On twisted tensor product group embeddings and the spin representation of symplectic groups

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Abstract

We describe an alternative perspective for at least some of the subgroups in Aschbacher’s class $C_9$. Then we show that under the twisted tensor product embedding of the groups $PSp_2(q^t)$ and $PSp_4(q^t)$ in orthogonal groups over $GF(q)$, $t \geq 3$, $q \geq 4$ even, an intermediate embedding of type $C_3$ occurs.

Keywords: Twisted tensor product representation, spin representation, partial ovoid, alternating product

1 Introduction

Let $V$ be a vector space over $GF(q)$ and let $G$ be a classical group associated with $V$. In [1] Aschbacher describes a family $C$ of eight “geometric” classes of subgroups of $G$ and shows that any subgroup of $G$ either lies in one of these classes or has the form $H = N_G(X)$, for some quasisimple subgroup $X$ of $G$ satisfying some extra conditions. Given such a group $H$ not lying in one of the eight classes of $C$, the main problem is to determine whether or not $H$ is maximal in $G$. If not, there exists a quasisimple subgroup $K$ with $X < K < G$ and one wants to study such configurations. For more details, see [23]. Here the focus is on a geometric description of $X$ and $K$.

Let $G$ be a finite classical group with natural module $V_0$ of dimension $n \geq 2$ over the Galois field $GF(q^t)$. Let $V_0^{q^i}$ denote the $G$–module $V_0$ with group action given by $g \cdot v = g^{q^i}(v)$, where $g^{q^i}$ denotes the matrix $g$ with its entries raised to the $q^i$–th power, $i = 0, \ldots, t - 1$. Then one can form the twisted tensor product module $V_0 \otimes V_0^{q^i} \otimes \cdots \otimes V_0^{q^{t-1}}$, a module which can be realized over the field $GF(q)$. This gives rise to an absolutely irreducible representation of the group $G$ on an $n^t$–dimensional natural module over $GF(q)$. Such representations are given by Steinberg [25] and further studied by Seitz [23]. See also [6]. We refer to this class of subgroups as $C_9$, as suggested by Seitz.

In [7] we studied the geometry of two classes of twisted tensor product group embeddings: $PSp_2(q^t) \leq P\Omega_{2t}^+(q)$, where $t \geq 2$ and $q$ is even; and $PSp_{2m}(q^t) \leq P\Omega_{2m'}(q)$ with $q$ even. [Note that $Sp_2(q) = SL_2(q)$ in all cases. We shall use $Sp_2(q)$ although some references will use $SL_2(q)$]. We found that our embedding of $PSL_2(q^t)$ is associated with an embedding of the projective line $PG(1, q^t)$ as a complete partial ovoid of a quadric in $PG(2t -$
1, q) (i.e., a maximal set of pairwise non-orthogonal points of the quadric); if \( t \geq 3 \), then the quadric is hyperbolic. Such partial ovoids are of interest because their size attains the Blokhuis–Moorhouse bound [2]. In particular, when \( t = 3 \) and \( q \geq 4 \), the embedding yields another description of the Desarguesian ovoid of the hyperbolic quadric of \( PG(7, q) \), [17]. Similarly, the embedding of \( PSp_{2m}(q^t) \), \( q \) even, in \( P\Omega_{(2m)^t}(q) \) has a particular application when \( m = 2 \) in the embeddings of symplectic ovoids of \( PG(3, q^t) \) as partial ovoids of hyperbolic quadrics of \( PG(4^t - 1, q) \) again with the size attaining the Blokhuis–Moorhouse bound.

In this paper we investigate further these twisted tensor product group embeddings, but from a different perspective. We show how the \( n\)-dimensional module over \( GF(q) \) for \( G \) may be viewed projectively as a subspace of the projective space \( PG((nt)^t - 1, q) \) containing the Grassmannian of \((t - 1)\)-subspaces of \( PG(nt - 1, q) \). From this viewpoint \( G \) preserves the intersection of the subspace and the Grassmannian. When \( n = 2m \leq 4 \), this approach enables us to address some questions on maximality. We prove that under the twisted tensor product group embedding of \( PSp_{2m}(q^t) \), \( m \leq 2 \), \( q \) even, an intermediate embedding of type \( C_3 \) occurs: \( PSp_{2m}(q^t) < PSp_{2m}(q) < P\Omega_{(2m)^t}(q) \). The partial ovoid referred to above lies on a unique quadric in \( PG((2m)^t - 1, q) \). It turns out that this quadric is precisely that arising from the spin representation of \( Sp_{2t}(q) \).

