On some maximal subgroups of unitary groups

Antonio Cossidente
Dipartimento di Matematica - Università della Basilicata
I-85100 Potenza - Italy
cossidente@unibas.it

Oliver H. King
Department of Mathematics - University of Newcastle
Newcastle Upon Tyne
NE1 7RU, U.K.
O.H.King@ncl.ac.uk

Abstract

The maximality of certain symplectic subgroups of unitary groups $PSU_n(K)$, $n \geq 4$, ($K$ any field admitting a non-trivial involutory automorphism) belonging to the class $C_5$ of Aschbacher is proved. Furthermore some related geometry in the case $n = 4$ and $K$ finite is investigated.

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1 Introduction

The seminal contribution to the classification of the maximal subgroups of the finite classical groups was Aschbacher’s theorem [3]. Aschbacher defines eight “geometric” classes $C_1 \ldots, C_8$, of subgroups of the finite classical groups and proves that a maximal subgroup either belongs to one of these classes or has a non-abelian simple group as its generalized Fitting subgroup.

In their book [12], P.B. Kleidman and M.W. Liebeck have identified the members of the eight classes for modules with dimension greater than 12,
and P.B. Kleidman [11] completed the work for modules with dimension up to 12, but without furnishing proof of his results. However, their analysis relies heavily upon the classification of finite simple groups. Various authors, too many to be quoted here, have used Aschbacher’s theorem to elucidate much of the maximal subgroup structure on the finite classical groups.

At least seven of the eight Aschbacher’s classes can be described as stabilizers of geometric configurations. Consequently, one might prefer a direct approach to the classification of maximal subgroups, which is free of the classification of finite simple groups, using the natural representations of classical groups. Certainly, this is the approach adopted by R.H. Dye and O.H. King, see for instance [8] and papers quoted therein, [10],[5]. They elucidate in many cases how maximal subgroups of finite classical groups and geometry are closely related.

Shangzhi Li obtained several results on maximal subgroups of classical groups, allowing the ground field to be infinite. Mainly he adopted an elementary but rather technical matrix approach, see [13].

In this paper, we are interested in Aschbacher’s class $C_5$. For a classical group $G$ acting on an $n$–dimensional vector space $V$ over a field $F$, the class $C_5$ is the collection of normalizers of the classical groups acting on the $n$–dimensional vector spaces $V_K$ over maximal subfields $K$ of $F$ such that $V = F \otimes_K V_K$.

Apart from the work of Kleidman and Liebeck, very little has been done for subgroups belonging to this class. As far as we know there are just three papers by Li [14], [15] and Li and Zha [16] devoted to this case.

Here, we study maximal symplectic subgroups belonging to the class $C_5$ of the unitary group $PSU_n(K)$, $n \geq 4$ even, $K$ any field admitting a non-trivial involutory automorphism. When the ground field is finite, our result has also been obtained by Li and Zha in [16] using suitable subgroups of unitary transvections.

Then, we concentrate on the case $n = 4$ and $K$ finite. It is well known that the special unitary group $PSU_4(q^2)$ is isomorphic to the orthogonal group $P\Omega^-_6(q)$ [6], [7] and this isomorphism allows us to investigate some geometry of symplectic subgeometries embedded in a Hermitian surface of $PG(3, q^2)$.

Finally, we remark again that the whole philosophy of our approach is to make maximum use of the underlying geometry.
2 The embedding $Sp_n(K_0) \leq SU_n(K)$

Let $K$ be a commutative field admitting a non-trivial involutory automorphism $\lambda \mapsto \bar{\lambda}$, with $K_0$ the fixed subfield. In this Section we prove our main result on the maximality of certain symplectic groups inside the unitary group $PSU_n(K)$.

