SFY0001 Basic Mathematics
2015/16
Lecture Notes

Contents

0 Introduction

Timetable

SFY0001 Lectures

Monday	5.00	Merz Court L302
Wednesday	12.00	Merz Court L302
Thursday	1.00	Merz Court L303
Friday	4.00	Percy Building G.05

SFY0001 Weekly Tutorials

Friday	9.00	Stephenson Building T10	Engineering Foundation Year + G101
Friday	11.00	Daysh Building G.05	GIS / SMS

N.B. *Each student attends only one weekly tutorial: see your personal timetable to find your tutorial.*

Exceptional arrangements
The lectures on Wednesday 7 and Thursday 8 October are replaced by lectures on

- Monday 5 October 2.00 in Merz Court L302,
- Friday 9 October at 10.00 in Merz Court L303.

Organisation

1. SFY0001 runs for the first six weeks of the first term. Weekly exercises are set and marked but do not count towards the overall SFY0001 mark (though your marks are copied to your department who will be monitoring your progress). The marked exercises are to be collected during Office hours on Mondays (see below). There are 2 tests: Test 1 at the end of week 3, and Test 2 at the end of week 6. Calculators are not allowed in the first test but are allowed in the second test and the exam.

2. Tests and Exam

   (a) If you pass **both** tests separately (the pass mark for each test is 40) then: (i) you pass SFY0001; (ii) your SFY0001 mark is the average of the two test marks; (iii) you don't take the January exam.

   (b) If you fail one or both tests, then you have to take the January exam, and your SFY0001 mark will be the exam mark. If you still fail then you resit the exam in August (and this will be your final attempt).

   (c) If your first attempt at SFY0001 was in 2014/15 and 2015/16 counts as your second attempt, then you must take the January exam (you cannot pass the module just by passing both tests).

   - Q: *Suppose I fail Test 1 with a mark of* 39 *but do spectacularly well in Test* 2, *with a mark of* 100. *Do I still have to take the exam?*

   - A: Yes,you still have to take the exam.

   - Q: *Suppose I fail Test 1, so I will have to take the exam. Can I safely skip Test 2?*

   - A: You should still take Test 2 because it will provide you with valuable feedback on your progress and it will be practice for the exam.

3. You'll find sample tests on Blackboard. Past exam papers are at **www.ncl.ac.uk/exam.papers/**.

4. The university has strict rules governing the use of calculators in examinations. Candidates may use a calculator in an examination only if that particular calculator appears on the University's approved list. No other calculator or electronic device may be used in an examination. The approved list currently consists of the models listed below (but check for any changes between now and the exam):-

   Casio FX-83GTPLUS   Casio FX-85GTPLUS   Casio FX-115MS

   plus any discontinued versions of the FX-83, FX-85 or FX-115.

N.B. You are not allowed to use a calculator in Test 1, but you are allowed (and will need) to use one in Test 2.

5. You will notice that there are gaps in these lecture notes. This is deliberate. You fill in these gaps during the lectures. The material omitted consists largely of solutions to examples.

6. This course is about solving problems and knowing relevant techniques. You don't need to explain the theory, but you should aim to write coherent answers. There are some formulae you will need to remember, and often remembering a diagram or construction will allow you to reconstruct the formula. Sometimes we include an explanation of why a technique works. You will not have a formula sheet for Test 1, but a very short list of formulae will be provided for Test 2 and the exam.

7. HELP! I NEED HELP! Where can I get it? There are four sources of individual help.

   (a) If you have a question or a problem, you can ask about it at your weekly tutorial.

   (b) You can see me during my office hours. These are times when I guarantee to be in my office to answer questions. You don't need an appointment — just drop in. You can collect your marked exercises and receive feedback during the Monday office hours. My office is Room 3.08 on Level 3 of the Herschel Building, and my office hours for SFY0001 will be:

Day	Hour	Room	Building
Mon	1.00 - 2.00	Room 3.08	Herschel Building
Mon	3.00 - 4.00	Room 3.08	Herschel Building
Tue	9.00 - 10.00	Room 3.08	Herschel Building

   (c) You can email me at Oli.King@ncl.ac.uk. Sometimes I can answer a short point by email. At other times it will be more appropriate to arrange an appointment to discuss the point in my office.

   (d) You can use the University's Maths-Aid drop-in centre. This is a terrific service: you can get one-on-one help with a specific point or help with a broad subject area; they can give you booklets and CD-ROMs that support the lectures. And the service is completely confidential. For details, see **http://www.ncl.ac.uk/students/mathsaid/** . You'll get better service if you make contact with Maths-Aid early in the semester. They get inundated in the run-up to the end-of-semester exams.

8. Copies of lecture notes, exercise sheets, handouts, etc. will be posted on Blackboard.

9. Lectures will recorded on ReCap and will be made available for viewing a few days after the lecture. You will be able to see the presentation slides (which duplicate the notes handed out) and hear my spoken explanations, but you will

not be able to see my writing on the board. For this reason, ReCap will not replace lectures but will be suitable for listening to lectures a second time with your notes in front of you.

10. You can practice using the Numbas system at **https://moodle.mas.ncl.ac.uk/** . You log in using your normal University user name and password. Assuming that you are registered for SFY0001, you should find that you have access to a Getting Started module and SFY0001. You can look at Getting Started, but the mathematics is more advanced than SFY0001 for the most part. If you click on SFY0001, you will find a number of assignments that correspond to written assignments, but they are not assessments, you can practice as much as you like.

1 The Laws of Arithmetic

We shall go over the rules for evaluating arithmetic expressions. We shall consider only examples involving whole numbers. This will keep the calculations simple and allow us to concentrate on the rules. Fractions will be introduced in the next chapter.

The evaluation rules are obeyed by calculators, so keep your calculator handy. You can use it to give a quick check that you are performing a calculation correctly.

Note, by the way, that calculators are not allowed in Test 1. (We know that your calculator can apply the rules correctly. The question is: Can you?) Calculators will be allowed in Test 2.

1.1 Expressions Without Brackets

1.1.1 Expressions Involving Only Additions and Subtractions.

To work out the value of an expression containing only additions and subtractions, you process it from left to right, keeping a running total (the total 'so far').

Example 1. Evaluate

$$12 - 7 + 15 - 26 + 2.$$

ANSWER

The final running total gives you the value of the original expression.

Feed this expression into your calculator and check that it gives the same answer.

1.1.2 Expressions Involving Only Multiplications and Divisions

To work out the value of an expression containing only multiplications and divisions, you process it from left to right, keeping a running total.

Example 2. Evaluate

$$12 \times 3 \div 4 \times 5 \div 9.$$

ANSWER

Therefore the value of the original expression is *ANSWER* .
Feed this expression into your calculator and check that it gives the same answer.

1.1.3 Expressions Involving Only Additions, Subtractions, Multiplications and Division

Now ask your calculator to evaluate some expressions involving operations of both types, for example
$3 \times 4 + 5$ — your calculator should give you the *ANSWER* ,
and
$3 + 4 \times 5$ — your calculator should give you the *ANSWER* .

Note that if you process the first expression from left to right you get the correct answer, but if you do the same with the second expression you get **35**, **which is the wrong answer**! The moral?

YOU DON'T ALWAYS PROCESS SUCH EXPRESSIONS FROM LEFT TO RIGHT!

What is going on here?

The golden rule is that DIVISIONS and MULTIPLICATIONS are worked out before ADDITIONS and SUBTRACTIONS. In technical language, we say that *division and multiplication TAKE PRECEDENCE over addition and subtraction*. Thus

$3 \times 4 + 5 =$
ANSWER
while
$3 + 4 \times 5 =$
ANSWER

Another example:

Example 3. Evaluate $\qquad 20 - 4 \times 3 \div 6 + 2 \times 2 \times 3$

ANSWER

What we have observed is part of a larger rule called BIDMAS. It stands for

<u>B</u>RACKETS
before
<u>I</u>NDICES
before
<u>D</u>IVISION and <u>M</u>ULTIPLICATION
before
<u>A</u>DDITION and <u>S</u>UBTRACTION.

Let us ignore the brackets and indices for now; we shall come to them shortly. For the moment the rule amounts to DMAS, i.e., divisions and multiplications take precedence over additions and subtractions.

We might ask: what about precedence between division and multiplication? The answer is that there is none, we simply process divisions and multiplications from left to right, leaving additions and subtractions.

A calculator sees the expression

$$20 - 4 \times 3 \div 6 + 2 \times 2 \times 3$$

as $\boxed{20} - \boxed{4 \times 3 \div 6} + \boxed{2 \times 2 \times 3}$ and evaluates the boxes as in Section 1.1.2. It then evaluates $20 - 2 + 12$ as in Section 1.1.1.

1.1.4 Indices (Powers)

The I in BIDMAS stands for **indices** (also called **powers** or **orders** or **exponents**, leading to BODMAS or BEDMAS). Indices became popular in the seventeenth century as a convenient shorthand for multiplying a number by itself repeatedly.

For example we write
3×3 as 3^2 (often read as '3 to the power 2' or '3 squared')
$3 \times 3 \times 3$ as 3^3 ('3 to the power 3' or '3 cubed')
$3 \times 3 \times 3 \times 3$ as 3^4 ('3 to the power 4' or '3 to the 4th')
$3 \times 3 \times 3 \times 3 \times 3$ as 3^5 ('3 to the power 5' or '3 to the 5th')
$3 \times 3 \times 3 \times 3 \times 3 \times 3$ as 3^6 ('3 to the power 6' or '3 to the 6th')
and so on ...... The indices or powers are the superscripts.

Example 4. Calculate 2^n for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10$. [In other words, calculate $2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, 2^9, 2^{10}$.]

ANSWER

How do we evaluate expressions which contain indices? For instance, when we encounter the expression 3×4^2, do we perform the multiplication or the power 2 first?

If we perform multiplication first we get $12^2 = 144$.

If we perform the square first we get $3 \times 16 = 48$.

If we use a calculator we get *ANSWER* .

BIDMAS gives us the explanation. The I in BIDMAS takes precedence over DMAS, so we square before we add, subtract, multiply or divide. The correct answer is *ANSWER* .

Example 5. Evaluate

$$36 \div 3^2$$

ANSWER

Example 6. Evaluate

$$5 \times 2^4 + 2^2 \times 3^3.$$

ANSWER

Index notation comes into its own when we have to work with numbers which are very big or very small - as happens in physics and chemistry. For example the mass of the Sun in grams is

$$2000000000000000000000000000000000g.$$

That's 2 followed by 33 zeros. We can write this more succinctly in index notation as 2×10^{33} grams. Similarly, the mass of an electron can be written neatly as 9×10^{-28} grams. Don't worry about the negative power – all will be explained shortly.

1.2 Brackets

The B in BIDMAS stands for BRACKETS. We have seen how to evaluate expressions like

$$2 + 3 \times 4.$$

You DON'T do the addition first:

$$2 + 3 \times 4 = 5 \times 4 = 20 \ \mathbf{WRONG}$$

You do multiplication first:

$$2 + 3 \times 4 = 2 + 12 = 14 \ \mathbf{RIGHT}$$

But let's suppose that you DO want to add 3 and 2 and multiply the resulting sum by 4. How do you indicate that? You use brackets. The golden rule with brackets is

OPERATIONS INSIDE A BRACKET
TAKE PRECEDENCE OVER
OPERATIONS OUTSIDE THE BRACKET

Examples 7. Evaluate the following:

(a) $(2 + 3) \times 4$
ANSWER

(b) $2 + (3 \times 4)$
ANSWER

(c) $(2 \times 3) + 4$
ANSWER

(d) $2 \times (3 + 4)$
ANSWER

(e) $4 \times (8 - 3)$
ANSWER

(f) $5 + 6 \div 2$
ANSWER

We restate the complete BIDMAS rule:

<u>B</u>RACKETS
before
<u>I</u>NDICES
before
<u>D</u>IVISION and <u>M</u>ULTIPLICATION
before
<u>A</u>DDITION and <u>S</u>UBTRACTION.

Sometimes brackets occur inside other brackets. When this happens we say that the brackets are **nested**. Here the rule is: WORK FROM THE INSIDE OUT.

Examples 8. Evaluate the following:

(a) $((6 - 4) \times 5) + 8$
ANSWER

(b) $2 \times [(4 \cdot 2 - 3 \cdot 4)(2 \cdot 4 - 1 \cdot 7) + (0 \cdot 88 \times 0 \cdot 5)] + 3 \cdot 0$
ANSWER

Sometimes brackets are **absent but implied**: for example, $\frac{7+5}{2}$ is understood to be $\frac{(7+5)}{2}$, so

$$\frac{7+5}{2} = (7+5) \div 2 = 12 \div 2 = \mathbf{6}.$$

[Note that $\frac{7+5}{2}$ is not the same as $7 + 5/2 = 7 + 2 \cdot 5 = 9 \cdot 5$.]

We shall talk about square roots a little later on. However it is worth noting for future reference that brackets are implied under square root signs:

$$\sqrt{4+5} = \sqrt{(4+5)} = \sqrt{9} = \mathbf{3}.$$

If we calculate $\sqrt{4}+\sqrt{5}$ we get $2+2{\cdot}236\ldots = 4{\cdot}236\ldots$. This is clearly different.

1.3 Two Minuses Make A Plus

This rule refers to three different situations.

(A) SUBTRACTING A NEGATIVE

$\boxed{a - (-b) = a + b}$

$2 - (-3) = 2 + 3 = \mathbf{5}$

$3 - (4 - 5) = 3 - 4 + 5 = \mathbf{4}$. (Here $- - 5 = +5$.)

(B) MULTIPLYING TWO NEGATIVES

$\boxed{(-a)(-b) = ab}$

$(-6) \times (-3) = 6 \times 3 = \mathbf{18}$.

(C) DIVIDING ONE NEGATIVE BY ANOTHER

$\boxed{(-a)/(-b) = a/b}$

$(-6) \div (-3) = 6 \div 3 = \mathbf{2}$.

(Here the minus signs cancel.)

Example 9. Evaluate $(-2)^3$, $(-2)^4$ and -2^4.

ANSWER

N.B. Note that we never write expressions like 15×-3 or $15 \div -3$. That is, we never put a multiplication or division sign alongside a negative sign. We always wrap the negative sign in brackets. So we write $15 \times (-3)$ or $15 \div (-3)$.

2 Fractions

2.1 Introduction

A **numerical fraction** is a number expressed in the form

$$\frac{p}{q}$$

where p and q are whole numbers (and of course $q \neq 0$). We call p the **numerator** and q the **denominator** (or sometimes just **top** and **bottom**). The formal mathematical term for a number that can be written as a fraction is a **rational number**.

Here are some fractions:

$$\frac{5}{8}, \frac{-3}{8}, \frac{2}{8}, \frac{4}{16}, \frac{-1}{-4}, \frac{-8}{1}, \frac{1}{1}, \frac{0}{1}, \frac{6}{2}.$$

Each of the numbers listed can be written in decimal form, for example by dividing the top number by the bottom:

$$\frac{5}{8} = 0 \cdot 625, \ \frac{-3}{8} = -0 \cdot 375, \ \frac{2}{8} = 0 \cdot 25, \ \frac{4}{16} = 0 \cdot 25,$$

$$\frac{-1}{-4} = 0 \cdot 25, \ \frac{-8}{1} = -8, \ \frac{1}{1} = 1, \ \frac{0}{1} = 0, \ \frac{6}{2} = 3.$$

Notice the following:

- Depending on our calculator, entering $-3 \div 8$ might or might not present problems. Entering $0 - 3 \div 8$ should not present any problems because from Section 1 we know that this is $0 - 0 \cdot 375$ (division first), which equals $-0 \cdot 375$. But we should really calculate $(-3) \div 8$.