The class \( C_9 \) has also been studied by Schaffer in [22], where he uses representation theory techniques (and the Classification of Finite Simple Groups). He eliminates a number of possibilities, largely when \( t \) is composite, and show that the remaining subgroups in this class are maximal except in a small number of cases. The main exceptions are precisely \( PSp_2(q^t) \leq P\Omega_{2t}^-(q) \) and \( PSp_4(q^t) \leq P\Omega_{4t}^+(q) \) with \( q \) even. In this paper we demonstrate the non-maximality without reliance on the Classification of Finite Simple Groups and in a way that shows geometrically how intermediate subgroups arise; the construction is not restricted to prime values of \( t \).

2 A perspective on \( C_9 \)

In this section we describe an alternative perspective for at least some of the subgroups in the Aschbacher’s class \( C_9 \). Such a viewpoint is used in the following section where, in certain cases, overgroups are identified in a natural geometric context. The perspective we identify arises from considering
subspaces of a certain Grassmannian.

Let $E_i, 1 \leq i \leq t$, be $n$–dimensional vector spaces over $GF(q^t)$ and let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_t$. Suppose that for each $i$, $e_{i1}, e_{i2}, \ldots, e_{im}$ is a basis for $E_i$ and suppose that $H \leq GL(E_1)$. For $v = \sum_j \lambda_j e_{ij} \in E_i$ we write $v^\Psi = \sum_j \lambda^j e_{i+1j} \in E_{i+1}$ (with $i + 1$ interpreted modulo $t$), and for $h \in H$ we write $h^\Psi$ for the matrix $h$ with every entry raised to the power $q$. Hence, to any $v \in E_1$ there correspond “conjugate” vectors $v^{\Psi^{t-1}} \in E_i$ and $H$ acts on $E_i$ via $h^{\Psi^{t-1}}(v^{\Psi^{t-1}}) = (hv)^{\Psi^{t-1}}$. Therefore we have an action of $H$ on $E$ and $H$ preserves a fibration of $E$ into $t$–dimensional subspaces of the form $\langle v, v^\Psi, \ldots, v^{\Psi^{t-1}} \rangle$. In projective terms, $E$ corresponds to a projective space $\Sigma = PG(nt - 1, q^t)$ and $H$ preserves a partial spread of $(t - 1)$–subspaces of $\Sigma$. We may regard $\Psi$ as a semi–linear map on $E$. The vectors in $E$ fixed by $\Psi$ are precisely the vectors $v + v^\Psi + \cdots + v^{\Psi^{t-1}}$, where $v \in E_1$, and they form an $nt$–dimensional vector space $V$ over $GF(q)$ that spans $E$ and is preserved by $H$. In $\Sigma$ we have a set of points preserved by $H$ forming a subgeometry $\Sigma_0 = PG(nt - 1, q)$ and on restriction, the partial spread above becomes a spread $S$ of $\Sigma_0$ preserved by $H$. Suppose that $H$ preserves a non–degenerate alternating form $f_1$ on $E_1$, then $H$ preserves the alternating form $f_i$ on $E_i$ given by $f_i(u^{\Psi^{t-1}}, w^{\Psi^{t-1}}) = f_1(u, w)^{q^t-1}$ and an alternating form $f$ on $E$ in which $f_{E_i}$ is $f_i$ and in which $E_1 \oplus E_2 \oplus \cdots \oplus E_t$ is an orthogonal decomposition. Moreover the restriction of $f$ to $V$ is a non–degenerate alternating form on $V$. Thus $H$ acts as a subgroup of $Sp_n(q^t)$ embedded in $Sp_{nt}(q)$ on $\Sigma_0$ preserving a spread $S$ consisting now of totally isotropic $(t - 1)$–subspaces. In a similar vein, if $Q_1$ is a non–degenerate quadratic form on $E_1$ and we define $Q_i(u^{\Psi^{t-1}}) = Q_1(u)^{q^t-1}$, then we obtain a non–degenerate quadratic form on $E$ and from there an embedding of $O_n(q^t)$ in $O_{nt}(q)$ preserving a spread of $\Sigma_0$, although not all of the $(t - 1)$–subspaces in the spread are totally singular.