Suppose that $V$ is an $n$–dimensional vector space over $K_0$ and $A$ is a non–degenerate alternating bilinear form on $V$. Let $\omega$ be an element of $K \setminus K_0$. Then $K = K_0 \oplus K_0\omega$ and there is a vector space $W = V \otimes_{K_0} K = \{ (\alpha + \beta \omega)v | \alpha, \beta \in K_0, v \in V \}$. Any vector $w \in W$ can be written as $w = \sum v_i \otimes (a_i + b_i\omega) = \sum (v_i \otimes 1)a_i + \sum (v_i \otimes \omega) b_i = (\sum v_i a_i) \otimes 1 + (\sum v_i b_i) \otimes \omega = w_1 + w_2 \omega$. [Also if $\omega^2 = \gamma \omega + \delta$, then $(\alpha + \beta \omega)(w_1 + w_2 \omega) = (\alpha w_1 + \beta \delta w_2) + (\beta w_1 + \beta \gamma w_2 + \alpha w_2) \omega$.] There is a natural extension of $A$ to an anti-hermitian form $C$ on $W$ given by:

$$C(w_1 + w_2 \omega, v_1 + v_2 \omega) = A(w_1, v_1) + \omega \bar{A}(w_2, v_2) + \omega A(w_2, v_1) + \bar{\omega} A(w_1, v_2).$$

If $\text{char}K = 2$, then $C$ is already an hermitian form. In all cases there exists a $\tau \in K$ such that $\bar{\tau} = -\tau$ (as follows from Hilbert’s Theorem 90) and $\tau C$ is a hermitian form with the same group as $C$. We write $H$ for $\tau C, U_n(K)$ for the unitary group of $H$, $Sp_n(K_0)$ for the symplectic group of $A$. We obtain the embedding $Sp_n(K_0) \leq SU_n(K)$. Note that $H$ does not depend on the choice of $\omega$. Factoring out scalars, we get the embedding $PSp_n(K_0) \leq PSU_n(K)$.

Let $x = w_1 + w_2 \omega \in W$. Then, with respect to $H$, $x$ is isotropic if and only if $C(x, x) = 0$, i.e., if and only if $\omega A(w_2, w_1) + \bar{\omega} A(w_1, w_2) = 0$, i.e., if and only if $\omega A(w_2, w_1) = \bar{\omega} A(w_2, w_1)$, i.e., if and only if $A(w_1, w_2) = 0$. In particular every vector in $V$ is isotropic with respect to $H$. Suppose that $0 \neq v \in V$ and that $t$ is a unitary transvection centred on $v$. Then $t : x \mapsto x + \lambda H(x, v)v$ for some $\lambda \in K$ such that $\bar{\lambda} = -\lambda$. If $x \in V$, then $t(x) = x + \lambda \tau A(x, v)v$ with $\lambda \tau \in K_0$, so $t$ fixes $V$ globally and the restriction of $t$ to $V$ is a symplectic transvection, i.e., $t \in Sp(n, K_0)$.

Let $\mathcal{H}$ be the Hermitian variety of $PG(n-1, K)$ associated with $H$. Let $\Sigma$ be the set of points of the $PG(n-1, K_0)$ corresponding to $V$, considered as a subset of $\mathcal{H}$ inside $PG(n-1, K)$. We can regard $Sp(n, K_0)$ and $SU(n, K)$ as acting on $PG(n-1, K)$. Then $Sp(n, K_0)$ fixes $\mathcal{H}$ globally and has $\Sigma$ as one orbit. Suppose that $x$ and $y$ are isotropic vectors in $W$ corresponding to points of $\mathcal{H} \setminus \Sigma$. Then $x = w_1 + w_2 \omega$, $y = v_1 + v_2 \omega$, for some linearly
independent $w_1, w_2 \in V$ and some linearly independent $v_1, v_2 \in V$ and by Witt’s Theorem there is an element of $Sp(n, K_0)$ taking $w_i$ to $v_i$ for each $i$, i.e., taking $x$ to $y$. Hence $Sp(n, K_0)$ has exactly two orbits on $H$.