- Not every fraction can be easily written as a decimal. For example $\frac{1}{3}$ would be an infinite (unending) decimal $0 \cdot 3333 \ldots$.

- Calculating $\frac{-1}{-4}$ as $-1 \div -4$ is illegal because we have $\div$ followed by $-$. However we can either calculate $(-1) \div (-4)$ or we can use our knowledge that $\frac{-1}{-4}$ is the same as $\frac{1}{4}$.

- Notice that $\frac{2}{8}, \frac{4}{16}, \frac{-1}{-4}$ all represent the same number $0 \cdot 25$, and indeed $\frac{3}{12}, \frac{5}{20}, \frac{6}{24}, \frac{1}{4}$ also represent $0 \cdot 25$.

This leads us to an important principle:
If you multiply (or divide) the numerator and denominator by the same (non-zero) number then you don't change the value of the fraction.

If we wish to have a single way of expressing a number as a fraction, we can use a **standard form** for a fraction. The usual standard form is the **lowest terms** form. We say that a fraction

$$\frac{p}{q}$$

is in **lowest terms** if p and q are whole numbers, q is positive and q is as small as possible. We can always reduce a fraction to lowest terms by cancelling 'common factors' from the numerator and denominator. **It is a good idea to reduce all your fractions to lowest terms!**

Example 10. write $\frac{30}{42}$ in lowest terms.

ANSWER

Fractions obey the same BIDMAS rules as the whole numbers, but addition, subtraction, multiplication and division are more complicated. We consider each in turn.

2.2 Addition of Fractions

If two fractions happen to have the same denominator, then we can add the fractions by adding the numerators:

$$\boxed{\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}}.$$

Examples 11. Express each of the following as a fraction in lowest terms:

(a) $\frac{2}{7} + \frac{3}{7}$.
ANSWER

(b) $\frac{4}{9} + \frac{11}{9}$.
ANSWER

If two fractions do not have the same denominator, then we have to convert them to fractions with a common denominator. There are two ways of doing this.

METHOD ONE (CROSS-MULTIPLICATION): This method always works, but it sometimes results in large numbers in the numerator and denominator and we might need to do a bit of work in order to end up with lowest terms. Suppose we wish to write as a single fraction $\frac{a}{b} + \frac{c}{d}$. We observe that $\frac{a}{b} = \frac{ad}{bd}$ and $\frac{c}{d} = \frac{bc}{bd}$. Thus

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

We obtain the rule:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}}$$

Examples 12. Express each of the following as a single fraction in lowest terms:

(a) $\frac{3}{4} + \frac{5}{6}$.
ANSWER

(b) $\frac{2}{7} + \frac{3}{11}$.
ANSWER

METHOD TWO: Pick a number that is a common multiple of both denominators and convert each fraction to one with this new number as the denominator.

Examples 13. Express each of the following as a single fraction in lowest terms:

(a) $\frac{3}{4} + \frac{5}{6}$.
ANSWER

(b) $\frac{7}{6} + \frac{8}{10}$.
ANSWER

2.3 Subtraction of Fractions

If two fractions happen to have the same denominator, then we can subtract one from the other bysubtracting one numerator from the other:

$$\boxed{\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}}.$$

Examples 14. Express each of the following as a fraction in lowest terms:

(a) $\frac{5}{8} - \frac{3}{8}$.
ANSWER

(b) $\frac{11}{12} - \frac{5}{12}$.
ANSWER

If two fractions do not have the same denominator, then we have to convert them to fractions with a common denominator. There are two ways of doing this, essentially the same procedures as for addition. In particular the cross-multiplying rule is:

$$\boxed{\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}}$$

Examples 15. Express each of the following as a fraction in lowest terms:

(a) $\frac{2}{7} - \frac{3}{11}$.
ANSWER

(b) $\frac{5}{6} - \frac{1}{4}$.
ANSWER

(c) $\frac{11}{108} - \frac{5}{72}$.
ANSWER

2.4 Multiplication of Fractions

Multiplication of fractions is straightforward. In fact it is easier than addition and subtraction. You just multiply the two numerators and multiply the two denominators:

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}}$$

Examples 16. Express each of the following as a fraction in lowest terms:

(a) $\frac{5}{8} \times \frac{1}{3}$.
ANSWER

(b) $\frac{5}{9} \times \frac{3}{20}$.
ANSWER

(c) $\frac{1}{2} \times \frac{3}{8}$.
ANSWER

Reasons to be careful. If we misremember the adition rule and calculate

$$\frac{4}{7} + \frac{3}{5} = \frac{4+3}{7+5} = \frac{7}{12}$$

which cannot be correct since each of $\frac{4}{7}$ and $\frac{3}{5}$ is bigger than $\frac{1}{2}$, so their sum is bigger than 1. In fact

$$\frac{4}{7} + \frac{3}{5} = \frac{20}{35} + \frac{21}{35} = \frac{41}{35}.$$

2.5 Division of Fractions

To divide by a fraction, "invert the divisor and multiply":

$$\boxed{\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}}$$

Examples 17. Express each of the following as a fraction in lowest terms:

(a) $\frac{5}{8} \div \frac{1}{3}$.
ANSWER

(b) $\frac{2}{3}/\frac{3}{4}$.
ANSWER

(c) $\frac{14}{3} \div \frac{1}{3}$.
ANSWER

Example 18. Express the following as a fraction in lowest terms: $\dfrac{\dfrac{2}{3}-\dfrac{1}{5}}{\dfrac{1}{6}+\dfrac{1}{8}}$.

ANSWER

2.6 Cancellation of Fractions

Recall the important principle:
If you multiply (or divide) the numerator and denominator by the same (non-zero) number then you don't change the value of the fraction.

This is the principle we have been using in reducing fractions to lowest terms:

$$\boxed{\frac{ab}{ac}=\frac{b}{c}}.$$

We often refer to this procedure as **cancellation**. Each step in the following is an example of cancellation:

$$\frac{-120}{-180}=\frac{120}{180}=\frac{20}{30}=\frac{2}{3}.$$

3 Algebra

3.1 Introduction

Algebra is an extension of arithmetic in which we allow letters to stand for numbers. It is governed by the same BIDMAS rules as arithmetic. The only difference is in the notation. In algebra it is traditional to indicate multiplication by juxtaposition ("placing alongside") instead of using $\times$ or $*$; however we still use $\times$ for the multiplication of two numbers. Thus we write

$$\begin{gathered} 3x \text{ instead of } 3 \times x \\ ab \text{ instead of } a \times b \\ 3x^2 \text{ instead of } 3 \times x^2 \\ 3 \times 2x \text{ instead of } 3 \times 2 \times x. \end{gathered}$$

Example 19. Evaluate the expression $3x^2 - 4xy + xyz$ when $x = 3$, $y = -2$, $z = 1$.

ANSWER

We apply our knowledge of BIDMAS and fractions to algebraic expressions. In principle everything can appear straightforward until we start working on examples. The difficulty is that although an algebraic expression such as $x + y$, x^2 or $\sqrt{x^2 + y^2}$ stands for a number, we don't know what the number is, and therefore operations such as cancellation become more complicated. If we perform an operation on algebraic expressions, the result should be something that is always correct (i.e., remains correct whatever numbers the letters stand for.

3.2 How To Expand a Bracket

So far all our expressions inside brackets have been built up using numbers, so we could perform the calculations and end up with a single number. This approach will not work with brackets containing algebraic expressions in which there are letters as well as numbers because we do not know what the letters stand for. We need a different technique. We **expand the brackets**, i.e., replace the bracket expressions by equivalent non-bracketed ones.

An important principle is that we can replace a multiple of a bracket with multiples of the individual terms:

$$x(a+b+c+d+\ldots) = xa + xb + xc + xd + \ldots$$

and

$$(a+b+c+d+\ldots)x = ax + bx + cx + dx + \ldots.$$

Examples 20. Expand the following brackets and simplify as far as possible.

(a) $3(x+y)$

ANSWER

(b) $4(x-y)$

ANSWER

(c) $(2x-3y+4z)6-5(3x-5y-6z)$

ANSWER

(d) $3(x-2y)-4(y-4x)$

ANSWER

To expand the product of two brackets, multiply every term in the first bracket with every term in the second bracket, e.g.,

$(a+b)(c+d+e)$

$= ac + ad + ae$ ($\longleftarrow$ multiply terms in second bracket by a)

$+bc + bd + be.$ ($\longleftarrow$ multiply terms in second bracket by b)

Examples 21. Expand the following brackets and simplify as far as possible.

(a) $(x + y - 2z)(2x - 3y + z)$

ANSWER

(b) $(x - y - 1)(x + y - 1) + (x + 2)(x + 3)$

ANSWER

COMMON ERROR

Applying a minus sign to the first term of a bracket but not to the terms that follow.

WRONG! $$-4(x - 2y + 3z) = -4x - 8y + 12z.$$

RIGHT! $$-4(x - 2y + 3z) = -4x + 8y - 12z.$$

3.3 Factorisation

Sometimes we need to **create brackets** via **factorisation**. We shall see this particularly in the context of solving equations. In the following examples we spot factors that are common to each term. In essence we are seeing terms as if they arose from removing brackets, and we are reinserting the brackets.

Examples 22. Factorise the following:

(a) $2a + 6b$.

ANSWER

(b) $x^2 + xy$.

ANSWER

(c) $2x^3 - 7x^2y + x^4y^2 - x^2$
ANSWER

3.4 Algebraic Fractions

Algebraic fractions are fractions in which the numerator and denominator are algebraic expressions rather than simply numbers. The rules we have for combining fractions are exactly the same as with numbers. In particular, for addition and subtraction, we need to construct common denominators (usually by cross-multiplying). A complication is that it less clear what we mean by 'lowest terms' - we come back to this shortly.

Examples 23. Express each of the following as a single fraction (not necessarily in lowest terms).

(a) $\dfrac{x}{x-1} + \dfrac{x-2}{x+1}$.
ANSWER

(b) $\dfrac{x}{x-1} - \dfrac{x-2}{x+1}$.

ANSWER

(c) $\dfrac{x}{x-1} \times \dfrac{x-2}{x+1}$.

ANSWER

(d) $\dfrac{x}{x-1} \div \dfrac{x-2}{x+1}$.

ANSWER

3.5 Cancellation (Simplifying Fractions)

Recall again the important principle for numerical fractions:
If you multiply (or divide) the numerator and denominator by the same (non-zero) number then you don't change the value of the fraction.

The same principle applies to algebraic fractions:
If you multiply (or divide) the numerator and denominator by the same (non-zero) algebraic expression then you don't change the value of the fraction.

The concept of multiplying top and bottom by the same expression is easier than division. An example would be the following:

$$\frac{x+1}{x+2} = \frac{(x-1)(x+1)}{(x-1)(x+2)} = \frac{x^2-1}{x^2+x-2}.$$

(You should check that you get the same answers when multiplying out $(x-1)(x+1)$ and $(x-1)(x+2)$.)

The concept of dividing top and bottom by the same (non-zero) algebraic expression amounts to the following. Given a fraction of the form $\frac{AB}{AC}$, where A, B, C are algebraic expressions, we can divide top and bottom by A to give:

$$\boxed{\frac{AB}{AC} = \frac{B}{C}}.$$

We shall often refer to this process as **cancellation**. The expression $\frac{B}{C}$ is simpler than $\frac{AB}{AC}$, so we also call the process **simplification**.

Example 24. Simplify the following expression: $\frac{x^2+xy}{x+xy+xz}$.
ANSWER

Example 25. Show that $(x-2)(x+3) = x^2+x-6$ and $(x-2)(x-3) = x^2-5x+6$. Hence simplify the following expression: $\frac{x^2+x-6}{x^2-5x+6}$.
ANSWER

The following appears to be a different approach. We are going to simplify $\frac{2x^2+4y}{6+6x}$. We identify a number or algebraic expression that divides each term of both the numerator and the denominator. The only possibility is the number 2. We get

$$\frac{2x^2+4y}{6+6x} = \frac{1\ \cancel{2}x^2+2\ \cancel{4}y}{3\ \cancel{6}+3\ \cancel{6}x} = \frac{x^2+2y}{3+3x}.$$

We cannot simplify any further. In fact this is essentially the same as the first approach, except that we are not explicitly writing down factorisations. We could have written

$$\frac{2x^2+4y}{6+6x} = \frac{2(x^2+2y)}{2(3+3x)} = \frac{x^2+2y}{3+3x}.$$

It is generally safest to factorise top and bottom as $\dfrac{AB}{AC}$ and cancel to give $\dfrac{B}{C}$, but it is acceptable to divide top and bottom by a number or algebraic expression, **provided that we divide each term**.

Examples 26. Identify the error(s) in each of the following:

(a) $\dfrac{1+2}{2+6} = \dfrac{1+1\ \cancel{2}}{1\ \cancel{2}+3\ \cancel{2}} = \dfrac{2}{4} = \dfrac{1}{2}.$

ANSWER

(b) $\dfrac{2+4}{10+6} = \dfrac{\cancel{2}+2\ \cancel{4}}{5\ \cancel{10}+3\ \cancel{2}} = \dfrac{2}{8} = \dfrac{1}{4}.$

ANSWER

(c) $\dfrac{1+x+xy}{x^2+y^2} = \dfrac{1+\ \cancel{x}+\ \cancel{x}y}{x\ \cancel{x}^2+y^2} = \dfrac{1+y}{x+y^2} = \dfrac{1+1\ \cancel{y}}{x+y\ \cancel{y}^2} = \dfrac{2}{x+y}.$

ANSWER

3.6 Three Formulae You Must Memorise

(A) $(x+y)^2 = x^2 + 2xy + y^2$.
EXPLANATION

(B) $(x-y)^2 = x^2 - 2xy + y^2$.
EXPLANATION

(C) $(x+y)(x-y) = x^2 - y^2$.
EXPLANATION

Examples 27. Expand the following brackets using the formulas above:

(a) $(2x+3y)^2$
ANSWER

(b) $(3x - 4y)^2$

ANSWER

(c) $(2x + 3y)(2x - 3y)$

ANSWER

Example 28. Calculate 103×97 by using the third formula above.
ANSWER

3.7 Indices (Powers) and the Index Laws

Recall from Section 1 that

$$2^5 = 2 \times 2 \times 2 \times 2 \times 2 = 32$$

and

$$3^4 = 3 \times 3 \times 3 \times 3 = 81.$$

If n is a positive whole number then we write the product of n copies of x as

$$x.x.x.\ldots.x \text{ (n factors)}$$

or simply as

$$x^n.$$

We call x the **base** and we call n the **index** or **power** (or **exponent**). Note that x^1 is just x. Computation with powers is simplified by using the **INDEX LAWS**.