Consider the $t$–fold alternating product of $E$, $\bigwedge^t(E)$, an $H$–module of dimension $\binom{nt}{t}$ over $GF(q^t)$. If $A \oplus B$ is any decomposition for $E$, then

$$\bigwedge^t(E) = \bigoplus_{i+j=t} \left( \bigwedge^i(A) \otimes \bigwedge^j(B) \right).$$

Thus $\bigwedge^t(E)$ has a subspace $\bigwedge^1(E_1) \otimes \bigwedge^{t-1}(E_2 \oplus E_3 \oplus \cdots \oplus E_t)$ and, by iteration, a subspace $\bigwedge^1(E_1) \otimes \bigwedge^1(E_2) \otimes \cdots \otimes \bigwedge^1(E_t)$, i.e., $E_1 \otimes E_2 \otimes \cdots \otimes E_t$. This latter subspace is preserved by $H$. The $t$–dimensional subspaces of $E$ correspond to the $1$–dimensional subspaces of $\bigwedge^t(E)$. Each $t$–dimensional
$GF(q)$–subspace of $V$ determines a $t$–dimensional $GF(q^t)$–subspace of $E$ and so $\wedge^t(V)$ may be regarded as a $GF(q)$–subspace of $\wedge^t(E)$. For any $v \in E_t$, the $t$–subspace $\langle v, v^\psi, \ldots, v^{\psi^{t-1}} \rangle$ is mapped to the $1$–dimensional subspace corresponding to $v \wedge v^\psi \wedge \cdots \wedge v^{\psi^{t-1}} \in E_1 \otimes E_2 \otimes \cdots \otimes E_t$. In projective terms $PG(\binom{n}{t} - 1, q^t)$ contains the Grassmannian $G$ of $(t-1)$–subspaces of $\Sigma$ and $E$ terms $t$ the $\wedge \mathbb{GF}$ corresponding to $\lambda v$ subspace $\Delta$ irreducibly on $E$. The $t-1$–subspaces of $\Sigma$ form a Grassmannian $G_0$ lying in a projective space $PG(\binom{n}{t} - 1, q^t)$ that is a subgeometry of $PG(\binom{n}{t} - 1, q^t)$. Each of the subspaces of $S$ is mapped into $\Delta \cap G_0$.

Now we recall the twisted tensor product construction for embedding $PG(n - 1, q^t)$ in $PG(n^t - 1, q)$. As described in the introduction, a classical group $G$ having an absolutely irreducible representation on a vector space $V_0$ over $GF(q^t)$ has an absolutely irreducible $n^t$–dimensional module over $GF(q^t)$. We showed in [6, 2.4.1] that if $v, v_2, \ldots, v_n$ is a basis for $V_0$ and if $\phi$ is the semi–linear transformation defined by $\phi(\lambda u_1 \otimes u_2 \otimes \cdots \otimes u_t) = \lambda^q u_1 \otimes u_2 \otimes \cdots \otimes u_{t-1}$, where $u_i \in \{v, v_2, \ldots, v_n\}$ (extended linearly over $GF(q^t)$), then the set $W$ of vectors in $V_0 \otimes V_0^q \otimes \cdots \otimes V_0^{q^{t-1}}$ fixed by $\phi$ is precisely the $n^t$–dimensional module over $GF(q^t)$ for $G$. For any $0 \neq v \in V_0$, the vector $w = v \otimes v^q \otimes \cdots \otimes v^{q^{t-1}}$ lies in $W$, and for any $0 \neq \lambda \in GF(q^t)$, we see that $\lambda w = (\lambda^q \cdots \lambda^{q^{t-1}})w$, i.e., a $GF(q^t)$–scalar multiple of $w$. Hence the points of $PG(n - 1, q^t)$ may be represented as points of $PG(n^t - 1, q)$. Given that $G$ preserves the set of all such points and that $G$ acts irreducibly, these points must span $PG(n^t - 1, q)$.