Let $G_n$ denote the stabilizer of $\Sigma$ in $SU(n, K)$ and let $F$ be a subgroup of $SU(n, K)$ such that $G_n < F$. Then $F$ has a single orbit of points on $H$. If $t$ is any unitary transvection in $SU(n, K)$, centred on $y$ say, then there exists $f \in F$ such that $f(y) \in V$ and $ftf^{-1}$ is a transvection centred on $f(y)$. Thus $ftf^{-1} \in Sp(n, K_0)$ and $t \in F$. It is well known that $SU_n(K)$, $n \geq 4$, is generated by its transvections \cite{segre1947permutable}, \cite{segre1948permutable} and so $F = SU_n(K)$, and $G_n$ is maximal in $SU_n(K)$. By the standard theorem for subgroups of quotients groups, the stabilizer $\overline{G_n}$ of $\Sigma$ in $PSU_n(K)$ is maximal in $PSU_n(K)$.

It is of some interest to know the structure of $G_n$. Suppose that $g \in G_n$ and that $v_1, \ldots, v_m, v_{m+1}, \ldots, v_n$ (with $n = 2m$) is a symplectic basis for $V$ with respect to $A$ (i.e., $A(v_i, v_{m+j}) = \delta_{ij}$). Then $A(g(v_i), g(v_{m+j})) = 0$ if and only if $i \neq j$ and by Witt’s Theorem there exists $h_1 \in Sp(n, K_0)$ such that $h_1 g(v_i) = \lambda_i v_i$ for some $\lambda_i \in K \ (1 \leq i \leq n)$. As $h_1 g$ fixes $\Sigma$, it follows that for all $i > 1$, $\lambda_i = \beta_i \lambda_i$ for some $\beta_i \in K_0$. Hence $h_1 g = \lambda_1 I_n h_2$, where $h_2 \in GL(n, K_0)$ and fixes $\Sigma$, i.e., $h_2 \in GSp(n, K_0)$ (the general symplectic group, consisting of elements of $GL(n, K_0)$ that preserve $A$ up to a scalar).

It is now clear that $g$ can be expressed as the product of a scalar matrix and an element of $GSp(n, K_0)$. Indeed all such products stabilize $\Sigma$. Hence $G_n$ consists of all such products lying in $SU(n, K)$. The image of $G_n$ in $PGL(n, K)$ is then simply $PGL(n, K_0) \cap SU(n, K)$.

We conclude that:

**Theorem 2.1.** $G_n$ is a maximal subgroup of $SU_n(K)$ containing $Sp(n, K_0)$ and $G_n = (GSp(n, K_0), G_K) \cap SU(n, K)$ where $G_K$ is the group of scalar matrices in $GL(n, K)$. The stabilizer $\overline{G_n}$ of $\Sigma$ in $PSU_n(K)$ is a maximal subgroup of $PSU_n(K)$ containing $PSP(n, K_0)$ and $\overline{G_n} = PGP(n, K_0) \cap PSU(n, K)$.

### 3 Permutable polarities

When the ground field $K$ is finite, a natural approach in proving the maximality of $\overline{G_n}$ in $PSU_n(q^2)$ seems to be Segre’s theory on permutable polarities as described in his celebrated paper \cite{segre1952permutable}. Here, we briefly discuss this theory.
In $\text{PG}(n-1, q^2)$ a non–singular Hermitian variety is defined to be the set of all absolute points of a non–degenerate unitary polarity, and is denoted by $\mathcal{H}(n-1, q^2)$.

For an Hermitian variety $\mathcal{H} = \mathcal{H}(n-1, q^2)$ we have that [19] :

1. the number of points is $[q^n + (-1)^{n-1}][q^{n-1} - (-1)^{n-1}]/(q^2 - 1)$.
2. the number of generators (maximal totally singular subspaces) is $(q^3 + 1)(q^5 + 1)\ldots(q^{2m+1} + 1)$, if $n = 2m + 1$, and $(q + 1)(q^3 + 1)\ldots(q^{2m+1} + 1)$, if $n = 2m + 2$.