FIRST INDEX LAW

$$\boxed{x^m.x^n = x^{m+n}}$$

e.g., $2^2 \times 2^3 = 2^{2+3} = 2^5$ (i.e., $4 \times 8 = 32$).
EXPLANATION

SECOND INDEX LAW

$$\boxed{(x^m)^n = x^{mn}}$$

e.g., $(2^2)^3 = 2^{2\times 3} = 2^6 = 64$ (i.e., $4^3 = 64$).
EXPLANATION

THIRD INDEX LAW

$$\boxed{(xy)^n = x^n.y^n}$$

e.g., $6^3 = (2 \times 3)^3 = 2^3 \times 3^3 = 8 \times 27 = 216$.
EXPLANATION

Examples 29. (a) Using the Index Laws (as appropriate), express $5^3 \times 5^8$ as a single power of 5.

ANSWER

(b) Using the Index Laws (as appropriate), express $(3^4)^5$ as a single power of 3.

ANSWER

(c) Using the Index Laws (as appropriate), express $2^7 \times 5^7$ as a power of a single number.

ANSWER

We shall extend the idea of powers of x to allow for negative and fractional exponents, i.e., we will consider powers x^n where n is negative or fractional, or both. We shall be careful to do this so that BIDMAS and the Index Laws still apply in this wider context.

NEGATIVE AND ZERO POWERS
If $a \neq 0$ and n is a positive integer, then

$$\boxed{a^{-n} \textbf{ means } \frac{1}{a^n}}$$

e.g., $2^{-3} = \frac{1}{2^3} = \frac{1}{8}$ and $\frac{a}{b^n} = ab^{-n}$. Also, if $a \neq 0$ then

$$\boxed{a^0 \textbf{ means } 1}$$

Thus we have defined a^n for every whole number n. With these definitions, **the three Index Laws apply**.

A particular application is that

$$\frac{a^m}{a^n} = a^m \times a^{-n} = a^{m+(-n)} = a^{m-n}.$$

Examples 30. Express each of the following as a single power and evaluate that power.

(a) $\frac{3^8}{3^5}$
ANSWER

(b) $(2^{-2})^{-2}$
ANSWER

(c) $3^5 \div 3^{-5}$

ANSWER

FRACTIONAL POWERS

If n is a nonzero whole number, what meaning should we attach to $a^{1/n}$? In fact, the guiding principle is to define $a^{1/n}$ so that the Index Laws still apply.

For that, we must have

$$(a^{1/n})^n = a^{(1/n)\times n} = a^1 = a.$$

In other words $a^{1/n}$ is the nth root of a, often written $\sqrt[n]{a}$. For example, $9^{1/2} = \sqrt{9} = 3$ (because $3^2 = 9$) and $8^{1/3} = \sqrt[3]{8} = 2$ (because $2^3 = 8$).

If there is an apparent choice, then $a^{1/n}$ is positive. For example $3^2 = 9$ and $(-3)^2 = 9$, but $\sqrt{9} = 3$ (not -3).

Examples 31. Evaluate each of the following.

(a) $121^{1/2}$

ANSWER

(b) $(-125)^{1/3}$

ANSWER

(c) $32^{1/5}$

ANSWER

(d) $16^{-1/2}$

ANSWER

More generally, in accordance with the Index Laws, we have

$$\boxed{a^{m/n} = (a^{1/n})^m = (a^m)^{1/n} = (\sqrt[n]{a})^m = \sqrt[n]{(a^m)}}$$

Examples 32. Express each of the following as a single power of 2.

(a) $2^2 \times 4^{-3} \times \sqrt{8}$

ANSWER

(b) $\dfrac{2^{3/2} \times 16 \times (1/2)^2}{\sqrt[3]{4}}$

ANSWER

Examples 33. Express each of the following as a single power of a.

(a) $\dfrac{\sqrt{a^3}.a^{-5/3}}{a^{-2}}$

ANSWER

(b) $\sqrt[3]{\dfrac{a^5.(a^3)^2}{a^2}}$

ANSWER

Examples 34. (a) Simplify $\dfrac{18^3 \times \sqrt{24}}{12^4 \times 16^2}$ (i.e., write it in the form $2^\alpha 3^\beta$).

ANSWER

(b) Simplify $\dfrac{(xy)^5}{(x^2)^2 y}$ (i.e., write it in the form $x^\alpha y^\beta$).

ANSWER

3.8 BIDMAS Revisited

Brackets
Indices
Division
Multiplication
Addition
Subtraction

- If an expression involves just $+$ and $-$, process from left to right.

  e.g., $3+4-5-6+7+9=12$.

- If an expression involves just $\times$ and $\div$, process from left to right.

  e.g., $3 \div 4 \times 5 \times 12 = 45$.

- If an expression involves $+$, $-$, $\times$ and $\div$, first process $\times$ and $\div$, then process $+$ and $-$.

  e.g., $3 \times 4 - 7 \times 12 \div 14 + 5 \div 2 \times 6 = 12 - 6 + 15 = 21$.

- If an expression involves $+$, $-$, $\times$, $\div$ **and powers**, process the powers first, then $\times$ and $\div$, and finally $+$ and $-$.

  e.g., $4 \times 3^2 = 4 \times 9 = 36$ (and not 12^2).

  e.g., $3x^2$ means "3 times the square of x", and not the square of $3x$, which is $(3x).(3x) = 9x^2$.

- If an expression contains brackets, simplify each bracket before combining it with other terms.

  e.g., $6 \times (2+3\times 5)^2 - (5 \times 3 - 3) \div 6$
  $= 6 \times (2+15)^2 - (15-3) \div 6$
  $= 6 \times 17^2 - 12 \div 6$
  $= 6 \times 289 - 2 = 1734 - 2$
  $= 1732$.

4 Equations

4.1 One Linear Equation in One Unknown

An **equation** in the 'unknown' x takes the form

$$E_1 = E_2$$

where E_1 and E_2 are algebraic expressions and one or both involves x. We sometimes call x a 'variable'. We could use a different letter. We could have more than one variable (so more than one letter).

Examples are:

- $x + 1 = 3$.
- $3x + 2 = 5x - 2y$.
- $x^2 + x = 6$.
- $xy + 3x + y + 1 = fx + gh$.

In Chapter 3 we looked at algebraic expressions where the letters stood for numbers, generally any numbers or combination of numbers. Now we ask what numbers or combination of numbers give the same value for the two expressions in the equation.

For example $x + 1$ and 3 have the same value if $x = 2$ (and in fact not for other values of x).

If we start from $3x + 2 = 5x - 2y$ we would find that $x = y + 1$: this means that the combinations of values of x and y for which the equation holds are those arrived at by choosing any value for y and then choosing x as $y + 1$ (for example, $y = 0, x = 1$ or $y = 2, x = 3$); we could alternatively arrive at $y = x - 1$, meaning that the combinations of values of x and y for which the equation holds are those arrived at by choosing any value for x and then choosing y as $x - 1$ (for example, $x = 1, y = 0$ or $x = 3, y = 2$).

For $x^2 + x = 6$, we might observe that $x = 2$ works, but actually so does $x = -3$: are they the only possibilities?

In the fourth example we might aim to write x in terms of y, f, g and h without knowing the values taken by these letters. In other words, if we knew the values of y, f, g and h, do we know what x would have to be?

To **solve** an equation in x means to determine the values of x for which the equation is true. These values could be algebraic expressions. In many cases, we can achieve this by "making x the subject", that is, by rearranging the equation so that it takes the form $x = E$, where E is a number or an algebraic expression not involving x. For example, from the equation $xy + 3x + y + 1 = fx + gh$ we would obtain $x = \dfrac{gh - y - 1}{y + 3 - f}$.

A **linear equation in one unknown** x is of the form

$$ax + b = cx + d$$

where a, b, c, d are fixed numbers or expressions not involving x.

To solve a linear equation (or indeed any other sort of equation), we may do the following:

(I) Add the same value to each side.

(II) Subtract the same value from each side.

(III) Multiply each side by the same (non-zero) value.

(IV) Divide each side by the same (non-zero) value.

In performing these operations we obtain an equivalent linear equation, one which has the same solution.

From $ax + b = cx + d$:

Subtract cx from each side: $ax + b - cx = cx + d - cx$, i.e., $ax + b - cx = d$. We have moved the term involving x on the right to the left (and changed sign in the process). All the terms involving x are now on the left.

Subtract b from each side: $ax + b - cx - b = d - b$, i.e., $ax - cx = d - b$. We have moved the term not involving x on the left to the right (and changed sign in the process). All the terms not involving x are now on the right.

On the left, take out x as a factor: $x(a - c) = d - b$.

Divide both sides by $a - c$ (assuming it is $\neq 0$). We end up with

$$x = \frac{d - b}{a - c}.$$

Example 35. Solve the equation $5x + 7 = 3 - x$ for x.
ANSWER

Example 36. Solve the equation $17 - 2T = 3T + 2$ for T.
ANSWER

Example 37. Solve the equation $3x + 2 = 5x - 2y$ for x.
ANSWER

Example 38. Solve the equation $7\theta + r - 2h = 4\theta - 3r + 2h + rh$ for θ.
ANSWER

Examples 39. Solve the following equations for x.

(a) $4(x-3) = 3(x-2)$
ANSWER

(b) $(2x+1)(x-2) = (x+3)(2x-3)$
ANSWER

(c) $\dfrac{2x+4}{4} = \dfrac{x-2}{3}$

ANSWER

(d) $\dfrac{2}{x+2} = \dfrac{1}{x-17}$

ANSWER

(e) $\dfrac{3x+1}{x+2} = \dfrac{6x-5}{2x+1}$
ANSWER

4.2 Two Simultaneous Linear Equations in Two Unknowns

We now consider simultaneous equations in two unknowns (or we might say 'two variables'). An example would be

$$\begin{array}{ccccl} 3x & + & 4y & = & 11 \quad (1) \\ 2x & + & 3y & = & 8 \quad (2) \end{array}.$$

We find the values of x and y that satisfy both equations at the same time. The numbers (1) and (2) are just labels. Such systems can be solved by **elimination**: eliminate one of the unknowns to get a single equation in the other unknown — and we saw in the last section how to solve that.
The variables are not always necessarily x, y. They could just as easily be a, b or r, h or P, Q or x_1, x_2.

ELIMINATION METHOD (Eliminating y)
We shall explain this method by reference to the example above.

$$\begin{array}{ccccl} 3x & + & 4y & = & 11 \quad (1) \\ 2x & + & 3y & = & 8 \quad (2) \end{array}.$$

Step One: Multiply (1) and (2) by suitable numbers so that the coefficients of y have the same size (though possibly of opposite sign).
Here we multiply (1) by 3 and (2) by 4:

$$\begin{array}{lllll} 9x & + & 12y & = & 33 \quad (3) = (1) \times 3 \\ 8x & + & 12y & = & 32 \quad (4) = (2) \times 4 \end{array}.$$

Notice that each equation has three terms (two on the left and one on the right). We have multiplied each term by 3 in equation (1) and each term by 4 in equation (2).
Notice also that we have new numbers for the new equations.
The principle is that equations (3) and (4) are equivalent to (1) and (2) in that there are exactly the same solutions for x, y.

Step Two: If coefficients of y have the same sign, subtract (3) from (4); if they have opposite sign, add (3) to (4). Here we subtract to get:

$$\begin{array}{llllll} 9x & + & 12y & = & 33 & (3) \\ -x & & & = & -1 & (5) = (4) - (3) \end{array}.$$

Notice that we have subtracted corresponding terms.
Again the principle is that equations (3) and (5) have the same solutions as (1) and (2).

Step Three: Solve (5) for x. Here $-x = -1$ so $x = 1$.
Step Four: Substitute this value for x in (3), and solve the resulting equation for y:
$9 \times 1 + 12y = 33$
whence $12y = 33 - 9 = 24$, so $y = 2$.
Solution: $x = 1, y = 2$.
Check! Substitute these values into the original equations (1) and (2) to check that they work.

ELIMINATION METHOD (Eliminating x)
We shall look at the same example, but instead eliminate x. We should get the same answer!

$$\begin{array}{llllll} 3x & + & 4y & = & 11 & (1) \\ 2x & + & 3y & = & 8 & (2) \end{array}.$$

Step One: Multiply (1) and (2) by suitable numbers so that the coefficients of x have the same size (though possibly of opposite sign).
Here we multiply (1) by 2 and (2) by 3:

$$\begin{array}{lllll} 6x & + & 8y & = & 22 \quad (3) = (1) \times 2 \\ 6x & + & 9y & = & 24 \quad (4) = (2) \times 3 \end{array}.$$

Step Two:

$$\begin{array}{rcrcll} 6x & + & 8y & = & 22 & (3) \\ & & y & = & 2 & (5) = (4) - (3) \end{array}.$$

Step Three: Solve (5) for y. Here $y = 2$.
Step Four: Substitute this value for y in (3), and solve the resulting equation for x:
$6x + 8 \times 2 = 22$
whence $6x = 22 - 16 = 6$, so $x = 1$.
Solution: $x = 1, y = 2$.
Check!

Examples 40. Solve the following simultaneous equations:

(a)

$$\begin{array}{rcrcrl} 4x & + & 5y & = & 8 & (1) \\ 3x & - & 2y & = & 29 & (2) \end{array}.$$

ANSWER

(b)

$$\begin{array}{rcrcl} 2x & + & 3y & = & 17 \\ -x & + & 5y & = & 11 \end{array}.$$

ANSWER

(c)

$$\begin{array}{rcrcr} s & + & 4t & = & 7 \\ 5s & - & 2t & = & 13 \end{array}.$$

ANSWER

4.3 The Two Degenerate Cases

We have seen how to solve one linear equation in one unknown. To find the values of two unknowns we need two equations. Normally, there will be a unique solution. But there are two degenerate cases, where there are two equations but there is not a unique solution.

First Degenerate Case
Consider the equations:

$$\begin{array}{cccccl} 2x & + & 3y & = & 6 & (1) \\ 4x & + & 6y & = & 12 & (2) \end{array}.$$

Step 1:

$$\begin{array}{cccccl} 4x & + & 6y & = & 12 & (3) = (1) \times 2 \\ 4x & + & 6y & = & 12 & (2) \end{array}.$$

We have the same equation twice. This system has **infinitely many solutions**: for each number c, if we take $x = c$, then we have $4c + 6y = 12$, so $6y = 12 - 4c$ and therefore $y = \dfrac{12 - 4c}{6} = \dfrac{6 - 2c}{3}$. In other words we have a 'general solution':

$$x = c, y = (6 - 2c)/3.$$

Second Degenerate Case
Consider the equations:

$$\begin{array}{cccccl} 2x & + & 3y & = & 6 & (1) \\ 2x & + & 3y & = & 5 & (2) \end{array}.$$

We don't need Step 1. Step 2:

$$\begin{array}{cccccl} 2x & + & 3y & = & 6 & (1) \\ & & 0 & = & -1 & (3) = (2) - (1) \end{array}.$$

The outcome $0 = -1$ is not possible. This system has **no solutions**. Looking at the original equations we see that any solutions x and y would have to make the left hand side add up to 5 and also to 6, which is impossible.

5 Coordinate Geometry

5.1 The Coordinate Plane

The **coordinate plane** is a flat surface, like a tabletop, but extending to infinity in all directions. On it are drawn two straight lines, called the x-axis and the y-axis, which meet at right angles as shown in a point O, called the **origin**.

