a rise of 10% is a rise of $\frac{10}{100}$ or $0{\cdot}10$, so the starting value must be multiplied by $1{\cdot}10$

Examples

1. 17 out of 85 is $17 \div \frac{85}{100} = \frac{17}{85} \times 100 = 20\%$
2. 3 out of 4 is $\frac{3}{4} \times 100 = 75\%$
3. 80 changing to 280 is a rise of 200, or $\frac{200}{80} \times 100 = 250\%$
4. 15% of 67·4 is $67\cdot 4 \times \frac{15}{100} = 10\cdot 11$
5. 60 changing to 30 is a decrease of 50%, but 30 increasing to 60 is an increase of 100% *of what is there now.*

Exercises

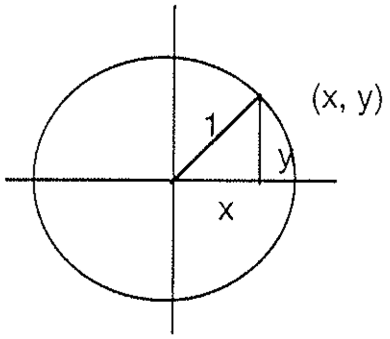
Find the percentage in each case:

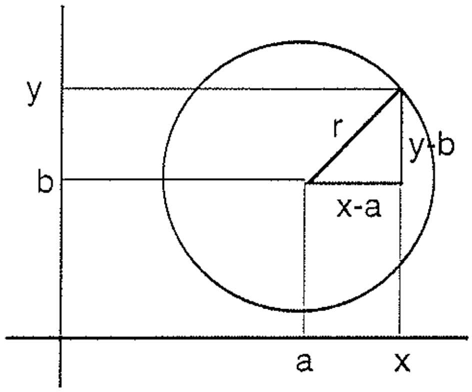
1. 6 out of 15
2. 3·5 out of 7·5
3. If 27·5 increases to 41·3, what is the rise in %?
4. If 80 decreases to 64, what is the change in %?
5. If a bank deposit of £50 increases by 5% each year for three years, find the result. Note that the 5% is of the *total* present in the account (*compound interest).*

SKILLS 321,421 EQUATION OF A CIRCLE

Explanation

A circle is a shape where every point is a fixed distance from its centre. Suppose the centre is the origin (0, 0), and the radius is 1. If a point on the circle has coordinates (x, y), then, by Pythagoras theorem (Skill 221), we must have $x^2 + y^2 = 1$





We now say that, any values x and y that satisfy this equation must give the coordinates of a point on this circle, or, simply, that this is the equation of the circle. (Note that this is an *implicit equation;* we do not have y in terms of x. If we re-arrange to get $y = \pm\sqrt{1 - x^2}$ we then have to give both values; y is not a function of x since a function has just one value.)

A circle centred at the origin, but with radius r has equation $x^2 + y^2 = r^2$. A circle centred at a point (a,b) and with radius r has equation $(x-a)^2 + (y-b)^2 = r^2$. Try to remember the connection between the circle and Pythagoras theorem.

Examples

1. A circle with centre (0,0), radius 3 has equation $x^2 + y^2 = 9$
2. Circle centre (1,2), radius 2, has equation $(x-1)^2 + (y-2)^2 = 4$.
3. Circle centre (-3,0), radius 1, has equation
   $(x-(-3))^2 + y^2 = 1$, or $x^2 + 6x + y^2 + 8 = 0$. Note the form (x-(-3)).
4. Given the equation $x^2 + 2x + y^2 - 4y - 4 = 0$, we can *complete the square* (skill 407) to get $(x+1)^2 + (y-2)^2 = 9$, and hence the centre is (-1,2) and the radius is 3.

Exercises

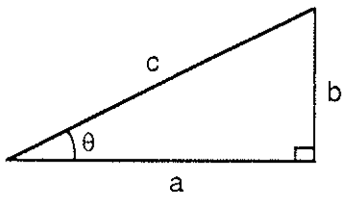
Write down the equation of the circle (simplifying if appropriate) with

1. Centre (0,0), radius 2,
2. Centre (2,0), radius 1,
3. Centre (-2,3), radius 3.
4. Find the centre and radius of the circle with equation $x^2 - 6x + y^2 = 0$

SKILL 322 SINE/COSINE RELATIONSHIP

Explanation

For the right-angled triangle shown, $\sin\theta = b / c$, $\cos\theta = a / c$. We also know that $c^2 = a^2 + b^2$.



Therefore $(\sin\theta)^2 + (\cos\theta)^2 = b^2 / c^2 + a^2 / c^2$
$= (b^2 + a^2) / c^2 = 1$ since $c^2 = a^2 + b^2$

We have shown that

$$(\sin\theta)^2 + (\cos\theta)^2 = 1$$

It is worth remembering that this is really just an alternative version of Pythagoras theorem (Skill 221).

For speed, we often write instead

$$\sin^2\theta + \cos^2\theta = 1$$

but this is confusing because we really want the square of the *whole* of $\sin\theta$ or $\cos\theta$. (Note that when you see $\sin^{-1} x$ later, it means something *completely* different!).

This result is also true for angles outside 0^0 to 90^0 (or 0 to $\frac{\pi}{2}$ radians).

Note also that $\sin\theta^2$ means, "take the sine of the square of θ", which is *not* the same.

Example

If $\cos\theta = \frac{1}{2}$, we can find $\sin\theta = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2} = \pm\sqrt{\frac{3}{4}} = \pm 0 \cdot 866$ - two possible answers.