Let us return to $S$ and its image in $\Delta \cap G_0$. We have seen that these points in $\Delta \cap G_0$ may be represented by $v \otimes v^q \otimes \cdots \otimes v^{q^{t-1}}$ as $v$ varies in $E_t$. Moreover we may take $H$ to be the group $SL_n(q^t)$ acting absolutely irreducibly on $E_t$. Hence the points corresponding to $S$ generate a $GF(q^t)$–subspace $\Delta_0$ of projective dimension $n^t - 1$. It follows that the $GF(q^t)$–span of $S$ is precisely $\Delta$. Hence we see the twisted tensor product module for $SL_n(q^t)$ as the subspace $\Delta_0$ of $PG(\binom{n}{t} - 1, q)$.

Observe that in one setting we have $H$ acting as a subgroup of $GL_n(q^t)$ on $PG(nt - 1, q)$, so here it is an Aschbacher $C_3$ group. In a second setting it is a subgroup of $GL_n(q^t)$ and lies in Aschbacher class $C_0$.

We have proved the following theorem.

**Theorem 2.1.** Let $G$ be a finite classical group with natural module $V$ of dimension $n$ over $GF(q^t)$. Then the twisted tensor product module for $G$,
in its projective form, is an \((n^t - 1)\)-subspace \(\Delta_0\) of the projective space \(\text{PG}(\binom{n^t}{t} - 1, q)\) spanned by the Grassmannian \(G_0\) of \((t - 1)\)-subspaces of \(\text{PG}(nt - 1, q)\). Moreover there is a spread \(S\) of \((t - 1)\)-subspaces of \(\text{PG}(nt - 1, q)\) whose image in \(G_0\) lies inside \(\Delta_0\) and is preserved by \(G\).

3 The embedding \(PSp_{2t}(q^t) < P\Omega_{2t}^+(q)\), \(q\) even, \(t \geq 3\)

In this section we assume that \(n = 2\), that \(q\) is even and \(t \geq 3\). We consider a vector space \(V\) of dimension \(2t\) and the corresponding projective space \(\Sigma_0 = \text{PG}(2t - 1, q)\).

In proving the main result in this section we use a construction, given by Brouwer [4], of the spin module for \(PSp_{2t}(q)\) as a quotient module of a submodule of \(\wedge^t V\). An equivalent formulation is given by Gow [15]. See also [3], [20], [21], [19].

Let \(I\) be the set of all totally isotropic \((t - 1)\)-subspaces of \(\Sigma_0\) with respect to a non–degenerate alternating form \(f\) and let \(S\) be a regular spread of \(\Sigma_0\) (with elements in \(I\)). Then the Grassmannian, \(G_0\), of \((t - 1)\)-subspaces of \(\Sigma_0\) has dimension \(\binom{2t}{t} - 1\) and the image of \(I\) in \(G_0\) spans a subspace \(F_t\) of dimension \(\binom{2t}{t} - \binom{2t}{t-2} - 1\). The vector space equivalent of \(F_t\) is the Weyl module of \(Sp_{2t}(q)\) for the fundamental weight \(\lambda_t\). In the current setting, where \(q\) is even, \(F_t\) has a unique maximal subspace fixed by \(PSp_{2t}(q)\), denoted \(N_t\). The quotient space \(M_t = F_t/N_t\) has dimension \(2^t - 1\), and corresponds to the spin module for \(Sp_{2t}(q)\). We write \(\theta\) for the Grassman map: \(I \to F_t\) and \(\iota_t\) for the corresponding map \(I \to M_t\) and note that for an element \(g \in PSp_{2t}(q)\) we have \(g(\theta(U)) = \theta(g(U))\), for \(U \in I\). We give an explicit proof that the mapping \(\iota_t\) is injective.

**Lemma 3.1.** If \(U \in I\) then \(\theta(U) \not\in N_t\).

**Proof.** The group \(PSp_{2t}(q)\) acts transitively on \(I\) so for any given \(U \in I\), \(\theta(U) \in N_t\) if and only if \(\theta(W) \in N_t\) for all \(W \in I\). However the \(\theta(W)\)'s, \(W \in I\), span \(F_t\). Hence \(\theta(U) \not\in N_t\), for all \(U \in I\). \(\square\)

**Lemma 3.2.** Suppose that the image of \(S\) in \(F_t\) spans a subspace \(L_t\) of dimension \(2^t - 1\) such that \(L_t \cap N_t = \emptyset\). Suppose also that \(\theta_S\) is injective. Then the following occur:
(a) If \( U, W \in \mathcal{I} \) and if \( \theta(U), \theta(W) \) have the same projection onto \( L_t \), then for any \( g \in PSp_{2t}(q) \), the points \( \theta(g(U)), \theta(g(W)) \) have the same projection onto \( L_t \).