PICTURE:

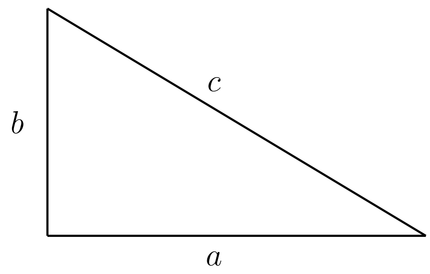
The position of any point P in the coordinate plane can be completely specified by 2 numbers:

$x =$ the distance of P in the direction of the x-axis from the origin
$y =$ the distance of P in the direction of the y-axis from the origin,

where x is taken to be positive to the right of the y-axis and negative to the left, with y taken to be positive above the x-axis and negative below the x-axis. Some examples are plotted on the picture above. The point P is denoted by (x, y), and the plane is also called the xy-plane. These coordinates are sometimes called Cartesian coordinates.

5.2 The Distance Formula

We are going to calculate the distance between two points. Recall Pythagoras' Theorem: In a right-angled triangle the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides.



$$a^2 + b^2 = c^2$$

We want a formula for the distance d between two points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ in the plane. We plot the two points and then construct a third point so that we have a right-angled triangle.

PICTURE

Denoting the distance between P and Q by d (the length of the hypotenuse), we calculate the other two side lengths as $x_2 - x_1$ and $y_2 - y_1$. Then, by Pythagoras' Theorem,

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

and so (taking square roots on both sides)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Note that we do not have to remember the distance formula – if necessary we can reconstruct it using Pythagoras' Theorem.

Examples 41. Find the distances between the following pairs of points.

(a) $(4, 2)$ and $(7, 6)$.

ANSWER

(b) $(-1, -3)$ and $(4, 9)$.

ANSWER

5.3 The Equation of a Straight Line

An equation of the form

$$ax + by = c, \qquad (*)$$

where a, b, c are numbers, is **the equation of a straight line**, in the sense that if we take all the pairs of numbers x, y satisfying the equation (*), then the points (x, y) lie on a straight line.

Consider the straight line $4x + 3y = 11$. We can say that the points $(2, 1)$, $(5, -3)$ and $(-7, 13)$ all lie on the line, since
$4 \times 2 + 3 \times 1 = 8 + 3 = 11$,
$4 \times 5 + 3 \times (-3) = 20 - 9 = 11$, and
$4 \times (-7) + 3 \times 13 = -28 + 39 = 11$;
but $(-6, 2)$ does not, since $4 \times (-6) + 3 \times 2 = -24 + 6 = -18 \neq 11$.

5.4 The Form $y = mx + c$

It is often convenient to recast (*) in the form

$$y = mx + c$$

i.e., to rearrange (*) so that y is the subject of the equation. (**N.B.**: We can't always do this! See later.) The number c here is not generally the same as the number c in (*).

Examples 42. Rearrange each of the following into the form $y = mx + c$.

(a) $3x + y = 5$

ANSWER

(b) $7y - 3x = 5$

ANSWER

(c) $ax + by = c$

ANSWER

The form $y = mx + c$ is unique: you can't write the equation in this form in two different ways. This means that the numbers m and c tell you something about the line. We call m the **gradient** or **slope** of the line $y = mx + c$.

(i) If $m > 0$ then the line slopes upward.

(ii) If $m = 0$ then the line is level.

(iii) If $m < 0$ then the line slopes downward.

We call c the **intercept**. It tells you where the line crosses the y-axis, i.e., at the point $(0, c)$. (So the gradient of the line $ax + by = c$ is $-a/b$ and it crosses the y-axis at the point $(0, c/b)$.)

Examples 43. (a) Find the gradient of the line $2y + 7x = 3$ and the point where it crosses the y-axis.

ANSWER

(b) Find the equation of the line with gradient 4 which passes through the point $(1, -2)$.

ANSWER

(c) Find the equation of the line with gradient -2 which passes through the point $(-17, 23)$.

ANSWER

(d) Find the equation of the line with gradient 0 which passes through the point $(4, 5)$.

ANSWER

5.5 Vertical Lines

Some lines can't be put in the form $y = mx + c$. These are the **vertical lines**.

Example 44. Draw the straight line through the points $(2, -1)$ and $(2, 3)$.
ANSWER

We see that all the points lie on a vertical line, and they all have the same x-coordinate, 2. Thus each point on the line satisfies the equation $x = 2$. In general a vertical line has equation of the form $x = k$ (where k is a number). We don't define a gradient for such lines, i.e., they don't have a gradient.

5.6 The Gradient of the Line Through $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

Case 1: $x_1 \neq x_2$ **so** $x_1 - x_2 \neq 0$
Let PQ have equation $y = mx + c$. Then, since both P and Q lie on the line, we have

$$y_1 = mx_1 + c \quad (1)$$

$$y_2 = mx_2 + c \ \ (2).$$

Therefore $(2) - (1)$ gives us $y_2 - y_1 = mx_2 - mx_1 = m(x_2 - x_1)$. Thus

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\text{difference in } y}{\text{difference in } x}$$

always provided the differences are calculated in the same order. (Note that we can only divide by $x_2 - x_1$ because $x_2 - x_1 \neq 0$.)

Examples 45. Find the gradients of the lines passing through the following pairs of points.

(a) $(2, -3)$ and $(1, 4)$

ANSWER

(b) $(4, 3)$ and $(1, 2)$

ANSWER

(c) $(2, 5)$ and $(-1, 5)$

ANSWER

(d) $(-3, -8)$ and $(-1, -2)$.

ANSWER

Case 2**:** $x_1 = x_2$
The line is vertical and the gradient is undefined.

5.7 The Equation of the Line Through $P = (x_1, y_1)$ and $Q = (x_2, y_2)$

Case 1**:** $x_1 \neq x_2$**.**

- Calculate $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.
- We shall find the equation of the line in the form $y = mx + c$. We have found m, now we have to find c.
- We can use either of the points $P = (x_1, y_1)$ or $Q = (x_2, y_2)$ to find c.

Examples 46. Obtain the equations of the lines passing through the following pairs of points.

(a) $(1, 5)$ and $(-2, -1)$

ANSWER

(b) $(2, -1)$ and $(6, 1)$

ANSWER

Case 2**:** $x_1 = x_2$
Then the line is vertical and has equation $x = x_1$ (or $x = x_2$, which is the same thing).

Example 47. Obtain the equation of the line passing through the pair of points: $(4, 3)$ and $(4, -1)$.
ANSWER

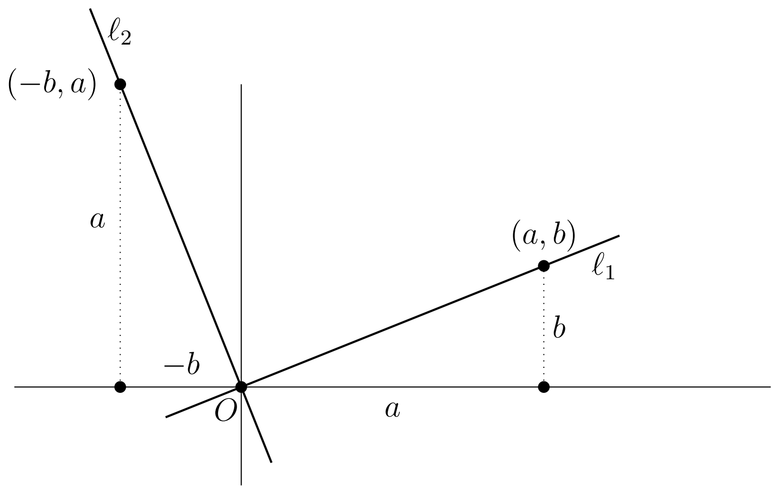
5.8 The Equation of the Line Through a Given Point and Parallel to a Given Line

Two lines are said to be **parallel** if they have the same gradient or if both are vertical. Alternatively, we could say that two lines are parallel if either they never meet or if they are the same line.

Example 48. Find the equation of the line that is parallel to $2x + 3y = 7$ and passes through $(1, 4)$.
ANSWER

5.9 When Are Two Lines Perpendicular?

Two lines are perpendicular if they cross at right angles. Consider the perpendicular lines ℓ_1 and ℓ_2 below.



The gradient of ℓ_1 is $\dfrac{b}{a}$ and the gradient of ℓ_2 is $\dfrac{a}{-b} = \dfrac{-a}{b}$. Notice that when we multiply these together we get $\dfrac{b}{a} \times \dfrac{-a}{b} = -1$.

In general, if we are given a line ℓ of gradient $m \neq 0$, then any line perpendicular to ℓ will have gradient $\dfrac{-1}{m}$. If ℓ has gradient 0, then any perpendicular line will be vertical. If ℓ is a vertical line, then any perpendicular line will be horizontal.

5.10 The Equation of the Line Through a Given Point and Perpendicular to a Given Line

Example 49. Find the equation of the line that is perpendicular to $y = 4x + 3$ and passes through $(-1, 3)$.
ANSWER

5.11 The Point of Intersection of Two Lines

Suppose we are given the equations of two lines: $3x + 4y = 5$ and $2x - 3y = 9$, and we wish to find the point at which they meet. In other words we want to find the values of x and y that satisfy both equations at the same time. Then we need to solve a pair of simultaneous equations:

$$\begin{array}{rcrcrl} 3x & + & 4y & = & 5 & (1) \\ 2x & - & 3y & = & 9 & (2) \end{array}.$$

The solution is as follows:
ANSWER

5.12 The Degenerate Cases Explained

When we considered pairs of simultaneous equations, we found that there were two degenerate cases. The degenerate cases can now be explained geometrically.

(I) **INFINITELY MANY SOLUTIONS** For example,
$x + y = 1$
$2x + 2y = 2.$

The "two" lines are really the same line. Therefore EVERY point on this line is a "point of intersection" (i.e., lies on both lines). So there are infinitely many points of intersection.

(II) **NO SOLUTIONS** For example,
$x + y = 1$
$x + y = 2.$

The two lines are distinct and parallel, so they never meet. There is no point of intersection.

5.13 Distance of a Point from a Line

Suppose that we wish to find the (shortest) distance of a point from a line. We consider an example.

Example 50. Find the distance of the point $(1, 1)$ from the line $5y + 2x = 10$.
ANSWER (Part 1) Sketch the line $5y + 2x = 10$*, the point* $(1, 1)$ *and the line through* $(1, 1)$ *that is perpendicular to* $5y + 2x = 10$.

ANSWER (Part 2) Calculate the equation of the perpendicular line constructed above and its point of intersection P *with* $5y + 2x = 10$*. Then calculate the distance from* $(1, 1)$ *to* P*.*

6 Quadratic Equations

6.1 Introduction

Quadratic equations in x are equations of the form

$$ax^2 + bx + c = 0$$

where $a \neq 0$. (If $a = 0$, the equation is linear.) The word "quadratic" indicates that the square is the highest power of the variable that occurs in the equation.

We shall see that there are several possible approaches to solving quadratic equations. The choice of approach depends to some extent on the nature of the equation, and to some extent on personal preference.

6.2 Completing the Square (1)

We start with a technique known as "Completing the Square". Later we shall see a formula – it is obtained by completing the square.

Completing the Square is a technique for converting an expression of the form

$$x^2 + bx + c,$$

where b and c are constants, to one of the form

$$(x + B)^2 + C,$$

where B and C are constants.

First recall the formula: $(x + y)^2 = x^2 + 2xy + y^2$. If we apply this to $(x + B)^2$ we get $x^2 + 2Bx + B^2$. Similarly $(x - B)^2 = x^2 - 2Bx + B^2$.

An example
Suppose we are given $x^2 + 4x$ and we wish to write it in the form $(x + B)^2 + C$.

- We might remember that $(x + 2)^2 = x^2 + 4x + 4$.
- That means that $x^2 + 4x = (x + 2)^2 - 4$.
- In other words $B = 2$ and $C = -4$.

Another look at the example
Suppose we are given $x^2 + 4x$ and we wish to write it in the form $(x + B)^2 + C$.

- Calculate $(x + B)^2 = x^2 + 2Bx + B^2$. Compare this to $x^2 + 4x$: both start with x^2 and we compare $2Bx$ with $4x$. This tells us $2B = 4$, i.e., $B = 2$.
- As before we have $(x + 2)^2 = x^2 + 4x + 4$, so $x^2 + 4x = (x + 2)^2 - 4$.

THE TECHNIQUE PART 1: $x^2 + bx$

(I) Supose we are given x^2+bx and we wish to write it in the form $(x+B)^2+C$.

(II) Calculate $(x + B)^2 = x^2 + 2Bx + B^2$. Compare this to $x^2 + bx$: both start with x^2 and we compare $2Bx$ with bx. This tells us $2B = b$, i.e., $B = \frac{b}{2}$.

(III) Now $\left(x + \frac{b}{2}\right)^2 = x^2 + 2 \times \frac{b}{2}x + \left(\frac{b}{2}\right)^2 = x^2 + bx + \left(\frac{b}{2}\right)^2$, so $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

Examples 51. Complete the square for each of the following.

(a) $x^2 + 6x$

ANSWER

(b) $x^2 - 8x$

ANSWER

(c) $x^2 - 13x$

ANSWER

THE TECHNIQUE PART 2: $x^2 + bx + c$
Complete the square for $x^2 + bx$ and add the number c to the result.

Examples 52. Complete the square for each of the following.

(a) $x^2 + 4x + 6$

ANSWER

(b) $x^2 - 6x + 8$

ANSWER

(c) $x^2 - 4x - 1$

ANSWER

(d) $x^2 + 3x + 2$

ANSWER

(e) $x^2 - 7x + 11$

ANSWER

6.3 Completing the Square (2)

We often need to complete the square with expressions of the more general form $ax^2 + bx + c$, i.e., convert to the form $a(x + B)^2 + C$. We take out factor a and then proceed as before. The process is best seen by means of examples.

Examples 53. Complete the square for each of the following.

(a) $2x^2 + 3x + 4$

ANSWER

(b) $3x^2 + 10x - 5$

ANSWER

Completing the square is a technique which has many applications. Here is just one. Suppose we need to find the smallest value that $3x^2 + 10x - 5$ can take, as x ranges across all possible values.

By completing the square, we can rewrite it as

$$3\left(x + \frac{5}{3}\right)^2 - \frac{40}{3}.$$

The square part is always ≥ 0, so the smallest possible value is $-\frac{40}{3}$, and this value is attained when $x = -\frac{5}{3}$.

6.4 The Quadratic Equation Formula

We shall now see how to obtain a formula for the roots of the quadratic equation $ax^2 + bx + c = 0$. We shall consider the relatively easy case where $a = 1$, i.e., the equation $x^2 + bx + c = 0$. If we complete the square for $x^2 + bx + c$, our equation becomes

$$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0.$$

Starting from

$$\left(x + \frac{b}{2}\right)^2 - \frac{b^2}{4} + c = 0$$

we rearrange this equation to get

$$\left(x + \frac{b}{2}\right)^2 = \left(\frac{b^2}{4} - c\right) = \frac{b^2 - 4c}{4}.$$

Taking square roots we get

$$x + \frac{b}{2} = \pm\sqrt{\frac{b^2 - 4c}{4}} = \pm\frac{\sqrt{b^2 - 4c}}{\sqrt{4}} = \pm\frac{\sqrt{b^2 - 4c}}{2}.$$

Hence

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

For the general quadratic equation $ax^2 + bx + c = 0$, the procedure is similar but more complicated. It ends up with the following formula. It is simplest to MEMORISE THE FORMULA!

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

"Minus b plus or minus the square root of $b^2 - 4ac$

ALL OVER $2a$"

Note the ALL OVER.