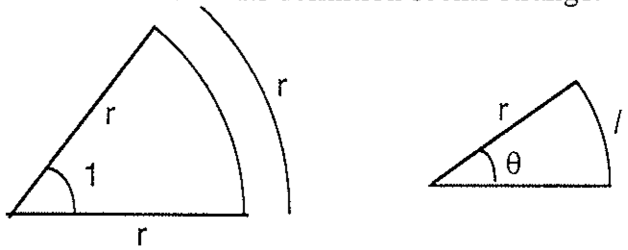
Exercises

1. With your calculator in *degree* mode, calculate $(\sin\theta)^2 + (\cos\theta)^2$ for several angles such as 0^2, 30^0, 135^0, $273 \cdot 1^0$etc.
2. In *radian* mode, check the result for angles 0, 1, $0 \cdot 1$, π etc.
3. If $\sin\theta = -\frac{1}{\sqrt{2}} = -0 \cdot 7071$, find two possible values of $\cos\theta$.with or without your calculator.

SKILL 323 RADIANS

Explanation

When we use degrees to measure angles, we are using a convenient unit, but we could choose any other. For example, we could divide the circle into 400 instead of 360 if we preferred. (This *is* done, and the units are called *grads.*) For mathematical work, and especially once we think of e.g. a *sine wave* rather than ratios of sides of triangles, the most convenient unit is the *radian*. Its definition seems strange.



A sector of a circle with angle 1 radian has arc length equal to the radius. A semicircle has arc length πr, so the angle 180^0 must be π radians, and the whole circle is 360^o or 2π radians. For any sector, we can say $l = r\theta$ or $\theta = l/r$ if θ is measured in radians.

We can also relate angle to area. The whole circle has angle 2π and area πr^2, hence the area of a sector is $\frac{1}{2}\theta r^2$. We do not use any symbol for radians; we may say $\theta = 1$ or $\theta = \frac{\pi}{2}$ etc.

Examples

1. 90^0 is one quarter of the circle, so 90^0 is $\frac{2\pi}{4} = \frac{\pi}{2}$ radians.
2. A sector with angle $\frac{1}{2}$ and radius 2 cm has arc length $2 \times \frac{1}{2} = 1$cm.
3. A sector with radius 3 m and arc 6 m has angle $6/3 = 2$ radians. Its area is $\frac{1}{2} \times 2 \times 3^2 = 9\text{m}^2$.

Exercises

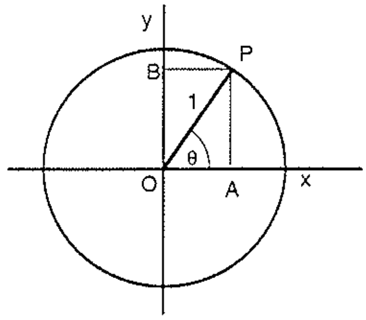
1. What are 30^0, 45^0, 60^0 in radians?
2. What are 120^0, 270^0 in radians?
3. What are the angles 1, 2, $\frac{\pi}{12}$, 0 in degrees?
4. A sector has radius 2 m and angle 20^0. By converting the angle to radians, find the arc length and area.

SKILL422 SINE AND COSINE FUNCTIONS

Explanation

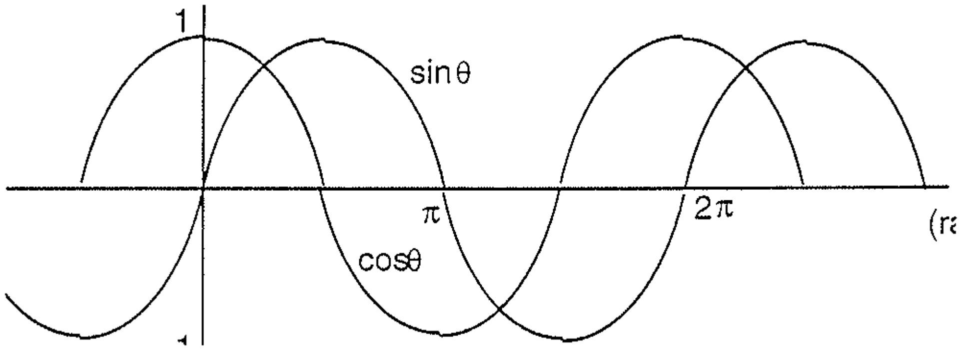
Sine and Cosine were defined as ratios of sides in triangles, but the idea can be extended to any angle. Using a circle of radius 1:

For any angle θ, measured anticlockwise from the +ve x -axis, we say
$\sin\theta = AP/OP = AP/1 = AP = OB$
and $\cos\theta = OA/OP = OA$
Therefore, $(\cos\theta, \sin\theta)$ are the coordinates of P.



If θ is not within $(0^o, 90^o)$ or $(0, \frac{\pi}{2})$ rads, one or both of OA, OB are negative, and $\sin\theta$, $\cos\theta$ may be negative. There is no need to remember any 'rules' for the sign - it can be seen from the diagram.

If we work in radians (to get away from any idea of triangles, and for other reasons) and keep increasing or decreasing θ, we get $\sin\theta$ and $\cos\theta$ for *any* θ, and hence functions of θ. The graphs look like this:



Each function repeats itself if θ increases or decreases by 2π rads (360^o) - we say that $\sin\theta$ and $\cos\theta$ are *period functions*, and their *period* is 2π.

Examples

1. $\sin(2\theta)$ has a period of π. rads (180^0) since if θ increases by π, $2\,\theta$ increases by $2\,\pi$.   2. $\cos(\theta/4)$ has period $8\,\pi$.   3. $\sin(2\pi t/0\cdot 1)$ has period $0\cdot 1$ - an oscillation of 10 Hertz.   4. $\sin[w(t-to)]$ has period $2\pi/w$.

Exercises

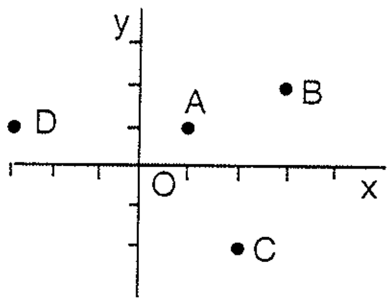
1. What is the period (in radians) of: (a) $\sin(3t)$, (b) $(t/2)$, (c) $\sin(2\pi t)$?
2. Sketch, by identifying peaks and roots, on the same diagram:
   (a) $\sin(2x)$, (b) $\cos(2x)$, (c) $\sin(4x)$ for values of x from $-\pi$ to $+3\,\pi$.