(b) If there exist \( U, W \in \mathcal{I} \) with \( U \neq W \) and \( U \cap W \neq \emptyset \), such that \( \theta(U), \theta(W) \) have the same projection onto \( L_t \), then there exists \( U', W' \in \mathcal{I} \) with \( U' \cap W' = \emptyset \), such that \( \theta(U'), \theta(W') \) have the same projection onto \( L_t \).

**Proof.**

(a) To say that \( \theta(U), \theta(W) \) have the same projection onto \( L_t \) is equivalent to saying that the line \( \theta(U)\theta(W) \) meets \( N_t \) in a point. If \( \theta(U)\theta(W) \) meets \( N_t \) in a point then so does \( g(\theta(U))g(\theta(W)) \), i.e., so does \( \theta(g(U))\theta(g(W)) \) and hence \( \theta(g(U)), \theta(g(W)) \) have the same projection onto \( L_t \).

(b) Suppose first that \( \dim(U \cap W) > t/2 - 1 \). Let \( X, Y \) be disjoint elements of \( \mathcal{I} \) and choose \( X_1, X_2 \) to be disjoint subspaces of \( X \) of dimension \( t \cdot \dim(U \cap W) - 1 \). For each \( i \), let \( Y_i = X_i^\perp \cap Y \) and let \( Z_i = X_i \oplus Y_i \). Then \( Z_1, Z_2 \in \mathcal{I} \) and \( \dim(Z_1 \cap Y) = \dim(Z_2 \cap Y) = \dim(U \cap W) \). It is always possible to find elements \( g_i \) of \( PSp_{2t}(q) \) such that \( g_i(U) = Z_i, g_i(W) = Y \) (an extension of Witt’s Theorem). Applying (a) we see that \( \theta(Z_1), \theta(Y) \) have the same projection onto \( L_t \) and so do \( \theta(Z_2), \theta(Y) \). Thus \( \theta(Z_1), \theta(Z_2) \) have the same projection onto \( L_t \) and \( Z_1 \cap Z_2 = Y_1 \cap Y_2 \) has smaller dimension than \( U \cap W \). Thus we may replace \( U, W \) by subspaces with a smaller intersection, until \( \dim(U \cap W) \leq t/2 - 1 \).

Now suppose \( \dim(U \cap W) \leq t/2 - 1 \) and again choose \( X, Y \) to be disjoint elements of \( \mathcal{I} \). Choose subspaces \( X_1 \) of \( X \) and \( Y_1 \) of \( X_1^\perp \cap Y \) such that \( X_1, Y_1 \), have dimension equal to \( \dim(U \cap W) \). Let \( X_2 \) be a complement to \( X_1 \) in \( Y_1^\perp \cap X \) and \( Y_2 \) a complement to \( Y_1 \) in \( X_1^\perp \cap Y \). Then \( X_2 + Y_2 \) is non–isotropic and contains a totally isotropic subspace \( Z_1 \) of dimension equal to \( \dim(X_2) \) but disjoint from both \( X_2, Y_2 \). Then \( Z = X_1 + Y_1 + Z_1 \in \mathcal{I} \) and \( \dim(X \cap Z) = \dim(Y \cap Z) = \dim(U \cap W) \). As above it is always possible to find elements \( g_1, g_2 \) of \( PSp_{2t}(q) \) such that \( g_1(U) = X, g_1(W) = Z, g_2(U) = Y, g_2(W) = Z \). Applying (b) we see that \( \theta(X), \theta(Z) \) have the same projection onto \( L_t \) and the same applies to \( \theta(Y), \theta(Z) \). Hence \( \theta(X), \theta(Y) \) have the same projection onto \( L_t \) and we may take \( U' = X, W' = Y \).