Examples 54. Solve the following quadratic equations using the formula.

(a) $6x^2 + x - 1 = 0$

ANSWER

(b) $x^2 - 4x - 21 = 0$

ANSWER

6.5 How Many Roots Does a Quadratic Equation Have?

The quadratic equations above have two roots (i.e., two solutions), since you can take either the positive or the negative of the square root: $\pm\sqrt{b^2 - 4ac}$. Let us consider two other examples.

Examples 55. Solve the following quadratic equations using the formula.

(a) $3x^2 + 6x + 3 = 0$

ANSWER

(b) $x^2 - x + 2 = 0$

ANSWER

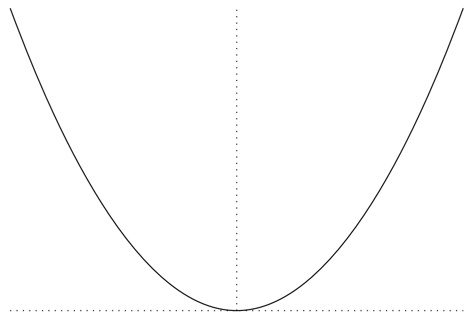
We see that sometimes there are two solutions, but there are two other possibilities: some equations have just one solution and others have no solution at all. It all depends on the number inside the square root.

The number $b^2 - 4ac$ is called the **discriminant** of the quadratic and is often denoted by Δ. Thus:

- If $b^2 - 4ac > 0$, there are two solutions to the quadratic equation.
- If $b^2 - 4ac = 0$, there is one solution to the quadratic equation (sometimes thought of as a repeated solution).
- If $b^2 - 4ac < 0$, there are no (real) solutions to the quadratic equation.

6.6 Geometrical Interpretation

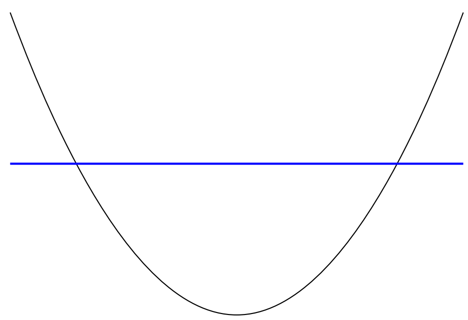
Given a quadratic expression ax^2+bx+c, the graph $y = ax^2+bx+c$ is obtained by plotting points: for each value of x, we calculate the value of $y = ax^2+bx+c$ and plot the point (x, y). For example, if $y = x^2$, we plot the points $(0, 0)$, $(1, 1)$, $(2, 4)$, $(-1, 1)$, $(-2, 4)$ and so on and we get



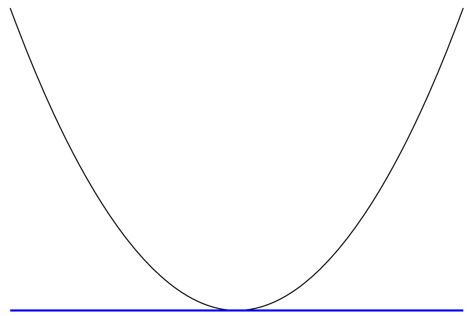
In general, if $a > 0$ we get a graph with the same general shape.

The line $y = 0$ is the x-axis. The points where $y = ax^2 + bx + c$ and $y = 0$ at the same time correspond to the values of x where $ax^2 + bx + c = 0$. Geometrically they are the points that lie on the parabola $y = ax^2 + bx + c$ and on the line $y = 0$, i.e., where the parabola and the line meet.

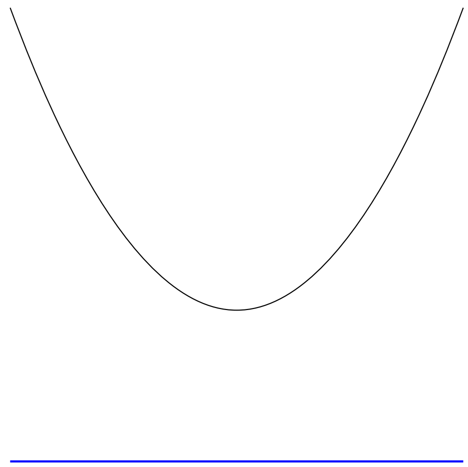
The following pictures illustrate ways in which the parabola and the line might occur (we assume that $a > 0$).



In the graph above there are two points of intersection, corresponding to two solutions to the quadratic equation $ax^2 + bx + c = 0$.



In the graph above there is one point of intersection, corresponding to one solution to the quadratic equation $ax^2 + bx + c = 0$.



In the graph above there is no intersection, corresponding to no solutions to the quadratic equation $ax^2 + bx + c = 0$.

6.7 Solution by Factorisation

Sometimes we can solve a quadratic equation more quickly by factorisation. Let us start with the quadratic equation

$$x^2 + bx + c = 0,$$

where b and c are numbers. We suppose that $x^2 + bx + c$ can be written as $(x + m)(x + n)$, where m and n are numbers that we do not yet know the values of.

If this is the case, then $x^2 + bx + c$ and $(x + m)(x + n)$ are the same expressions. Therefore if we multiply out $(x + m)(x + n) = x^2 + (m + n)x + mn$ we should still have $x^2 + bx + c$. Thus $m + n = b$ and $mn = c$. We look for numbers m, n that have these properties – we use intelligent trial and error.

Examples 56. Solve the following quadratic equations using factorisation.

(a) $x^2 + 8x + 15 = 0$

ANSWER

(b) $x^2 + 2x - 15 = 0$

ANSWER

If we are given a more general quadratic equation equation $ax^2 + bx + c = 0$, then we can divide both sides of the equation by a – remember that we have to divide each term on the left by a. However this is only worth doing if it results in an easy factorisation.
For example, if we are given the equation: $3x^2 + 6x - 45 = 0$, we could divide through by 3 to get $x^2 + 2x - 15 = 0$ (see above for the solution).

7 Circles

7.1 Equation of a Circle

Given a point (a, b) and a positive number r, a circle with centre (a, b) and radius r is the set of points (x, y) at distance r from (a, b).

To say that (x, y) is at distance r from (a, b) is equivalent to $\sqrt{(x-a)^2 + (y-b)^2} = r$, i.e.,

$$(x-a)^2 + (y-b)^2 = r^2.$$

This is one form of the **equation of a circle**.

If we expand the brackets, we get
$x^2 - 2ax + a^2 + y^2 - 2by + b^2 = r^2$.
i.e., $x^2 + y^2 - 2ax - 2by + (a^2 + b^2 - r^2) = 0$.
This is of the general form $\boxed{x^2 + y^2 + 2gx + 2fy + c = 0}$ which is the **standard form of the equation of a circle**.

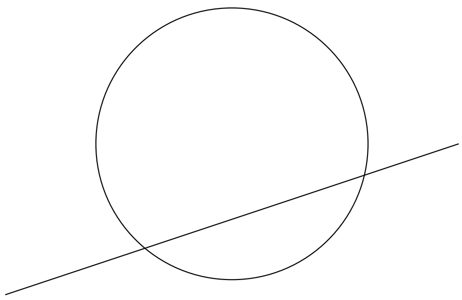
Example 57. Derive the equation of the circle with centre $(2, 3)$ and radius 5, expressing your answer in standard form.
ANSWER

Example 58. Find the centre and radius of the circle: $x^2 + y^2 - 2x - 4y - 4 = 0$.
ANSWER

We might ask: does an equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always respresent a circle?
The answer is: no, not always. For example, there are no points satisfying the equation $x^2 + y^2 + 1 = 0$ (because $x^2 + y^2 + 1 \geq 1$). There are other examples, but you just need to be aware that there is not always a circle.

7.2 How to Find the Points Where a Line Meets a Circle

We wish to find the points of intersection of the straight line $y = mx + d$ and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.



Thus we seek the points (x, y) which satisfy both of the equations

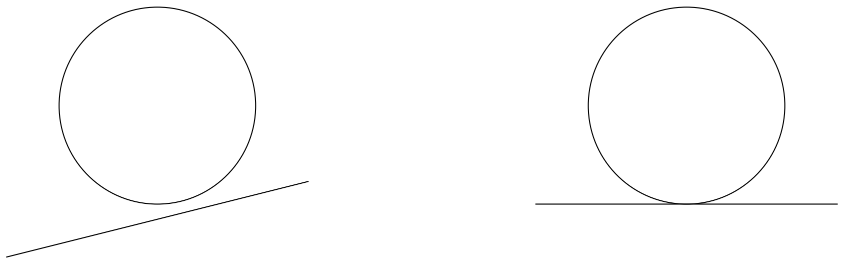
$$y = mx + d \text{ and } x^2 + y^2 + 2gx + 2fy + c = 0.$$

If we substitute $mx + d$ for y in the circle equation, we get a quadratic equation in x:

$$x^2 + (mx + d)^2 + 2gx + 2f(mx + d) + c = 0.$$

The roots of this quadratic equation are the x-coordinates of the points of intersection of the straight line and the circle.
We can then use the equation $y = mx + d$ to get the corresponding y-coordinates.

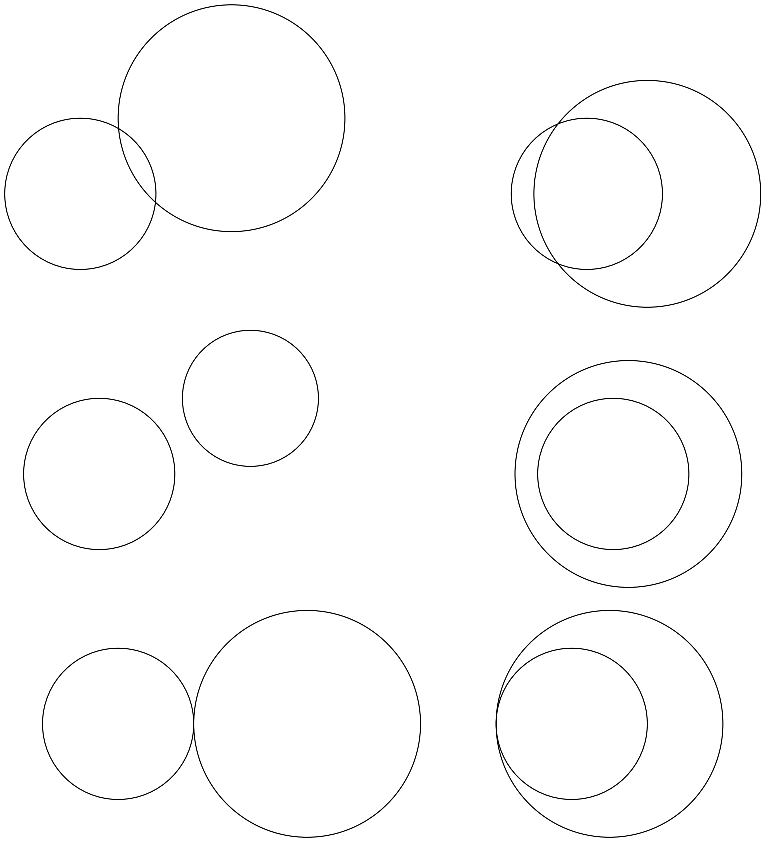
N.B. If the quadratic equation has NO SOLUTIONS, then this means that the straight line never meets the circle. If it has A SINGLE SOLUTION then the line is tangent to the circle.



Example 59. Find the points of intersection of the straight line $y = x + 3$ and the circle $x^2 + y^2 - 2x - 4y + 1 = 0$.
ANSWER

7.3 How To Find The Intersection Points of Two Circles

There are a number of ways in which two circles might or might not meet, as shown in the following pictures:



What we observe is that two circles meet in 2 points, in no points, or in 1 point.

We shall see how to find the points of intersection of two circles by means of an example.

Example 60. Find the points of intersection of the circles
$x^2 + y^2 + x - 3y - 10 = 0$
$2x^2 + 2y^2 - x - 2y - 15 = 0$
ANSWER

8 Trigonometry

8.1 What is Trigonometry?

Trigonometry is the study of triangles, the lengths of their sides and the sizes of their angles.

It builds on the fact that if two triangles have the same shape then (i) corresponding angles are equal, (ii) corresponding sides are in proportion.

In fact, the word "trigonometry" comes from the Greek: "tri"=three, "gon"=side, "metry"=measurement.

Trigonometry has its origins in ancient Babylon (modern Iraq). The Babylonians were the first to measure angles in degrees. They probably divided the circle into $360°$ because the annual progress of the Sun in its (apparent) orbit round the earth seemed to them to be divided into 360 days.

The ancient Egyptians were interested in triangles because they wanted to measure land for tax purposes.

Trigonometry as we know it began with the ancient Greeks, who used it in their astronomical and geographical studies (nothing in this chapter would be news to an ancient Greek, though the notation would certainly be unfamiliar).

Trigonometry is now an essential tool for physicists, chemists and surveyors. In 1852, it was used to calculate the height of Mount Everest to within 30 feet, although the mountain was not climbed until 1953. In fact the measurements used in the calculation were taken at a distance of 150 miles. And when human beings land on Mars, you can be sure that trigonometrical calculations will be involved.

We shall concentrate on one of the basic techniques of trigonometry: **solving a triangle**, that is, working out the lengths of its three sides and the sizes of its three angles.

8.2 Degrees or Radians?

Traditionally, angles were measured in degrees:
$\boxed{360° \longleftrightarrow \text{circle}}$ $\boxed{180° \longleftrightarrow \text{half-circle}}$ $\boxed{90° \longleftrightarrow \text{quarter-circle}}$

But measurement in degrees is ultimately based on the fact that the Earth takes approximately 360 days to orbit the Sun, so it would appear completely unnatural to anyone who was not an Earth dweller, for instance a Martian. We shall henceforth work with **radians**, a universal measure which does not depend on local conditions in the Solar System, but merely on the ratio of the circumference of a circle to its diameter.

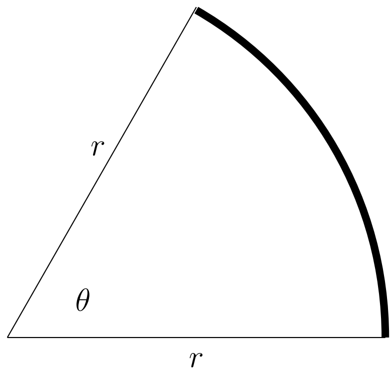
The rule is

$$2\pi \text{ radians } = 1 \textbf{ circle} = 360°,$$

so 1 radian is about $57.3°$. A circle of radius 1 has circumference of length 2π units.