SKILL 151 COORDINATES

Explanation

When we draw a diagram or map, we can refer to any point on it using two numbers or *coordinates*. First we must choose an *origin* or reference point, and two standard directions or *axes*. Then we can say how far in each direction one must go to reach any point. These two distances are the coordinates of the point; if either is negative it means that one must go in the direction opposite to the standard direction.

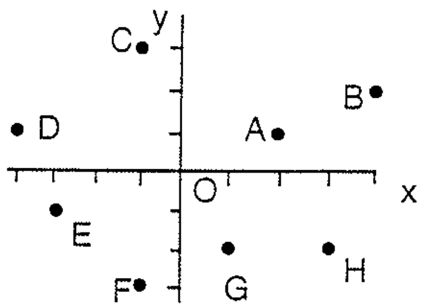
Examples



Point A is (1,1) - i.e. go a distance of 1 in both x and y directions from the origin O. B is (3,2) - *always* put the 'horizontal' distance first.
C is (2,-2), and D is (-3,1).

Exercises

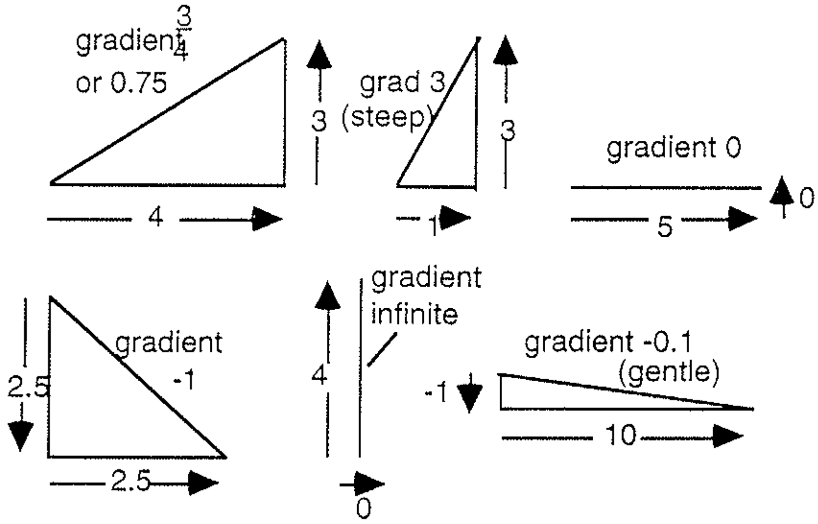
Give the coordinates of all the marked points,



SKILL 251 GRADIENT OF A STRAIGHT LINE

Explanation

Gradient means *steepness.* If y depends on x (y is a *function* of x), the gradient of the graph of y against x means the ratio of a change in y to the change in x, including the +/- signs. For example



It is the same as tan θ, where θ is the angle measured anti-clockwise from the positive x - axis. (It is *not* the same as the value used for roads, which is sin θ.)

If the end-points of the line are given, or any two points on it, we can find the changes in the two variables.

Example

The gradient of the line through the points (-1, 2) and (4, 7) is:

vertical change/horizontal change = (7 - 2) / (4 - (-1))
= 5 / 5 = 1

Notes

1. It does not matter which point is taken to be the 'start' of the line; the result is unchanged.
2. When coordinates are given, e.g. (3, -4), remember that the first is horizontal and the second vertical (often, but not always, x and y).

Exercises

Find the gradients of the lines through the points:

1. (0,0) and (3,8),
2. (2,-1) and (-3,4),
3. (-2,-4) and (-5,-11),

Draw sketches of graphs with gradients that are:

4. Large (steep) and negative,
5. Small (gentle) and positive.

SKILL 351 EQUATION OF A STRAIGHT LINE

Explanation

There are several ways of writing the equation of a straight line. For example,

$y = \frac{1}{2}x - 1, \quad x - 2y = 2, \quad (x = 2t, y = t - 1)$

all represent the same line (you can think of the third as giving x and y at the time t.)

The first is the most common, and is often written $y = mx + c$, where m is the *gradient* (Skill 251) and c is the *intercept* (value of y when $x = 0$). To find the equation of the line through two points, perhaps (1,2) and (3,8), we can either find the gradient and then the intercept, or find both together.

Using the first method, we get

$m = y \text{ change} / x \text{ change} = (8 - 2) / (3 - 1) = 3$

Then $y = 3x + c$. To pass through the point $(1,2), 2 = 3 \times 1 + c, \ c = -1$, so we get $y = 3x - 1$ (check that (3,8) also satifies this).

Using the second method, we use both points to get $2 = m + c, \ 8 = 3m + c$ and solve for m and c.

Our equation $y = 3x - 1$ can easily be rewritten in the second form:

$3x - y = 1$ (or $-3x + y = -1$, or $6x - 2y = 2$ etc. - all the same)

Examples

1. The line through (-1,2) and (-3,3) is:
   $m = (3 - 2) / (-3 - (-1)) = 1/(-2) = -\frac{1}{2}$, so $y = -\frac{1}{2}x + c$.
   At (-1,2), $2 = (-\frac{1}{2})(-1) + c, \ c = \frac{3}{2}, \ y = -\frac{1}{2}x + \frac{3}{2}$.
2. The gradient of the line $2x - 5y = 4$ is:
   $5y = 2x - 4, \ y = \frac{2}{5}x - \frac{4}{5}$, gradient is $\frac{2}{5}$
3. The line $x = 3t + 5$ crosses the x - axis at $x = 5$ (when $t = 0$). It crosses the t - axis when $x = 0$, i.e. $0 = 3t + 5$, $t = -5/3$.
4. $y = 3$ has zero gradient - it is parallel to the x - axis.
5. $x = -2$ has infinite gradient - it is parallel to the y - axis.