**Proposition 3.3.** Under the assumptions of Lemma 3.2, the mapping \( \iota_t \) is injective

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Proof. Suppose that the mapping $\iota_t$ from $I$ to $M_t$ is not injective. Then for some $U, W \in I$ with $U \neq W$ the points $\theta(U), \theta(W)$ have the same projection onto $L_t$. By Lemma 3.2, we may assume that $U \cap W = \emptyset$. Now the group $PSp_{2t}(q)$ acts transitively on pairs of disjoint totally isotropic $(t - 1)$–dimensional subspaces of $\Sigma_0$, so there must exist $g \in PSp_{2t}(q)$ such that $g(U), g(W) \in S$ and by Lemma 3.2 (a), $\theta(g(U)), \theta(g(W))$ have the same projection onto $L_t$. Here however, $\theta(g(U)), \theta(g(W)) \in L_t$ and we have a contradiction to the assumption that $\theta|_S$ is injective.

We recall here, in outline, the embedding when $q$ is even of $PG(1, q^t)$ as a partial ovoid $O$ on a quadric $Q^+(2^t - 1, q)$ given in [7]. Fix a basis for a 2–dimensional vector space over $GF(q^t)$ and map $(a, b)$ to $(a, b) \otimes (a^{q^t-1}, b^{q^t-1})$. The points on the partial ovoid have coordinates of the form $a_i^1 + b_i^1 + \ldots + a_i^{q^t-1} + b_i^{q^t-1}$ where $\{i_1, i_2, \ldots\} \cup \{j_1, j_2, \ldots\}$ represents a partition of $\{1, q, q^2, \ldots, q^{t-1}\}$, and all such possible partitions occur.

The following can be found in [1, 4.9] or, in part, in [18, Lemma 5.9], but a quick proof is more illuminating.

Proposition 3.4. Let $O$ be the partial ovoid described above. Then $O$ lies on a unique quadric in $PG(2^t - 1, q)$.

Proof. As shown in [7], the partial ovoid lies on a quadric $Q$, say with quadratic form $Q'$ and associated alternating form having matrix $A$, and $O$ admits $PSp_{2t}(q^t)$ as an absolutely irreducible automorphism group. If $Q'$ is any quadric containing $O$, with corresponding $Q', A'$, then $A^{-1}A'$ commutes with $PSp_{2t}(q^t)$ and by Schur's Lemma is therefore a scalar matrix. Given that $O$ must span $PG(2^t - 1, q)$, $Q'$ is completely determined by taking 0 on $O$ and by its alternating form. Thus $Q'$ is a scalar multiple of $Q$ and $Q' = Q$.

We are now in a position to prove our main result.

Theorem 3.5. Under the twisted tensor product group embedding $PSp_{2t}(q^t) \leq P\Omega_{2t}^+(q), q \geq 4$ even, $t \geq 3$, an intermediate $C_3$–embedding occurs: $PSp_{2t}(q^t) \leq PSp_{2t}(q) < P\Omega_{2t}^+(q)$.

Proof. Recall from Section 2 the construction of a spread $S$ of totally isotropic $(t - 1)$–subspaces of $\Sigma_0 = PG(2t - 1, q)$. We have seen that $S$ is preserved by a group $PSp_{2t}(q^t)$ acting on $\Sigma_0$ (regarding $Sp_{2t}(q^t)$ as $PSp_{2t}(q^t)$)
in this projective context). From [9, Theorems 1, 3], the full stabilizer of $S$ in $PSp_{2t}(q)$ is the group $PSp_2(q^t) \cdot \langle \Psi \rangle$ where $\langle \Psi \rangle = \Gal(GF(q^t), GF(q))$, and $S$ is a regular spread. The construction given in Section 2 can be described as follows, noting that the subspaces $E_i$ now correspond to lines in $\Sigma = PG(2t - 1, q^t)$. There exists a non–isotropic line $\ell$ of $\Sigma$ such that, denoting by $\ell^\Psi$, $i = 1, \ldots, t - 1$, the images of $\ell$ under a semilinear map corresponding to $\Psi^t$ (sometimes referred to as conjugate of $\ell$ with respect to $GF(q^t)$), each fibre $\pi$ of $S$ is of the type $\langle P_0, P_1, \ldots, P_{t-1} \rangle$, where $P_i \in \ell^\Psi$ and $P_1^\Psi$.