Let $\mathcal{A}$ be a symplectic polarity permuting with the Hermitian polarity $\mathcal{U}$ associated with $\mathcal{H}(n-1, q^2)$. Set $\mathcal{V} = \mathcal{AU} = \mathcal{UA}$. Then $\mathcal{V}$ is a non–linear collineation and $\mathcal{V}$, $\mathcal{A}$ and $\mathcal{U}$ together with the identity map form a four–group. From [19, pg. 132], the points and lines fixed by $\mathcal{V}$ form a configuration $\mathcal{W}$ on $\mathcal{H}(n-1, q^2)$. As B. Segre pointed out [19, p. 128, 132], $\mathcal{V}$ fixes $((q^n + 1)/(q-1))$ points on $\mathcal{H}(n-1, q^2)$ but no point outside $\mathcal{H}(n-1, q^2)$, and leaves $((q^n - 1)(q^{n/2} - 1))/((q - 1)(q^2 - 1))$ lines of $\mathcal{H}(n-1, q^2)$ invariant so that each fixed point is incident with $((q^{n/2} - 1)/(q-1))$ invariant lines and each invariant line is incident with $q + 1$ fixed points. This symmetric configuration extends to a $(n-1)$–dimensional projective space $\Sigma \cong \text{PG}(n-1, q)$. In this context, $\Sigma$ is naturally equipped with the symplectic polarity $\mathcal{A}$ whose absolute lines are the lines of the above symmetric configuration. If $\mathcal{H}(n-1, q^2)$ has canonical equation $\sum_{i=0}^{n-1} X_i^{q+1} = 0$, then $\Sigma$ can be described as the subset of points of $\mathcal{H}(n-1, q^2)$ whose coordinates are of the form $x_{2i-1} = \rho x_{2i}^q$, $i = 0, 1, \ldots, (n-2)/2$, where $\rho \in GF(q^2)$ such that $\rho^{q+1} = -1$.

Finally, since $\mathcal{A}$ and $\mathcal{U}$ commute, a $\mathcal{V}$–fixed point $P$ on $\mathcal{H}(n-1, q^2)$ admits the same conjugate hyperplane $P^\perp$ with respect to both $\mathcal{A}$ and $\mathcal{U}$.

The stabilizer of $\Sigma$ in $\text{PSU}_n(q^2)$ turns out to be $\text{PSU}_n(q) \cdot ((2, q-1)(q+1, n/2))/((q + 1, n))$, see Section 2 and [12, Proposition 4.5.6].

With this geometric setting, the main result of Section 2 can be easily re–formulated in a more geometric way. For instance, to prove that the group $\overline{G}_n$ has exactly two orbits on totally isotropic points of $\mathcal{H}(n-1, q^2)$ one can argue as follows.

Let $P_1$ and $P_2$ be points of $\mathcal{H}(n-1, q^2) \setminus \Sigma$. The tangent hyperplanes $P_1^\perp$ and $P_2^\perp$ to $\mathcal{H}(n-1, q^2)$ at $P_1$ and $P_2$, meet $\Sigma$ in $(n-3)$–dimensional subspaces $\Sigma_1$ and $\Sigma_2$, respectively [20], and $\ell = \Sigma_1 \cap \Sigma_2$ is a line of $\Sigma$. Assume that $\ell$ is not isotropic. Then, $\ell^\perp$ with respect to $\mathcal{A}$ is a $(n-1)$–dimensional subspace of $\Sigma$ skew to $\ell$. Extend $\ell$ to a line $\overline{\ell}$ over $GF(q^2)$. Then $\overline{\ell} \subseteq P_1^\perp \cap P_2^\perp$ and so $\overline{\ell}^\perp$ passes through $P_1$ and $P_2$ and hence belongs to $P_1^\perp \cap P_2^\perp$, a contradiction. Hence, $\ell$ is a totally isotropic line. The stabilizer of $\ell$ in $\overline{G}_n$
acts on \(\ell\) as \(PSL_2(q)\) acts on \(PG(1, q) \subset PG(1, q^2)\). Since \(G_n\) acts transitively on totally isotropic lines of \(\Sigma\) and the number of totally isotropic lines of \(\Sigma\) is \(((q^n - 1)/(q - 1))((q^{n-2} - 1)/(q^2 - 1))\), we are done.

The maximality proof is essentially the same as in Section 2 rephrased in projective terms.