Exercises

Find the equation of the line through the points

1. (2,3) and (8,12), 2. (- 2,3) and (10,-1)

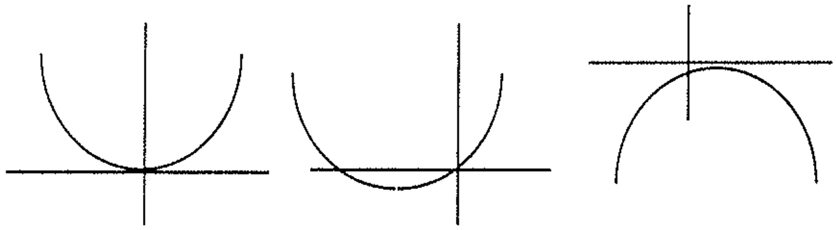
3. What is the gradient of the line $-3x + 2y = 1$, and where does it meet each axis?

4. Sketch roughly, on the same diagram, the graphs of
   $x + y = 1, \ x - y = 1, \ y = \frac{1}{2}x + 1, \ y = -x, \ y = 3, \ x = 2$

SKILL 451 - QUADRATIC GRAPHS

Explanation

The graph of a quadratic function, such as x^2, $2-3x+4x^2$, $1-u^2$, t^2-2tetc. always takes the form of a symmetrical parabola, with either a minimum point or a maximum point.



The curve may cut the horizontal axis at two points (*roots*), touch it (one root only), or not touch it. All such curves can be thought of as the basic curve of x^2(or t^2, u^2 etc.) or of $-x^2$, but *scaled* and *shifted* (or translated) in one or both directions.

Examples (see figure over)

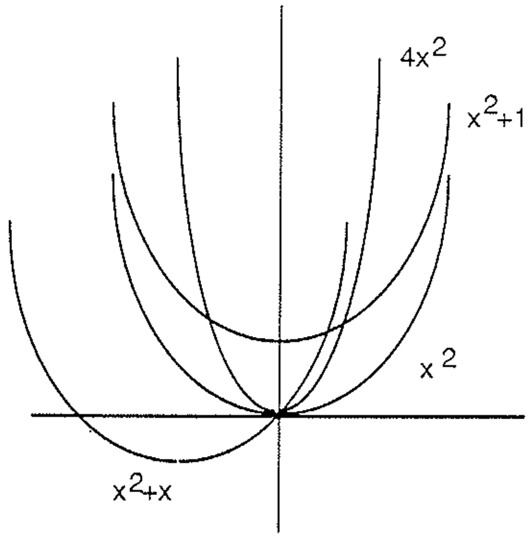
1. The graph of x^2+1 is the graph of x^2 but with all values increased by 1.
2. The graph of $4x^2$ is the graph of x^2 but with values multiplied by 4. Or think of it as $(2x)^2$, so that all x-values are scaled by a factor of 2.
3. The graph of $(x-1)^2$ has x-values shifted by 1. The minimum is now at x = +1, so the shift is *to the right.*
4. The graph of $f(x)=x^2+x$ can be obtained by completing the square to get

$$f(x)=\left(x+\frac{1}{2}\right)^2-\frac{1}{4}$$

so there is a shift of $\frac{1}{2}$ to the left and $\frac{1}{4}$ downwards. We can also write

$f(x)=x(x+1)$

and notice that $f=0$ if $x=0$ or -1



Exercises Sketch the following by considering scaling, shifting and/or roots and crossing points.

1. $f(x) = x^2 - 2$,
   $f(x) = (x+2)^2 - 3$
2. $f(x) = -2x^2$

DIAGNOSYS - BASIC MATHEMATICS FOLLOW-UP

SKILL 452 RECIPROCAL GRAPHS

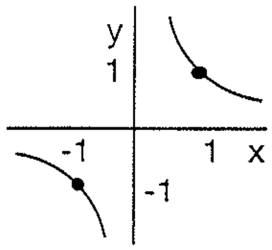
Explanation

Some functions have one variable *proportional* to another, *e.g.* $T_F - 32 = 9T_C/5$ (conversion between ^{0}F and ^{0}C). Others have one variable *inversely proportional* to another, e.g. $p = RT/V$, (pressure inversely proportional to volume in gases) or $p \propto 1/V$. We call $1/V$ the *reciprocal* of *V*. Graphs of such relationships show one variable decreasing as the other increases, with one becoming infinite as the other approaches a certain value. The *asymptotes* are straight lines approached ever more closely by the curve as one variable becomes infinite.

Examples

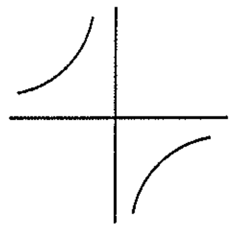
1. $y = \dfrac{1}{x}$

or $xy = 1$

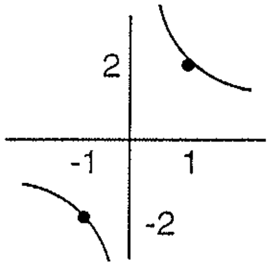


2. $y = -\dfrac{1}{x}$

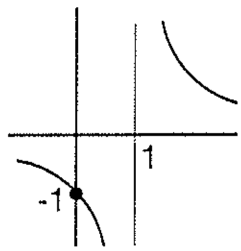
or $xy = -1$



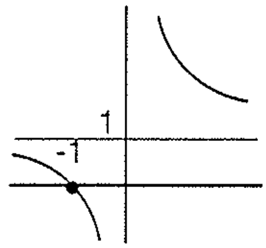
3. $s = \dfrac{2}{t}$



4. $y = \dfrac{1}{x-1}$



5. $y = \dfrac{1}{x} + 1$



Exercises

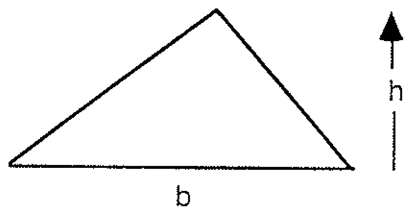
Sketch the graphs of the following functions using asymptotes, crossing points and only *one or two* values:

(a) $y = \dfrac{3}{x}$ (b) $y = 1 - \dfrac{1}{x}$, (c) $y = \dfrac{1}{x+1}$

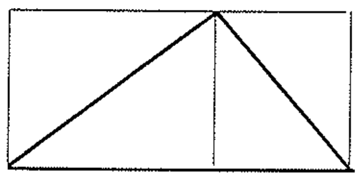
SKILL 161 AREA OF A TRIANGLE

Explanation

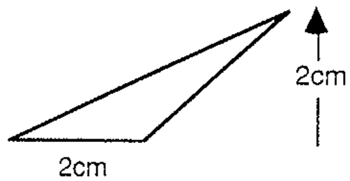
The area of a triangle is 'half base × height" or $A = bh/2$.



This can be seen by enclosing it in a rectangle, which clearly has twice the area.



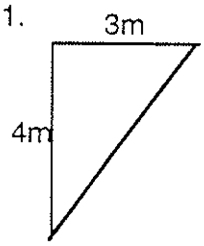
Example



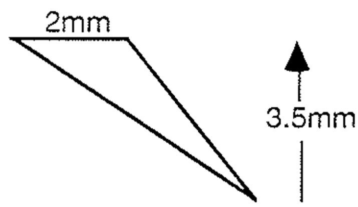
Area = $2cm^2$

Exercises Find the areas in each case.

1.



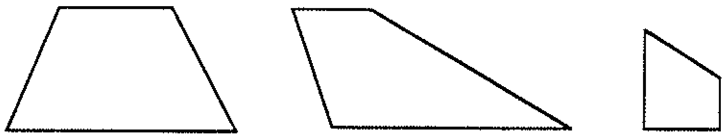
2.



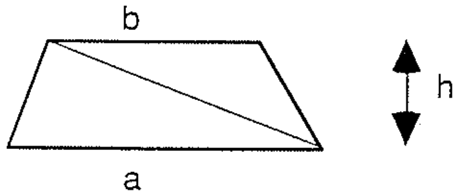
SKILL 261 AREA OF A TRAPEZIUM

Explanation

A *trapezium* is a shape with four straight sides (a *quadrilateral*), where two sides are parallel, but the others are not, e.g.



To find the area, we can divide it into two triangles as follows:



$$A = \frac{1}{2}ah + \frac{1}{2}bh = h(a+b)/2$$

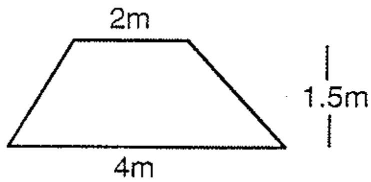
We can regard this as

Area = height × average length

where 'height' is between the parallel sides.

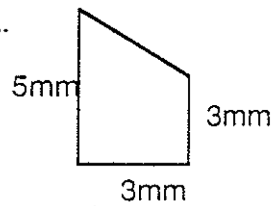
Examples

1.



$A = 1 \cdot 5(2+4)/2 = 4 \cdot 5m^2$

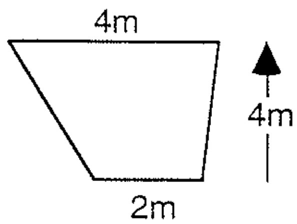
2.



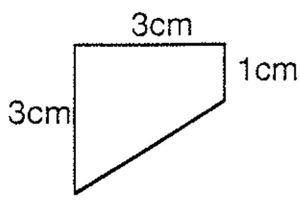
$A = 3(5+3)/2 = 12mm^2$

Exercises Find the area of:

1.



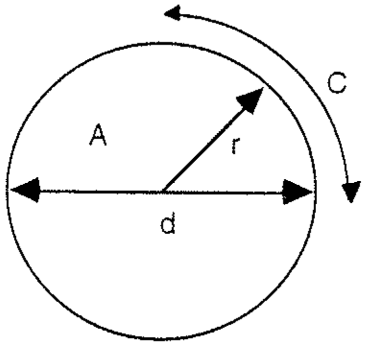
2.



SKILL 262 AREA AND CIRCUMFERENCE OF CIRCLE

Explanation

For any circle, we have the formulae $A = \pi r^2$ $C = \pi d$



where π is the name we give to the fixed number (constant) π=3.14159..... You may know the approximation $\pi = \frac{22}{7} = 3\frac{1}{7}$ - this is *not* exact.

Because $d = 2r$, we can also write these as

$$A = \pi\left(\frac{d}{2}\right)^2 = \tfrac{1}{4}\pi d^2, \quad C = \pi(2r) = 2\pi r$$

Other circular, or rounded, shapes and bodies have formulae for length, area and volume with π in them. For example, volume of sphere = $\frac{4}{3}\pi r^3$

Examples

1. If $r = 2\cdot 5m$, $A = 6\cdot 25\pi = 19\cdot 63m^2$, $C = 5\pi = 15\cdot 71m$
2. If $d = 3m$, $A = \pi(\frac{3}{2})^2 = 9\pi/4 = 7\cdot 069m^2$, $C = 3\pi = 9\cdot 425m$