We can choose coordinates for $\Sigma$ in such a way that $\pi \circ \ell$ and $\ell^\Psi$ are given by matrices having the coordinates of $X_0, X_1, \ldots, X_{t-1}$ as their rows. The Plücker coordinates are the $t \times t$ determinants of $t \times t$ submatrices, written in some pre–determined order. For the $(t - 1)$–subspaces of $S$ with simplex $P_0, P_1, \ldots, P_{t-1}$, we have Plücker coordinates given by determinants of submatrices of

$$
\begin{pmatrix}
  a & b & 0 & 0 & \ldots & \ldots & 0 & 0 \\
  0 & 0 & a^q & b^q & \ldots & \ldots & 0 & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
  0 & 0 & 0 & 0 & \ldots & \ldots & a^{q^{t-1}} & b^{q^{t-1}}
\end{pmatrix},
$$

for some $a, b \in GF(q^t)$ (not both zero). All but $2^t$ of the $t \times t$ submatrices necessarily have determinant zero. The remaining Plücker coordinates are precisely those of the partial ovoid $O$ described above. We know from [7] that these points lie on a quadric $Q^+(2^t - 1, q^t)$, that they lie in a subgeometry $PG(2^t - 1, q)$ and further they lie on a quadric $Q^+(2^t - 1, q)$ in that subgeometry.

As we have seen $G_0$ (whose points are $(t - 1)$–subspaces of $\Sigma_0$) is the restriction of $G$ to a subgeometry $PG((2^t) - 1, q)$. Hence the points of $G$ corresponding to $S$ lie in a $(2^t - 1)$–dimensional subspace $L_t$ of $PG((2^t) - 1, q)$. Moreover the action of $PSp_2(q^t)$ on $L_t$ is precisely that on the partial ovoid $O$ so it is absolutely irreducible. We know that $L_t$ lies in a subspace $F_t$ of
$PG\left(\binom{2t}{t} - 1, q\right)$ fixed by $PSp_{2t}(q)$ and further that $F_t$ has a unique maximal subspace $N_t$ fixed by $PSp_{2t}(q)$, with $F_t/N_t = M_t$ having dimension $2^t - 1$. By Lemma 3.1, $L_t$ cannot be a subspace of $N_t$. Further, $PSp_2(q^t) \leq PSp_{2t}(q)$ acts irreducibly on $L_t$, so $L_t \cap N_t = \emptyset$. The $(t-1)$–subspaces in $S$ correspond to distinct points in $L_t$. Hence Proposition 3.3 applies and the mapping $\iota_t$ from $I$ to $M_t$ is injective.

We know that $M_t$ corresponds to the spin module for $Sp_{2t}(q)$ and that therefore the totally isotropic $(t-1)$–subspaces of $\Sigma_0$ are mapped to points on a quadric in $M_t$. However $L_t$ is a complementary subspace to $N_t$ in $F_t$ so $M_t$ may be regarded as the projection of $F_t$ onto $L_t$ and the subspaces in $S$ correspond to the points with coordinates already given. Those points, as shown in Proposition 3.4, form a partial ovoid $O$ lying on a unique quadric.

It now follows that the embedding of $PSp_2(q^t)$ via the twisted tensor product as a subgroup of $POm_{2t}^+(q)$ may be obtained by first embedding $PSp_2(q^t)$ as an Aschbacher’s class $C_3$ subgroup of $PSp_2(q)$ and then embedding $PSp_{2t}(q)$ via its spin representation in $POm_{2t}^+(q)$.

**Corollary 3.6.** $N_{POm_{2t}^+}(PSp_2(q^t))$ is the stabilizer of $O$ in $POm_{2t}^+(q)$ and it is a maximal subgroup of $PSp_2(q)$ (spin representation) if and only if $t$ is a prime.

**Proof.** The first part follows from [7, Theorem 4.9]. The second part follows from the previous theorem and [10], [11], [12], [13].

**Remark 3.7.** A consequence of the arguments given in proving Theorem 3.5 is that the quadric on which $Sp_{2t}(q)$ acts via its spin representation may be recovered from a complete partial ovoid corresponding to a regular spread of totally isotropic $(t-1)$–subspaces of $PG(2t - 1, q)$. Once the partial ovoid is constructed in $PG(2^t - 1, q)$, the quadric is uniquely determined.

An argument similar to that in the proof of Theorem 3.5 applies to the twisted tensor product group embedding $PSp_4(q^t) < POm_{4t}(q)$, $q$ even, and we obtain the following theorem.