A circle of radius r has circumference of length $2\pi r$ units. Therefore a sector with angle θ radians has an arc length that is $\theta/2\pi$ of the circumference of the circle, so has arc length

$$\frac{\theta}{2\pi} \times 2\pi r = \theta r.$$



CONVERSION RULES

$$\boxed{\theta° \longrightarrow \theta \times \frac{\pi}{180} \text{ radians.}}$$

$$\boxed{\theta \text{ radians } \longrightarrow \theta \times \frac{180°}{\pi}.}$$

N.B. It is common to express radians as multiples of π.
$360° = 2\pi$ radians $180° = \pi$ radians (so the angles in a triangle add up to π)
$90° = \pi/2$ radians $60° = \pi/3$ radians $45° = \pi/4$ radians $30° = \pi/6$ radians

Examples 61. Convert each of the following to radians:

(a) 300°

ANSWER

(b) 160°

ANSWER

(c) -15°

ANSWER

Examples 62. Convert each of the following from radians to degrees:

(a) $2\pi/3$

ANSWER

(b) $-3\pi/8$

ANSWER

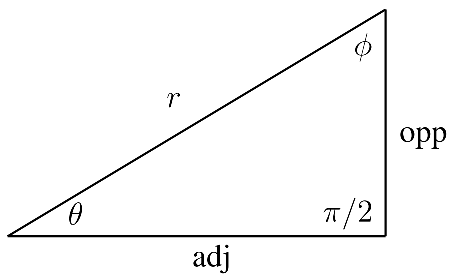
(c) $17\pi/15$

ANSWER

N.B. In the exams we will assume that all calculators are set to radians and all angle sizes will be measured in radians, so SET YOUR CALCULATOR TO RADIANS NOW. Every year some students lose valuable marks because their calculators are set to degrees. Why work in radians rather than in degrees? Because when you do calculus (in SFY0003), you must work in radians. It is impossible to work in degrees.

8.3 The Three Principal Trig Functions

Consider a right-angled triangle with angles θ, ϕ and $\pi/2$, as shown.



In the picture above, 'adj' is length of the side adjacent to θ, 'opp' is the length of the side opposite θ and r is the length of the hypotenuse.
Since the angles in a triangle add up to π, we have $\theta+\phi+\pi/2 = \pi$, so $\theta+\phi = \pi/2$. Thus

$$\theta = \pi/2 - \phi \text{ and } \phi = \pi/2 - \theta.$$

By Pythagoras' Theorem we have $r^2 = \text{adj}^2 + \text{opp}^2$. **Note though that it only applies to right-angled triangles.**

The three main trig functions are the **sine**, the **cosine** and the **tangent**. (The word "sine" comes from the Latin "sinus", meaning "bay", which is in turn derived from a Sanskrit word "jiva".) They are defined as follows:

$$\sin(\theta) = \frac{\text{opp}}{r}, \quad \cos(\theta) = \frac{\text{adj}}{r}, \quad \tan(\theta) = \frac{\text{opp}}{\text{adj}}.$$

It is usual to write simply $\sin\theta$, $\cos\theta$ and $\tan\theta$. Notice that if the bottom line is horizontal, then $\tan\theta$ is the gradient of the hypotenuse.

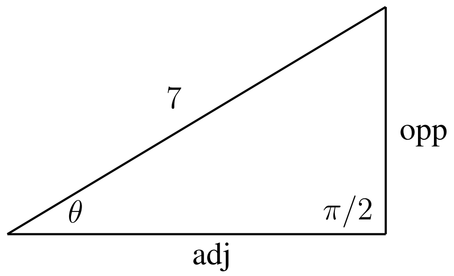
There is a mnemonic that some people find useful :

SOHCAHTOA "Sin=Opp/Hyp; Cos=Adj/Hyp; Tan=Opp/Adj."

Note that, since the hypotenuse is the longest side in a right-angled triangle, we have $0 < \sin\theta < 1$ and $0 < \cos\theta < 1$. But $0 < \tan\theta < \infty$.

Rearranging $\sin\theta = \dfrac{\text{opp}}{r}$ and $\cos\theta = \dfrac{\text{adj}}{r}$, we get opp $= r\sin\theta$ and adj $= r\cos\theta$. In other words the length of the adjacent side is $r\cos\theta$ and the length of the opposite side is $r\sin\theta$. This allows us to calculate side lengths in a right-angled triangle when we know the length of the hypotenuse and an angle.

Example 63. Find the lengths adj and opp in the triangle below when $\theta = 0 \cdot 45$ (radians).



ANSWER

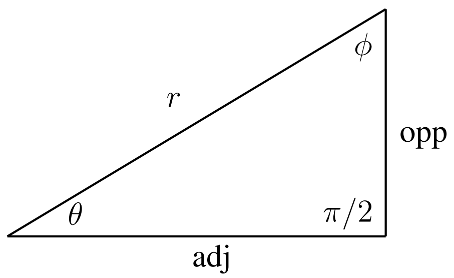
Observe that

$$\frac{\sin\theta}{\cos\theta} = \frac{\text{opp}}{r} \div \frac{\text{adj}}{r} = \frac{\text{opp}}{r} \times \frac{r}{\text{adj}} = \frac{\text{opp}}{\text{adj}} = \tan\theta$$

so

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

Let us look again at the triangle below, but this time consider the angle ϕ.



The side adjacent to ϕ has length opp, while the side opposite ϕ has length adj. Therefore

$$\sin\phi = \frac{\text{adj}}{r}, \;\; \cos\phi = \frac{\text{opp}}{r}, \;\; \tan\phi = \frac{\text{adj}}{\text{opp}}.$$

It follows that

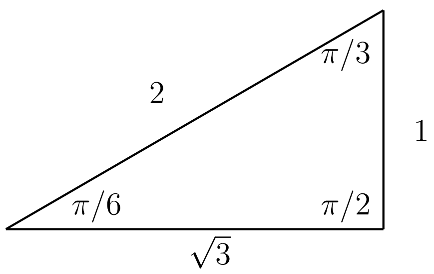
$$\sin\phi = \cos\theta, \;\; \cos\phi = \sin\theta, \;\; \tan\phi = \frac{1}{\tan\theta}.$$

Recall that $\phi = \dfrac{\pi}{2} - \theta$, so

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \text{ and } \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

There are two triangles for which $\sin$, $\cos$ and $\tan$ have nice values.

The 1–$\sqrt{3}$–2 triangle.



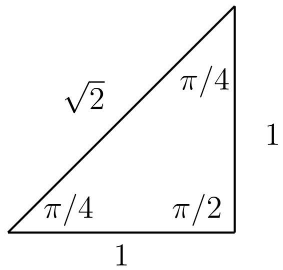
This triangle has angles of sizes $\pi/6, \pi/3, \pi/2$.

We see that

$\sin\dfrac{\pi}{6} = ANSWER$; $\cos\dfrac{\pi}{6} = ANSWER$; $\tan\dfrac{\pi}{6} = ANSWER$.

$\sin\dfrac{\pi}{3} = ANSWER$; $\cos\dfrac{\pi}{3} = ANSWER$; $\tan\dfrac{\pi}{3} = ANSWER$.

The 1–1–$\sqrt{2}$ triangle.



This triangle has angles of sizes $\pi/4, \pi/4, \pi/2$. We see that

$$\sin\frac{\pi}{4} = ANSWER \qquad ;\cos\frac{\pi}{4} = ANSWER \qquad ;\tan\frac{\pi}{4} = ANSWER \qquad .$$

8.4 Pythagoras Revisited

When we square $\sin\theta$, $\cos\theta$, and $\tan\theta$, there is a special notation:

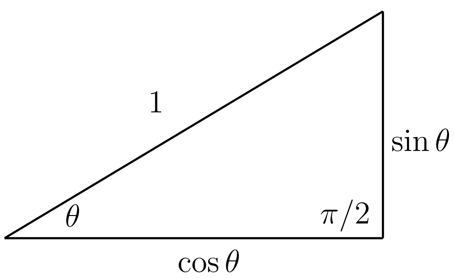
$$(\sin\theta)^2 \text{ is often written } \sin^2\theta$$

$$(\cos\theta)^2 \text{ is often written } \cos^2\theta$$

$$(\tan\theta)^2 \text{ is often written } \tan^2\theta.$$

The same principle applies to higher powers, for example $(\sin\theta)^3$ is often written $\sin^3\theta$, but we shall not need that in SFY0001.

Suppose we have a right-angled triangle with the hypotenuse of length 1 and angle θ as shown below. Then adj $= \cos\theta$ and opp $= \sin\theta$.

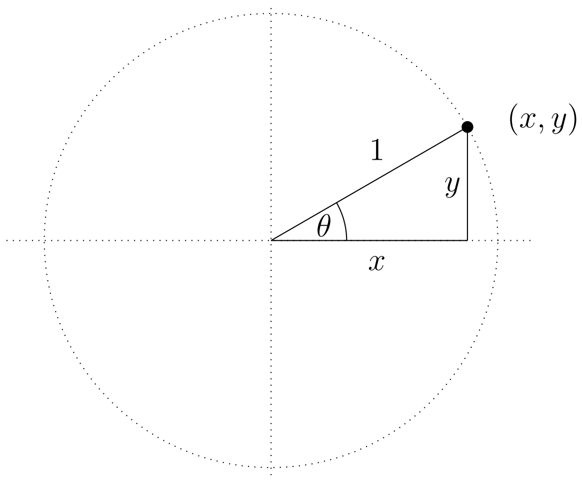


From Pythagoras's Theorem we obtain $\boxed{\sin^2\theta + \cos^2\theta = 1}$.

8.5 Trigonometric Functions for General Angles

So far we've only considered right-angled triangles and so we have defined $\sin\theta$, $\cos\theta$ and $\tan\theta$ only for angles with $0 < \theta < \pi/2$. We shall now consider other values of θ and consider what $\sin\theta$, $\cos\theta$ and $\tan\theta$ might mean.

We start with a circle of radius 1 about the origin in the xy-plane, and a point (x, y) on the circle, chosen with $x, y > 0$. Construct a right angled traingle as shown in the picture. We can label the lengths of the sides as shown, and calculate $\sin\theta, \cos\theta$ and $\tan\theta$ in terms of the coordinates.



$$0 < \theta < \frac{\pi}{2}$$

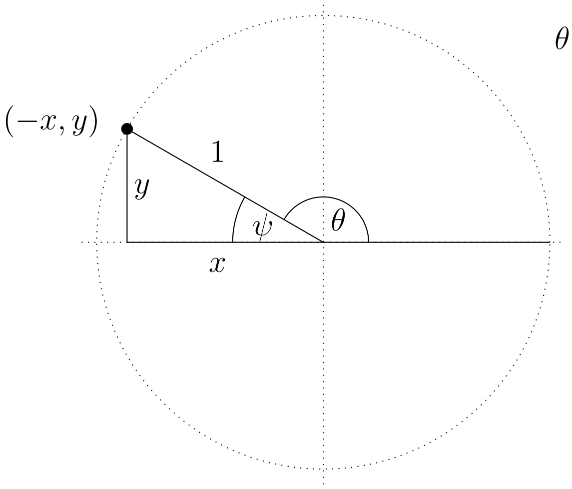
$$\sin\theta = y$$

$$\cos\theta = x$$

$$\tan\theta = \frac{y}{x}$$

This suggests how we might think of θ more generally. Essentially we take $\sin\theta$ and $\cos\theta$ as the coordinates of a point on the unit circle.

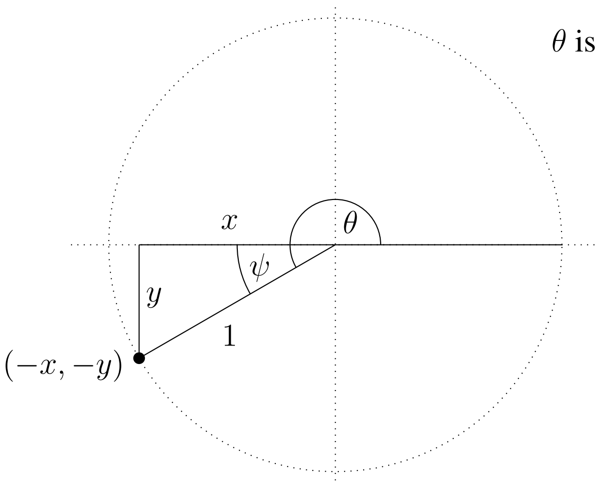
In the following pictures we shall assume that $x, y > 0$. Consider an angle θ with $\frac{\pi}{2} < \theta < \pi$ and let $\psi = \pi - \theta$:



θ is measured anti-clockwise from the positive x-axis

$$\sin\theta = y > 0$$
$$\cos\theta = -x < 0$$
$$\tan\theta = \frac{y}{-x} = \frac{-y}{x} < 0$$
$$\sin\psi = y = \sin\theta$$
$$\cos\psi = x = -\cos\theta$$
$$\tan\psi = \frac{y}{x} = -\tan\theta$$

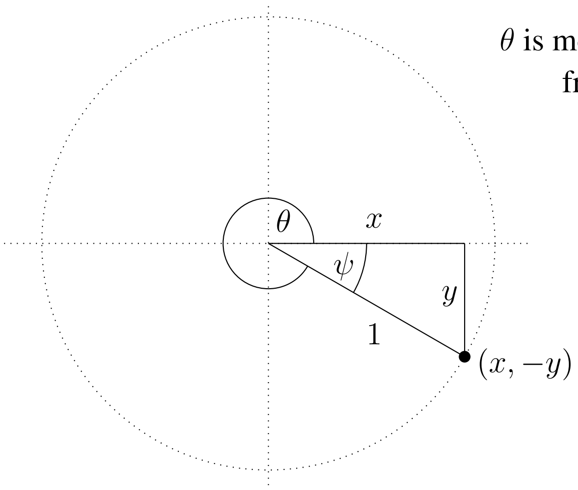
Next consider $\pi < \theta < \frac{3\pi}{2}$ and let $\psi = \theta - \pi$:



θ is measured anti-clockwise from the positive x-axis

$$\sin\theta = -y < 0$$
$$\cos\theta = -x < 0$$
$$\tan\theta = \frac{-y}{-x} = \frac{y}{x} > 0$$
$$\sin\psi = y = -\sin\theta$$
$$\cos\psi = x = -\cos\theta$$
$$\tan\psi = \frac{y}{x} = \tan\theta$$

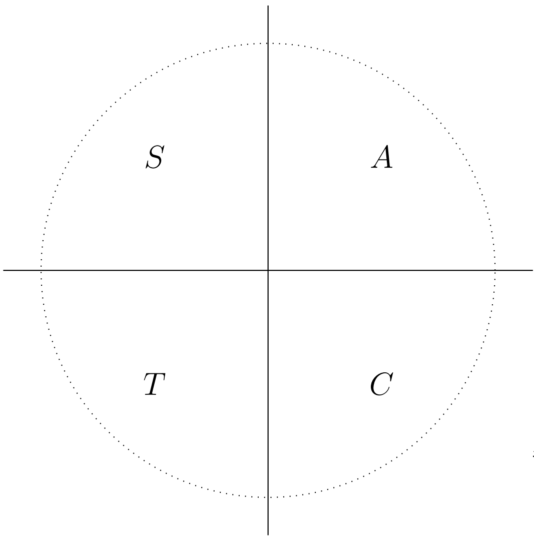
Now consider $\frac{3\pi}{2} < \theta < 2\pi$ and let $\psi = 2\pi - \theta$:



θ is measured anti-clockwise from the positive x-axis

$$\sin\theta = -y < 0$$
$$\cos\theta = x > 0$$
$$\tan\theta = \frac{-y}{x} < 0$$
$$\sin\psi = y = -\sin\theta$$
$$\cos\psi = x = \cos\theta$$
$$\tan\psi = \frac{y}{x} = -\tan\theta$$

The following picture describes which trig functions are positive and where:



$A =$All

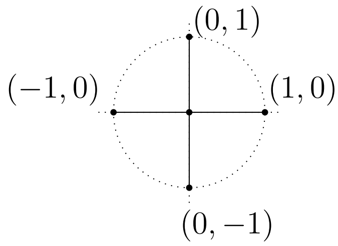
$S = \sin$

$T = \tan$

$C = \cos$

All Stations To Carlisle

Finally we consider points on the x and y axes:



$$\sin 0 = 0, \cos 0 = 1, \tan 0 = 0$$

$$\sin \pi/2 = 1, \cos \pi/2 = 0$$
$\tan \pi/2$ not defined

$$\sin \pi = 0, \cos \pi = -1, \tan \pi = 0$$

$$\sin 3\pi/2 = -1, \cos 3\pi/2 = 0$$
$\tan 3\pi/2$ not defined

Now suppose that we have any number θ. The main principle is that

$$\sin \theta = \sin(\theta + 2\pi),\ \cos \theta = \cos(\theta + 2\pi),\ \tan \theta = \tan(\theta + 2\pi).$$

This allows us to add or subtract 2π until we get a number between 0 and 2π. Remember that

- $0 : \sin 0 = 0, \cos 0 = 1, \tan 0 = 0$
- $\frac{\pi}{6} : \sin \pi/6 = \frac{1}{2},\ \cos \pi/6 = \frac{\sqrt{3}}{2},\ \tan \pi/6 = \frac{1}{\sqrt{3}}$
- $\frac{\pi}{4} : \sin \pi/4 = \cos \pi/4 = \frac{1}{\sqrt{2}},\ \tan \pi/4 = 1$
- $\frac{\pi}{3} : \sin \pi/3 = \frac{\sqrt{3}}{2},\ \cos \pi/3 = \frac{1}{2},\ \tan \pi/3 = \sqrt{3}$
- $\frac{\pi}{2} : \sin \pi/2 = 1,\ \cos \pi/2 = 0,\ \tan \pi/2$ is not defined.