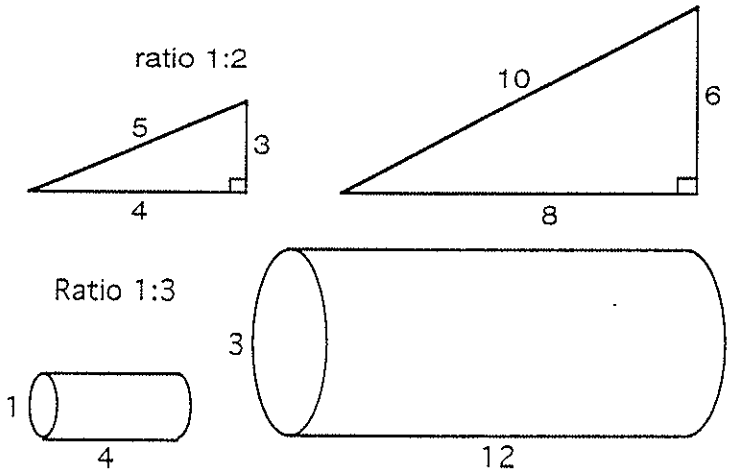
Exercises

1. If $r = 4m$, find A and C.
2. If $d = 1.5m$, find A and C.
3. Make sure you can write down the formulae, using r or d, from memory.

SKILLS 263, 362, 462 SIMILAR TRIANGLES etc.

Explanation and examples

Two triangles (or other shapes) are *similar* if they have exactly the same shape but different sizes. Therefore all angles are equal, but lengths are not. All corresponding lengths are in the same ratio.

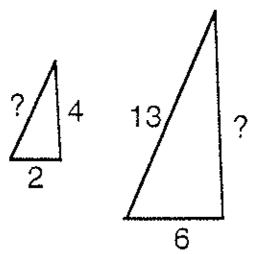


The areas (or surface-areas in 3D) are in a ratio that is the *square* of the length ratio. The ratio of areas of the triangles above is 4:1. The ratio of each part of the surface area, or the cross-section, for the cylinders is 9:1.

The volumes are in a ratio equal to the *cube* of the length ratio: This is obvious if we compare cubes of dimensions $1 \times 1 \times 1$ and $2 \times 2 \times 2$, but applies to any shape.

Exercises

1. Find the sides marked '?' for these similar triangles

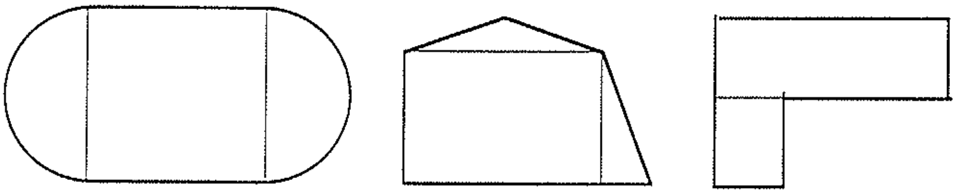


2. If two spheres have radii in the ratio 4:1, what are the ratios of their surface areas and volumes?

SKILL 363 AREA OF IRREGULAR SHAPES

Explanation

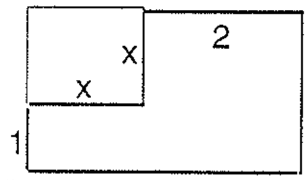
Any area that can be subdivided into rectangles, triangles or sectors of circles can be found using standard formulae, e.g.



Note that the last could also be found by subtracting a smaller area from a larger one.

Example

1. (as in the Test)



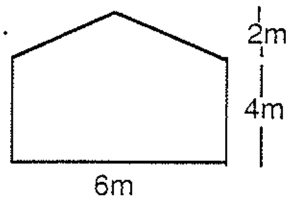
Area = Larger - smaller

$$= (2+x)(1+x) - x^2 = 2 + 2x + x + x^2 - x^2 = 2 + 3x$$

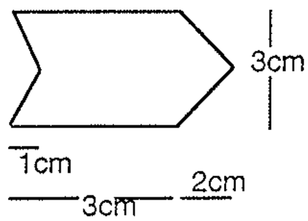
(Skill312 - expanding two brackets)

Exercises Find each area

1.



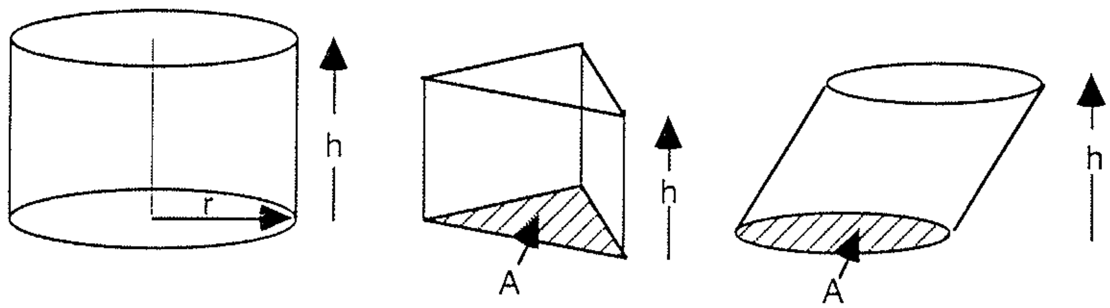
2.



SKILLS 361,461 VOLUME and SURFACE AREA OF CYLINDER

Explanation

A *cylinder* usually means a 3D body with circular cross-section, and with ends perpendicular to its long axis (first diagram):



Its volume is given by Area x Height, or $V = \pi r^2 h$

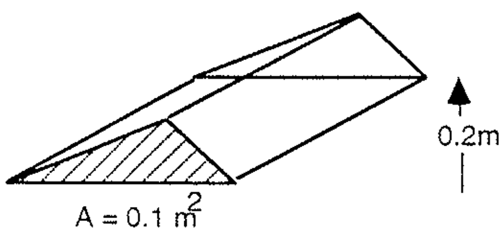
The same idea applies if it isn't a circular cylinder but a *prism*, which has the same cross-section throughout, or is *slanted*, for either shape, provided we take the height perpendicular to the ends, which are parallel. This last result is easy to appreciate if we view the cylinder as a pile of discs pushed over, but retaining the same volume of material. Naturally, for the cylinder the dimensions may be given as a diameter and length etc.