**Theorem 3.8.** Under the twisted tensor product embedding $PSp_4(q^t) < POm_{4t}(q)$, $q$ even, an intermediate $C_3$–embedding occurs: $PSp_4(q^t) < PSp_{4t}(q) < POm_{4t}^+(q)$.

**Remark 3.9.** More generally, the twisted tensor product construction embeds $PSp_{2m}(q^t)$ as a $C_3$–subgroup of $PSp_{2mt}(q)$ which in turn is embedded
in $PO^+_m(q)$ via the spin representation. As $2^{mt} = (2m)^t$ only for $m = 1$ or 2, there is no prospect of an intermediate $C_3$–embedding with respect to $PSp_{2m}(q^t)$ with $m > 2$.

In the following Proposition the case $q = 2$ is considered.

**Proposition 3.10.** Under the twisted tensor group embedding $PSp_2(2^t) \leq PO^+_2(2)$, an intermediate alternating group $A_{2t+1}$ occurs. The normalizer $N$ of $PSp_2(2^t)$ in $A_{2t+1}$ is $PSp_2(2^t) \cdot t$ and it is a maximal subgroup of $A_{2t+1}$.

**Proof.** The structure of $N$ has been determined in [7] and $N$ has been identified as the stabilizer of $O$ in $PO^+_2(2)$. The maximality of $N$ in $A_{2t+1}$ follows from the classification of 2–transitive finite permutation groups, see [5], [7, Result 4.4]. In fact $PSp_2(2^t) \cdot t$ can only be $P\Gamma L_2(2^t)$. \qed

**Remark 3.11.** It is a consequence of [16, Theorem I] that $Sp_6(q)$, $q$ even, in its spin representation, is a maximal subgroup of $PO^+_3(q)$ and from [14] we have that $A_{2t+1}$ is a maximal subgroup of $PO^+_2(2)$.

## 4 Some other examples

In [8] we consider an embedding of $O_n(q^2)$ in $O_{2n}(q)$ as a group preserving a spread of lines in $PG(2n-1, q)$. We noted there that a geometric construction of the spread arises from the so–called trace–trick, in essence the construction in Section 2.

We assume that $q$ is odd and $t = 2$. The construction in Section 2 yields a spread of lines corresponding to points of $PG(n-1, q^2)$, fixed by $O_n(q^2)$. We pick the cases $n = 3, 4$ as being of particular interest because of the existence of classical 1–systems [24].

Consider first the case $n = 3$. Here there are two possibilities: $O_3(q^2) \leq O_5^+(q)$ and $O_3(q^2) \leq O_6^-(q)$ and both occur (the sign is determined by the precise choice of the quadratic form $Q_1$ on $E_1$). We consider the embedding $O_3(q^2) \leq O_5^+(q)$. The spread $\mathcal{S}$ consists of $q^4 + q^2 + 1$ lines, but of these precisely $q^2 + 1$ are totally singular. The set $\mathcal{M}$ of totally singular lines in $\mathcal{S}$ form a 1–system of the Klein quadric, i.e., each plane of $Q^+(5, q)$ containing one line of $\mathcal{M}$ is disjoint from the others. The image of $\mathcal{M}$ under the Grassman map is a set of $q^2 + 1$ points in $PG(8, q)$ preserved by $O_3(q^2)$. From what we have seen in Section 2, the image of $\mathcal{M}$ is a subset of the embedding of $PG(2, q^2)$ in $PG(8, q)$ and the coordinates correspond to the twisted tensor
product embedding. Thus we see from [6] that the image of \( \mathcal{M} \) under the Grassman map is a partial ovoid of the parabolic quadric \( \mathcal{Q}(8, q) \) lying also on a rational curve in \( PG(8, q) \).

The case \( n = 4 \) is somewhat similar. Here we consider \( O^-_4(q^2) \leq O^-_8(q) \).

The spread \( \mathcal{S} \) contain \((q^2 + 1)(q^4 + 1)\) lines of which \( q^4 + 1 \) are totally singular and form a 1–system \( \mathcal{N} \) of an elliptic quadric \( \mathcal{Q}^-(7, q) \) of \( PG(7, q) \). This time the image of \( \mathcal{N} \) under the Grassman map is a set of \( q^4 + 1 \) points on a hyperbolic quadric \( \mathcal{Q}^+(15, q) \) of \( PG(15, q) \) and the given points form a partial ovoid.

References