Remember also that

- $\sin(\pi - \theta) = \sin \theta,\ \ \cos(\pi - \theta) = -\cos \theta,\ \ \tan(\pi - \theta) = -\tan \theta$
- $\sin(\pi + \theta) = -\sin \theta,\ \ \cos(\pi + \theta) = -\cos \theta,\ \ \tan(\pi + \theta) = \tan \theta$
- $\sin(2\pi - \theta) = -\sin \theta,\ \ \cos(2\pi - \theta) = \cos \theta,\ \ \tan(2\pi - \theta) = -\tan \theta$

Examples 64. Evaluate each of the following.

(a) $\cos 5\pi/6$

ANSWER

(b) $\sin 5\pi/4$

ANSWER

(c) $\tan 11\pi/6$

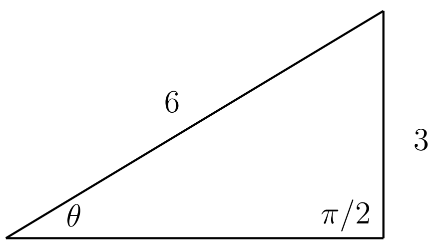
ANSWER

(d) $\sin 25\pi/6$

ANSWER

8.6 The Inverse Trigonometric Functions

Suppose that we know $\sin\theta$ or $\cos\theta$, but not θ itself. How can we work 'back' to θ? For example, suppose we are given the following right-angled triangle:



How can we find θ, given that $\sin\theta = \dfrac{3}{6} = 0 \cdot 5$? The answer is that we know $\sin\pi/6 = 0 \cdot 5$ so we conclude that $\theta = \pi/6$. We say that $\sin^{-1} 0 \cdot 5 = \pi/6$. More generally, if we are given a number x (with $0 \leq x \leq 1$), the angle whose $\sin$ is x is written $\sin^{-1} x$.

To find the angle using a calculator, we use the $\sin^{-1}$ button. On a typical (Casio) calculator, the function is obtained by pressing [SHIFT] followed by [sin].

Similarly if we are given a number x (with $0 \leq x \leq 1$), the angle whose $\cos$ is x is written $\cos^{-1} x$, and (for $0 \leq x < \infty$) the angle whose $\tan$ is x is written $\tan^{-1} x$. These are accessed on a calculator in a similar way as $\sin^{-} 1$.

Examples 65. Evaluate each of the following using a calculator. Express the answer to 4 dp.

(a) $\sin 1, \cos 1, \tan 1.$

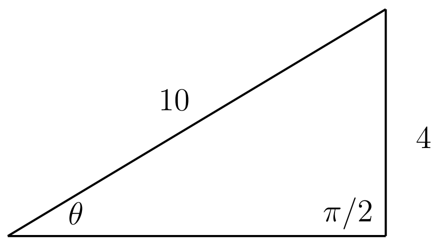
ANSWER

(b) $\sin^{-1} 0 \cdot 8415, \cos^{-1} 0 \cdot 5403, \tan^{-1} 1 \cdot 5574.$

ANSWER

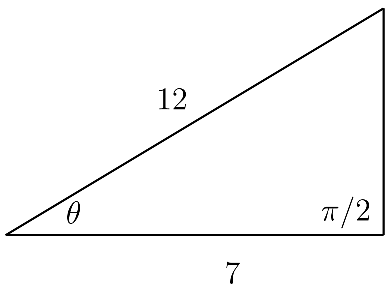
Examples 66. In each of the following, determine θ.

(a)



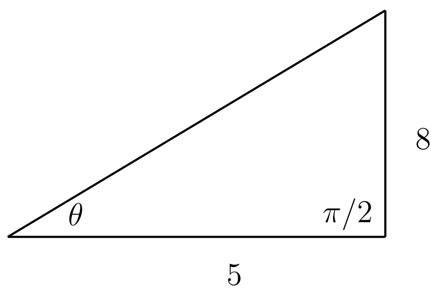
ANSWER

(b)



ANSWER

(c)



ANSWER

Now $\sin^{-1}$ is meant to be the opposite function to $\sin$.
Choose any number θ with $0 \leq \theta \leq \pi/2$. For example take $\theta = 0 \cdot 75$. Using the calculator:

- $\sin 0 \cdot 75 = 0 \cdot 68163876$

- $\sin^{-1} 0 \cdot 68163876 = 0 \cdot 75$.

It works the other way round to. Choose any number x with $0 \leq x \leq 1$. For example take $x = 0 \cdot 65$. Using the calculator:

- $\sin^{-1} 0 \cdot 65 = 0 \cdot 7075844367$.
- $\sin 0 \cdot 7075844367 = 0 \cdot 65$.

There is a problem though. For example, $\sin 2\pi = \sin 0 = 0$, but $\sin^{-1} 0 = 0$: we don't get back to 2π. Indeed we have specifically said that $\sin 0 = \sin 2\pi = \sin 4\pi$ etc. and similarly $\sin \pi/2 = \sin 5\pi/2 = \sin 9\pi/2$ etc. This is not often a problem when working with triangles.

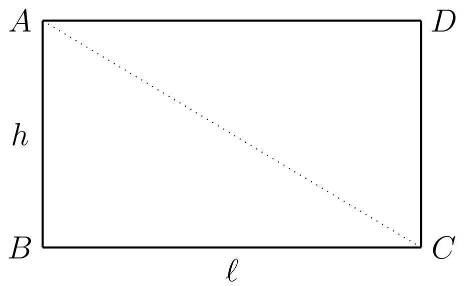
- For $-1 \leq x \leq 1$, $\sin^{-1} x$ is defined to be the unique number θ in the range $-\pi/2 \leq \theta \leq \pi/2$ for which $\sin \theta = x$.
- For $-1 \leq x \leq 1$, $\cos^{-1}(x)$ is defined to be the unique number θ with $0 \leq \theta \leq \pi$ for which $\cos \theta = x$.
- For any x, $\tan^{-1}(x)$ is defined to be the unique number θ with $-\pi/2 < \theta < \pi/2$ for which $\tan \theta = x$.

We shall see that $\cos^{-1}$ is sometimes more useful than $\sin^{-1}$ when we are working with triangles with an angle greater than $\pi/2$.

8.7 Area of a Triangle

THE AREA OF A RECTANGLE

Consider a rectangle $ABCD$ with base length ℓ and height h.



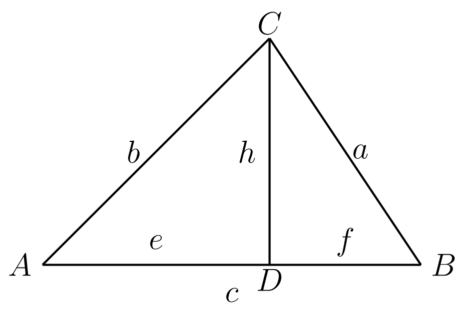
The area of a rectangle is the base times the height: $h \times \ell$.

THE AREA OF A RIGHT-ANGLED TRIANGLE
Suppose we are given a right-angled triangle ΔABC. Then we can construct a rectangle $ABCD$ with the same base and height is made up of two copies of the triangle, as shown in the picture above.

Therefore the area of ΔABC is half the area of the rectangle, i.e., $\frac{1}{2}h \times \ell$. Thus

THE AREA OF A RIGHT-ANGLED TRIANGLE $= \frac{1}{2}$ BASE $\times$ HEIGHT

THE AREA OF A GENERAL TRIANGLE
Now consider a general triangle. It is usual to let A, B, C denote both the vertices and also the sizes of the angles at these vertices, and to let a, b, c denote the corresponding **opposite** sides. We shall, in addition, let the line through C perpendicular to AB meet AB in the point D. Thus, $AB = c$ is the base of the triangle and $CD = h$ is its height.

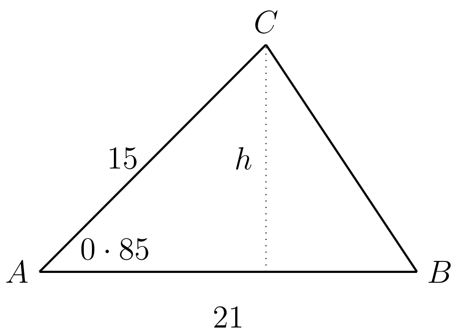


Then the area of ΔABC = Area of ΔADC + Area of ΔBCD
$= \frac{1}{2}e \times h + \frac{1}{2}f \times h = \frac{1}{2}(e + f) \times h = \frac{1}{2}c \times h = \frac{1}{2}$ base $\times$ height.

Thus

THE AREA OF A GENERAL TRIANGLE $= \frac{1}{2}$ BASE $\times$ HEIGHT

Example 67. Find the area of the triangle ABC shown below.



ANSWER

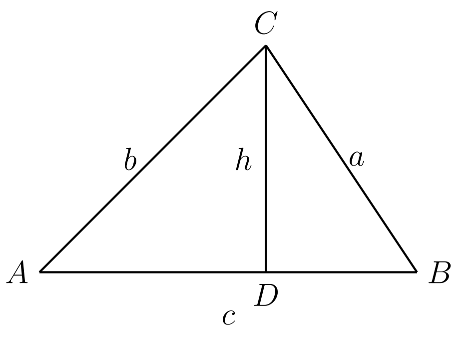
Warning:
The formulae

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \cos\theta = \frac{\text{adj}}{\text{hyp}}, \tan\theta = \frac{\text{opp}}{\text{adj}}$$

apply **only in right-angled triangles**, for the simple reason that the word "hypotenuse" (= side opposite the right angle) is meaningless in other triangles.

8.8 A Trigonometric Formula for the Area of a Triangle

Our consideration of the area of a general triangle leads to a trigonometric formula for the area of a triangle.



We saw that the area is $\frac{1}{2}ch$.

Using trigonometry we see that $h = b \sin A$ (and also $a \sin B$). Therefore the area is $\frac{1}{2}cb \sin A$. It is also $\frac{1}{2}ca \sin B$.

This gives us the required formula:

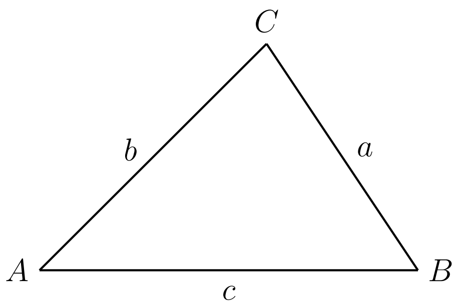
$$\boxed{\text{THE AREA OF A GENERAL TRIANGLE} = \frac{1}{2}bc \sin A}$$

If we drew the triangle with AC as the base we would obtain a formula for the area as $\frac{1}{2}ba \sin C$ (and also $\frac{1}{2}bc \sin A$). We therefore have three (equally good) formulae for the area of a triangle:

$$\frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ba \sin C.$$

8.9 The Sine Rule

We suppose we have a triangle ΔABC:



The **Sine Rule** is:

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$$

EXPLANATION
We know that

$$\frac{1}{2}bc \sin A = \frac{1}{2}ca \sin B = \frac{1}{2}ab \sin C.$$

Multiply through by 2 and then divide through by abc. We get

$$\frac{bc \sin A}{abc} = \frac{ca \sin B}{abc} = \frac{ba \sin C}{abc},$$

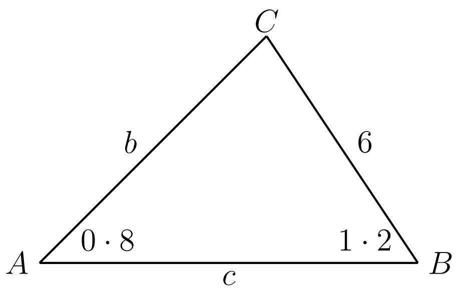
i.e.,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

Inverting these fractions (turning them upside down), we get the Sine Rule as stated above.

Suppose that we know: **Two Angles and a Side**. Then we can use the Sine Rule to solve the triangle, i.e., to calculate the remaining angles and sides.

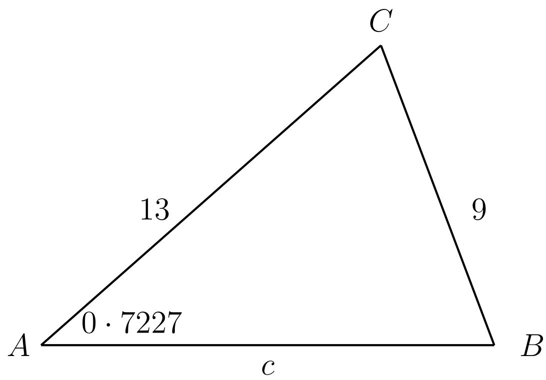
Example 68. In the triangle below, determine the length b, the angle C and the length c. Work to 4 decimal places.



ANSWER

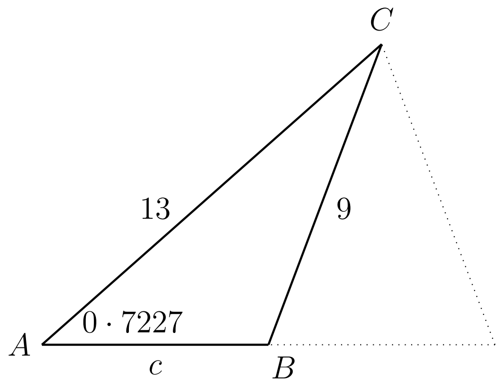
Suppose that we know: **Two Sides and a Angle**. Then we can *sometimes* use the Sine Rule to solve the triangle, i.e., to calculate the remaining angles and sides.

Example 69. In the triangle below, determine the angle B, the angle C and the length c. Work to 4 decimal places. Assume that the diagram is accurate in that $B, C < \pi/2$.