The *total* surface area can be calculated for non-slanted cylinders quite easily. For the usual circular cylinder, we need:

Area	=	Ends + curved side
	=	$2(\pi r^2)$ + circumference x height
	=	$2\pi r^2 + (2\pi r)h = 2\pi r(r+h)$

Exercises

1. Find the volume of a cylinder of diameter 0.3m and height 0.25 m
2. Find the volume of the triangular prism shown:



3. Find the total surface area of the cylinder in q1.

SKILL 343 GEOMETRIC PROGRESSIONS

Explanation

A *geometric progression* is a *series* in which *terms* are a fixed multiple of the previous terms, and the total is to be added up.
For example:

$S_1 = 1 + 2 + 4 + 8 + 16.$ (5 terms, ratio 2)

$S_2 = 5 - 15 + 45 - 135.$ (4 terms, ratio -3)

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ (infinite, ratio $\frac{1}{2}$)

To add up a series, we write it down in its original form, and also multiplied by its fixed ratio. Subtracting one from the other gives a simple result.

Examples

1.
$$\begin{aligned} S &= 1+2+4+8+16 \\ 2S &= 2+4+8+16+32 \quad \text{(multiply by ratio 2)} \\ S-2S &= 1+0+0+0+0-32 \\ -S &= -31 \\ S &= 31 \end{aligned}$$

2.
$$\begin{aligned} S &= 5-15+45-135 \\ -3S &= -15+45-135+405 \quad \text{(ratio -3)} \\ S-(-3S) &= 5-405 = -400 \\ S &= -100 \end{aligned}$$

3.
$$\begin{aligned} S &= a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \quad \text{(n terms, ratio r)} \\ rS &= ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n \\ S - rS &= a - ar^n \\ S &= \frac{a(1-r^n)}{1-r} \end{aligned}$$

4. In example 3, if $|r|<1$ and $n \to \infty$, then $r^n \to 0$ and $S \to a/(1-r)$ - the sum of a *convergent* infinite G.P.

Exercises

Find the sum using the method of Examples 1 and 2 of

1. $10 + 5 + 2{\cdot}5 + 1{\cdot}25.$
2. $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8}.$

Find the sum using the formulae of Examples 3 and 4 of

3. $3 - 9 + 27 - 81 + 243.$
4. $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

Principles of Differentiation *(not in the DIAGNOSYS test)*

Explanation

If an object accelerates at a rate $a = 2ms^{-2}$, we mean its velocity increases by $2ms^{-1}$ every second, so we get $v = 2t$ if it is stationary when $t = 0$. A graph of v against t is a straight line with a gradient of 2. *Acceleration is the rate of change of velocity with respect to time,* and for other values of a we'd have $v = at$ and a graph of gradient a (provided a was some fixed number).

If the acceleration changes, we no longer have $v = at$ nor a straight -line graph. Suppose we know that $v = t^2$, with a curved graph. What can we say about the acceleration, and about the gradient? We look at the change in v over a short time interval.

At $t = 2s$, for example, $v = 4ms^{-1}$. At $t = 2 \cdot 1s$, just slightly later, $v = 2 \cdot 1^2 = 4 \cdot 41ms^{-1}$. The change in v is $0 \cdot 41ms^{-1}$ in a time of $0 \cdot 1s$, so we can say that the acceleration is, on average,

$$a_{av} = \frac{\text{change in } v}{\text{time taken}} = \frac{0 \cdot 41}{0 \cdot 1} = 4 \cdot 1ms^{-2}$$

So, over the time interval $2s$ to $2 \cdot 1s$, this is the average acceleration. However, it is actually changing all the time.

Exercise

Find the velocity at $3s$ and at $3 \cdot 1s$ and calculate the average acceleration over this interval. Also try between $4s$ and $4 \cdot 1s$. Do you notice any rough pattern in the figures?

To find the acceleration nearer to $t = 2s$, we use a shorter interval, such as $t = 2s$ to $t = 2 \cdot 01s$, or $1 \cdot 99s$ to $2s$. For $2s$ to $2 \cdot 01s$ we get

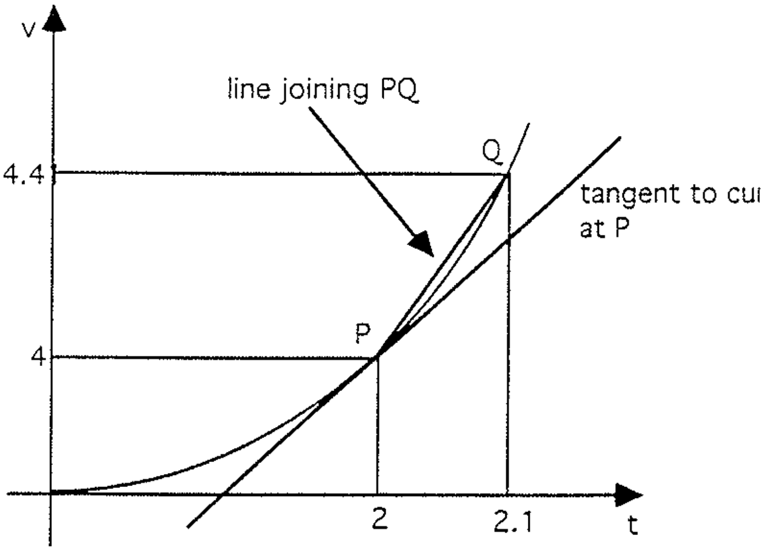
$$a_{av} = \frac{4 \cdot 0401 - 4}{0 \cdot 01} = \frac{0 \cdot 0401}{0 \cdot 01} = 4 \cdot 01ms^{-2}$$

and for $1 \cdot 99s$ to $2s$ we get

$$a_{av} = \frac{4 - 3 \cdot 9601}{0 \cdot 1} = \frac{0 \cdot 0399}{0 \cdot 01} = 3 \cdot 99ms^{-2}$$

This suggests that very close to $t = 2s$ the acceleration is $a = 4ms^{-2}$.