4 Symplectic subgeometries and their groups

In this Section we study some geometry of the embedding \(G_4 \leq PSU_4(q^2)\). This gives us information on possible intersection sizes of two symplectic groups \(PSp_4(q)\) inside the unitary group \(PSU_4(q^2)\). Notice that, if \(n = 4\), there are \(q^2(q^3 + 1)\) symplectic subgeometries embedded in \(H(3, q^2)\) [19].

**Theorem 4.1.** Two symplectic subgeometries in \(H(3, q^2)\) meet in 0, \(q + 1\) or \(2(q + 1)\) points. In the case of \(q + 1\) points, the points lie on a totally isotropic line. In the case of \(2(q + 1)\) points, the points lie on a hyperbolic pair. If \(q = 2\), no two disjoint symplectic subgeometries exist.

**Proof.**

We use the duality between \(H(3, q^2)\) and \(Q^{-}(5, q)\) [6], [7], [18]. Symplectic subgeometries of \(H(3, q^2)\) correspond to hyperplanes of \(PG(5, q)\) meeting \(Q^{-}(5, q)\) in a parabolic quadric. Thus the intersection of two subgeometries corresponds to the intersection of two such hyperplanes, which is a solid. There are three types of solid on \(Q(4, q)\), accordingly as the quadric is met in an elliptic quadric, a cone or a hyperbolic quadric. A solid meeting the quadric in an elliptic quadric contains 0 lines of the quadric; a solid meeting the quadric in a cone contains \(q + 1\) lines of the quadric and, finally, a solid meeting the quadric in a hyperbolic quadric contains \(2(q + 1)\) lines of the quadric. In the second case, the vertex of the cone is on all \(q + 1\) lines, so this case corresponds to two symplectic subgeometries meeting in \(q + 1\) points, all on a line. In the third case, each regulus of the hyperbolic quadric corresponds to a hyperbolic line, and as the reguli are opposite, the two hyperbolic lines are polar. The number of parabolic quadrics \(Q(4, q)\) on an elliptic quadric \(Q^{-}(3, q)\) in \(Q^{-}(5, q)\) is \(q - 1\) and \(q - 1 \geq 2\) if \(q > 2\). Thus the case of disjoint subgeometries does not occur if \(q = 2\).

The previous theorem yields information on possible intersection sizes of two copies of \(PSp_4(q)\) inside \(PSU_4(q^2)\). We have the following theorem.
Theorem 4.2. Let $G, G'$ be the stabilizers in $PSU_4(q^2)$ of two symplectic geometries embedded in $H(3, q^2)$. Set $K = G \cap G'$. Then one of the following cases occur. $K$ is either the stabilizer of an elliptic congruence or, the stabilizer of a totally isotropic line or, the stabilizer of a hyperbolic pair. In all cases $K$ is a maximal subgroup of $G$.

Proof. It follows from Theorem 4.1 and the classification of maximal subgroups of $PSp_4(q)$, [17], [9]. It turns out that the stabilizer of an elliptic congruence is actually the stabilizer of a complete $(q^2 + 1)$--span of $H(3, q^2)$ as shown in [1], [2].

Notice that $PSU_4(q^2)$ has one or two classes of subgroups isomorphic to $PSp_4(q)$ according as $q$ is even or odd. This depends on the fact that the group $PO_6^-(q)$ has either one one or two orbits on parabolic quadric sections of $Q^-(5, q)$.

Remark 4.3. It would be interesting to have information about the intersection of two distinct symplectic subgeometries embedded in $H(n - 1, q^2)$, for $n \geq 6$. For $n = 6$ and $q = 2$, we found, by computer, that two symplectic subgeometries embedded in $H(5, 4)$ can meet in 3, 9 or 15 points, corresponding to the points of a totally isotropic line, three pairwise conjugate lines, a subgeometry $PG(3, 2)$ through a totally isotropic line, respectively.

Remark 4.4. We note that the embedding $PSp_n(q) \leq PSU_n(q^2)$ is also interesting from a graph theory point of view since the permutation character of $1_{PSp_n(q^2)}$, $n$ even, is multiplicity free, and this property is closely related to distance transitive graphs [4].

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References