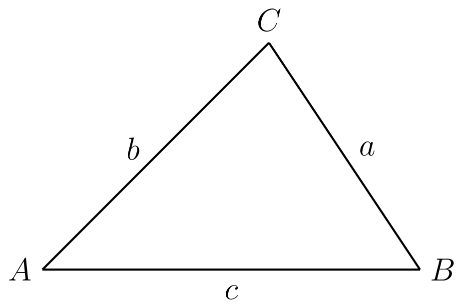
ANSWER

Remark.
Recall that $\sin(\pi - \theta) = \sin\theta$. In particular $\sin 1\cdot 2710 = \sin(\pi - 1\cdot 2710)$, i.e., $\sin 1\cdot 2710 = \sin 1\cdot 8706 = 0\cdot 9554$. Thus in the example above we can only be sure of the value of B by knowing that it lies between 0 and $\pi/2$. The following picture shows a second triangle having $a = 9, b = 13, A = 0\cdot 7227$, but with $B > \pi/2$. So, if we had not been told that $B < \pi/2$, there would have been two answers to the problem.



8.10 The Cosine Rule

We suppose we have a triangle ΔABC:

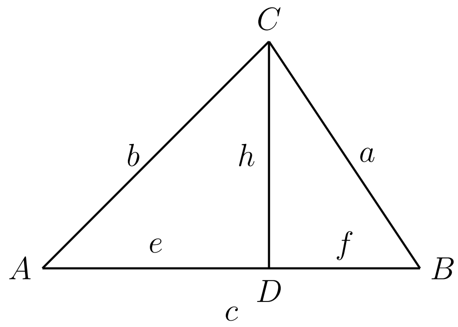


The **Cosine Rule** is: $\boxed{a^2 = b^2 + c^2 - 2bc\cos A}$
Notice that if A is a right-angle, then $\cos A = 0$ and the Cosine Rule becomes $a^2 = b^2 + c^2$, i.e., Pythagoras's Theorem.

If we rotate the triangle so that first BC and then CA is the base, we obtain alternative versions: $\boxed{b^2 = c^2 + a^2 - 2ac\cos B}$ and $\boxed{c^2 = a^2 + b^2 - 2ab\cos C.}$

EXPLANATION

Recall the picture below.



We note four pieces of information:

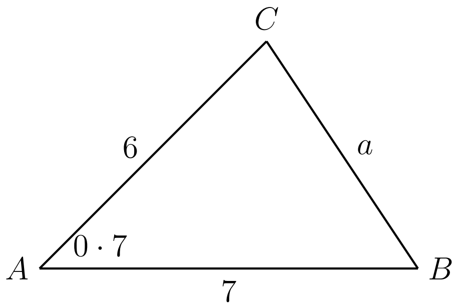
- $e = b\cos A$.
- $b^2 = h^2 + e^2$ (by Pythagoras's Theorem).
- $a^2 = h^2 + f^2$ (by Pythagoras's Theorem).
- $c = e + f$.

Then

$$\begin{aligned} a^2 &= h^2 + f^2 = h^2 + (c-e)^2 = h^2 + c^2 + e^2 - 2ec \\ &= (h^2 + e^2) + c^2 - 2ce = b^2 + c^2 - 2cb\cos A. \end{aligned}$$

If we know: **Two Sides and a Angle** where the sides are "b" and "c", and the angle "A" is between them, then the Cosine Rule tells you the size of the remaining side "a".

Example 70. In the triangle below, determine the length a. Work to 4 decimal places.



ANSWER

8.11 Using the Cosine Rule to find Angles

If we want to find an angle, we can rearrange the expression in the Cosine Rule to make $\cos A$ the subject:

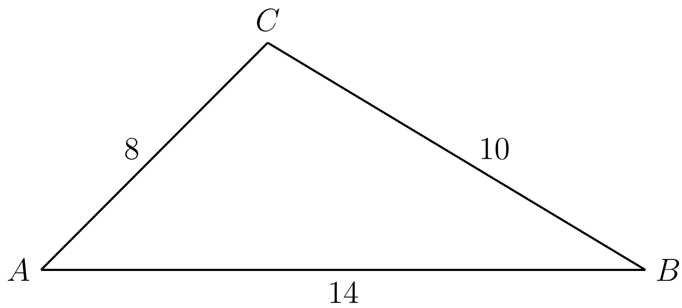
- Start from $a^2 = b^2 + c^2 - 2bc\cos A$ (add $2bc\cos A$ to each side)
- $a^2 + 2bc\cos A = b^2 + c^2$ (subtract a^2 from each side)
- $2bc\cos A = b^2 + c^2 - a^2$ (divide each side by $2bc$)
- We get $\boxed{\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}.}$

From the other two versions of the Cosine Rule, the same procedure gives us:

$$\boxed{\cos B = \frac{a^2 + c^2 - b^2}{2ac}} \text{ and } \boxed{\cos C = \frac{a^2 + b^2 - c^2}{2ab}.}$$

Having calculated the value of $\cos A$ (or $\cos B$ or $\cos C$) we can use $\cos^{-1}$ to find A (or B or C).

Example 71. In the triangle below, determine the angles A, B and C. Work to 4 decimal places.



ANSWER

Note that for each number x between -1 and 1, there is a unique angle θ between 0 and π with $\cos\theta = x$. This is different from the situation with $\sin$.

Examples 72. (a) In ΔABC, $a = 4, b = 10, c = 13$. Find A, B, C. (Work to 4 decimal places.)

ANSWER

(b) In ΔABC, $a = 4, b = 5, C = 2.3$. Find A, B, c.

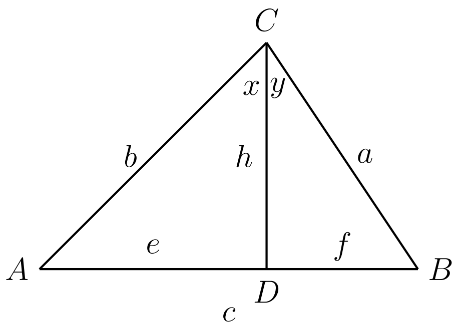
ANSWER

(c) In ΔABC, $A = \pi/3$, $B = 3\pi/7$ and $a = 6$. Find C, b, c.
ANSWER

8.12 The Compound Angle Formulas - Private Reading

This section will not be assessed in Test 2 or in the January exam. Read it to prepare for further study.

There are formulae for $\sin(x+y)$, $\sin(x-y)$, $\cos(x+y)$, $\cos(x-y)$ in terms of $\sin x$, $\cos x$, $\sin y$ and $\cos y$; and for $\tan(x+y), \tan(x-y)$ in terms of $\tan x$ and $\tan y$. (These formulae have many uses.) Consider the usual diagram, with the angles x and y as shown:



We have

$$\sin x \cos y + \cos x \sin y = \frac{e}{b} \cdot \frac{h}{a} + \frac{h}{b} \cdot \frac{f}{a} = \frac{h}{ab}(e+f) = \frac{hc}{ab}$$

$$= \frac{h}{b} \cdot \frac{c}{a} = \frac{b \sin A}{b} \cdot \frac{c}{a} = \frac{\sin A}{a} \times c = \frac{\sin C}{c} \times c = \sin C = \sin(x+y).$$

Thus

$$\boxed{\sin(x+y) = \sin x \cos y + \cos x \sin y}$$

If we replace y by $-y$, we get

$$\sin(x-y) = \sin x \cos(-y) + \cos x \sin(-y).$$

But $\cos(-y) = \cos y$ and $\sin(-y) = -\sin y$, so

$$\boxed{\sin(x-y) = \sin x \cos y - \cos x \sin y}$$

These two formulas are often summarised as

$$\boxed{\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y}$$

Now recall that $\sin(\pi/2 - x) = \cos x$ and $\cos(\pi/2 - x) = \sin x$. It follows that

$$\begin{aligned} \cos(x+y) &= \sin(\pi/2 - x - y) = \sin[(\pi/2 - x) - y] \\ &= \sin(\pi/2 - x)\cos y - \cos(\pi/2 - x)\sin y \\ &= \cos x \cos y - \sin x \sin y. \end{aligned}$$

Thus

$$\boxed{\cos(x+y) = \cos x \cos y - \sin x \sin y}$$

If we replace y by$-y$, we get

$$\boxed{\cos(x-y) = \cos x \cos y + \sin x \sin y}$$

These two formulas are often summarised as

$$\boxed{\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y}$$

Finally

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Now divide numerator and denominator by $\cos x \cos y$ to get

$$\frac{\left[\dfrac{\sin x \cos y}{\cos x \cos y} + \dfrac{\cos x \sin y}{\cos x \cos y}\right]}{\left[\dfrac{\cos x \cos y}{\cos x \cos y} - \dfrac{\sin x \sin y}{\cos x \cos y}\right]} = \frac{\left[\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}\right]}{\left[1 - \dfrac{\sin x \sin y}{\cos x \cos y}\right]} = \frac{\tan x + \tan y}{1 - \tan x \tan y},$$

i.e.,

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

Replacing y by $-y$ we obtain

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}.$$

These two formulas are often summarised as

$$\boxed{\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}}$$

8.13 Multiple Angle Formulas - Private Reading

This section will not be assessed in Test 2 or in the January exam. Read it to prepare for further study.

If we take the particular case $y = x$ in the formulae

$$\sin(x+y) = \sin x \cos y + \cos x \sin y, \ \cos(x+y) = \cos x \cos y - \sin x \sin y,$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

we obtain

$$\sin 2x = 2 \sin x \cos x,$$

$$\cos 2x = \cos^2 x - \sin^2 x,$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

Remember that $\sin^2 + \cos^2 x = 1$, so rearranging, we also have $\sin^2 x = 1 - \cos^2 x$ and $\cos^2 x = 1 - \sin^2 x$. Substituting each of these into the formula for $\cos 2x$ gives two more formulae:

$$\cos 2x = 2\cos^2 x - 1,$$

$$\cos 2x = 1 - 2\sin^2 x.$$

As an **example** of applications of these formulae, we use them to calculate $\sin \pi 12$ and $\cos \pi/12$. Recall that $\cos \pi/6 = \frac{\sqrt{3}}{2}$. If we take $x = \pi/12$, so that $2x = \pi/6$, then

$$\frac{\sqrt{3}}{2} = 2\cos^2 x - 1 = 1 - 2\sin^2 x,$$

so $2\cos^2 x = 1 + \frac{\sqrt{3}}{2} = \frac{2+\sqrt{3}}{2}$, which gives us $\cos^2 x = \frac{2+\sqrt{3}}{4}$ and therefore $\cos x = \sqrt{\frac{2+\sqrt{3}}{4}}$ (we take the positive square root because we know that $\cos \pi/12$ will be positive). Similarly $2\sin^2 x = 1 - \frac{\sqrt{3}}{2} = \frac{2-\sqrt{3}}{2}$, which gives us $\sin^2 x = \frac{2-\sqrt{3}}{4}$ and therefore $\sin x = \sqrt{\frac{2-\sqrt{3}}{4}}$ (again, we take the positive square root because we know that $\sin \pi/12$ will be positive).

8.14 The Addition Formulae - Private Reading

This section will not be assessed in Test 2 or in the January exam. Read it to prepare for further study.

Let $u = \dfrac{x+y}{2}$ and $v = \dfrac{x-y}{2}$, so $x = u + v$ and $y = u - v$. Then

$$\sin x + \sin y = \sin(u+v) + \sin(u-v) = [\sin u \cos v + \cos u \sin v] + [\sin u \cos v - \cos u \sin v]$$

$$= 2 \sin u \cos v = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right).$$

this gives us the first of four 'addition' formulae. The other three are obtained in a similar manner.

(a) $\sin x + \sin y = 2 \sin\left(\dfrac{x+y}{2}\right) \cos\left(\dfrac{x-y}{2}\right)$.

(b) $\sin x - \sin y = 2 \cos\left(\dfrac{x+y}{2}\right) \sin\left(\dfrac{x-y}{2}\right)$.

(c) $\cos x + \cos y = 2 \cos\left(\dfrac{x+y}{2}\right) \cos\left(\dfrac{x-y}{2}\right)$.

(d) $\cos x - \cos y = -2 \sin\left(\dfrac{x+y}{2}\right) \sin\left(\dfrac{x-y}{2}\right)$.

8.15 The Other Trig Functions - Private Reading

This section will not be assessed in Test 2 or in the January exam. Read it to prepare for further study.

Thrse are three more trig functions that you are likely to come across:

- the **cosecant**: $\csc x = \dfrac{1}{\sin x}$
- the **secant**: $\sec x = \dfrac{1}{\cos x}$
- the **cotangent**: $\cot x = \dfrac{1}{\tan x}$.

They can be seen as abbreviations, so you can always get by without them, but they often make formulae simpler.

8.16 Graphs of the Main Trig Functions - Private Reading

This section will not be assessed in Test 2 or in the January exam. Read it to prepare for further study.

The following pictures show the graphs of $\sin x, \cos x$ and $\tan x$.

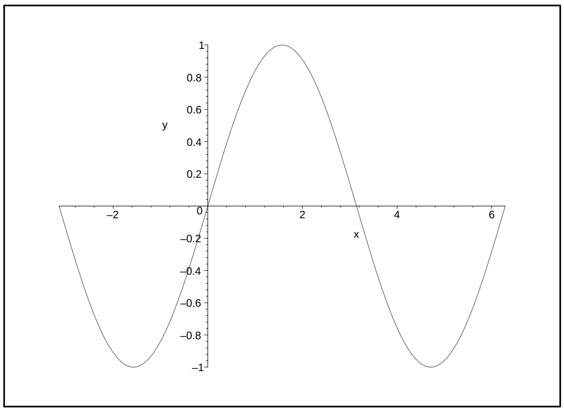


Figure 1: $y = \sin x$

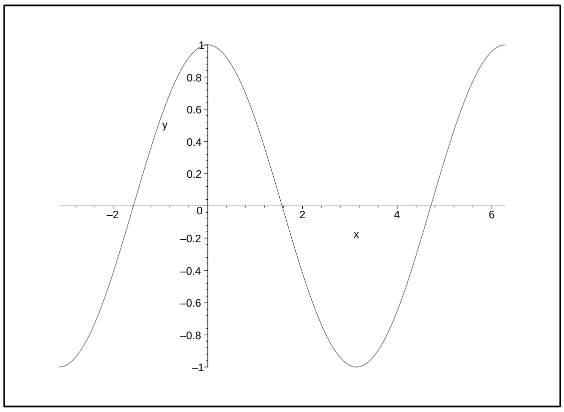


Figure 2: $y = \cos x$

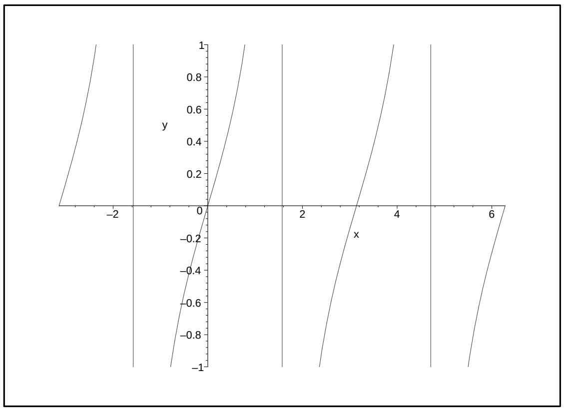


Figure 3: $y = \tan x$