We can also look at this on a graph.



Our first calculation gave the slope of the line PQ to be $0 \cdot 41 / 0 \cdot 1 = 4 \cdot 1$. When we reduced the interval to only $0 \cdot 01s$, we were in effect moving the point Q towards P along the curve (or having Q to the left of P when we used $1 \cdot 99s$ to $2s$). That brought the line PQ closer to the tangent to the curve at P. The slope of the tangent is exactly what we want for the acceleration when $t = 2s$.

Our results so far suggest that $a = 2t$ for all the values $t = 2s$, $t = 3s$ etc. We'll now confirm this formula, so that we can use it for any value of t without further calculations.

At time t, we have $v = t^2$.
At time $t + \delta t$ (a little bit later on), we have

$$v = (t + \delta t)^2$$

$$= t^2 + 2t(\delta t) + (\delta t)^2$$

Note dt means 'a bit of t'; it is *not* two things multiplied together.
Average acceleration over the time interval δt is therefore

$$\frac{\text{change in } v}{\text{time taken}} = \frac{\left[t^2 + 2t(\delta t) + (\delta t)^2\right] - \left[t^2\right]}{\delta t}$$

$$= \frac{2t(\delta t) + (\delta t)^2}{\delta t}$$

$$= 2t + \delta t$$

Now, as we make the time interval δt smaller and smaller, it is clear that the value of acceleration gets nearer and nearer to

$$a = 2t$$

Exercise

(i) If $v = 2t^2 - 1$, find the velocity at times $t = 1s$ and $t = 1 \cdot 1s$ and hence the average acceleration over the interval $1s$ to $1 \cdot 1s$.
Predict the formula for acceleration at any time, *t*, namely $a =$ ......$\times t$.

(ii) Find the velocity for time t and time $t + \delta t$. From these, write down the expression for the average acceleration over time interval δt, and simplify as before. Finally deduce the exact formula for acceleration.

If we write δv to mean a small change in velocity, then our average acceleration is

$$a_{av} = \frac{\delta v}{\delta t}$$

When we let δt become vanishingly small (we "take the *limit* $\delta t \to 0$"), we obtain the exact value of *a*, and we write

$$a = \frac{dv}{dt} = 2t \quad \text{(for our first example)}$$

SINE RULE and COSINE RULE (not in the Test)

Explanation

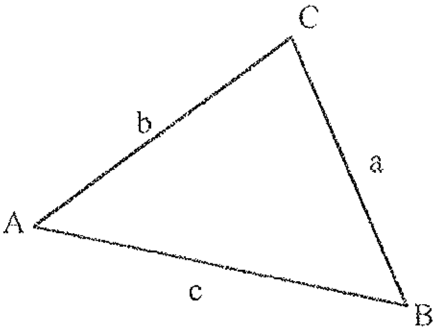
If we know two angles and one side of a triangle, or two sides and one angle, we can find the others, using the *Sine Rule* or the *Cosine Rule*, which relate sides and angles.

If the angles are A,B,C, and the opposite sides are a,b,c then we have:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad \text{Sine Rule}$$

and

$$a^2 = b^2 + c^2 - 2bc\cos A \qquad \text{Cosine Rule}$$



Naturally, the Cosine Rule can be re-written to apply to any of the three angles.

Examples

1. If $a = 4,\ b = 2,\ C = 45°$ find the other sides and angles.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$= 16 + 4 - 16\left(\frac{1}{\sqrt{2}}\right) = 8\cdot686 \text{ so } c = 2\cdot947$$

Then $\frac{2}{\sin B} = \frac{2\cdot947}{\sin C}$, $\sin B = \frac{2(0\cdot7071)}{(2\cdot947)} = 0\cdot4798$

$B = \arcsin(0\cdot4798)$ and a sketch confirms which value to take,

so $B = 28\cdot7°$ and finally $A = 106\cdot3°$.

2. If $a = 2,\ b = 3,\ A = 30°$ find all the other sides and angles.

$\frac{2}{\sin 30} = \frac{3}{\sin B}$ so $\sin B = \frac{3(\frac{1}{2})}{2} = \frac{3}{4}$, $B = \arcsin \frac{3}{4} = 48\cdot6°$ or $131\cdot4°$

Then $C = 180° - A - B = 101\cdot4°$ <u>or</u> $18\cdot6°$

Now $c^2 = a^2 + b^2 - 2ab\cos C$

$$= 4 + 9 - 12(-0\cdot198) = 15\cdot38 \text{ so } c = 3\cdot92$$

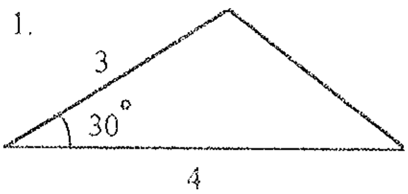
$$\text{or } = 4 + 9 - 12(0\cdot948) = 1\cdot627 \quad \text{so } c = 1\cdot28$$

So in this case the information given doesn't fix the answer uniquely.

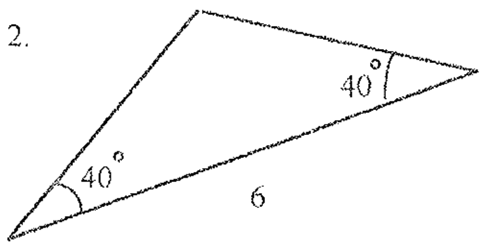
Exercises

'Solve the triangles' in each case:

1.



2.



(November 1995)