Maximal orthogonal subgroups of finite unitary groups

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Abstract

Certain orthogonal subgroups of finite unitary groups belonging to the fifth Aschbacher class, $C_5$, are studied and their maximality is proved using the geometry of permutable polarities.

Keywords: Finite unitary group, Finite orthogonal group, Maximal subgroup, Subgeometry.

1 Introduction

Aschbacher’s Theorem, proved in [1], states that a subgroup of a finite classical group either belongs to one of eight “geometric” classes $C_1, \ldots, C_8$, or has a non-abelian simple group as its generalized Fitting subgroup. At least seven of the eight Aschbacher’s classes can be described as stabilizers of geometric configurations. In consequence, one might prefer a direct geometric approach to the classification of maximal subgroups over one dependent on the classification of finite simple groups. This is the approach adopted by R.H. Dye and O.H. King in a number of papers (e.g., [7], [12] and [13]) and
is also the approach taken by the authors in [3] and [4].

As in [3], we are interested in Aschbacher’s class $C_5$. For a classical group $G$ acting on a $n$–dimensional vector space $V$ over a field $K$, the class $C_5$ is the collection of normalizers of the classical groups acting on the $n$–dimensional vector spaces $V_0$ over maximal subfields $K_0$ of $K$ such that $V = K \otimes_{K_0} V_0$. Apart from the work of Kleidman and Liebeck [14], very little has been done for subgroups belonging to this class. As far as we know there are just three other papers by Li [15], [16] and Li and Zha [17] devoted to this class.

This paper can be considered a continuation of [3], where the maximality of certain symplectic groups of the unitary groups was proved. Here we prove the maximality of certain orthogonal subgroups of the finite unitary group $PSU_n(q^2)$ for $n \geq 3$. Our main result is expressed in terms of, and our approach to the proof depends on, the geometry of permutable polarities in projective spaces, described in [19]. We prove:

**Theorem**

Suppose that $n \geq 3$ and that $q$ is odd. If $U$ a non–degenerate unitary polarity on $PG(n-1,q^2)$ and if $B$ is an orthogonal polarity commuting with $U$, then the set of absolute points of $U$ fixed by the non-linear collineation $V = UB$ forms a non-degenerate quadric in a subgeometry $PG(n-1,q)$. The stabilizer of the quadric in $PSU_n(q^2)$ is maximal except when $n = 3$ and $q = 3$ or 5 and when $n = 4$, $q = 3$ and the quadric is hyperbolic.

Our approach is essentially an induction argument in which a reduction to lower dimension is achieved via “hyperbolic rotations”. These are elements of order $q-1$ that “rotate” most of the points on a hyperbolic line and leave fixed two points of the line and the points on its orthogonal complement. To a certain extent, our result has also been obtained by Li and Zha in [16], using suitable subgroups of unitary transvections.

Throughout the paper we will assume that $q$ is a power of an odd prime $p$ and that $n \geq 3$. It is perhaps appropriate to comment that over even order fields, the orthogonal groups have symplectic groups as overgroups and so cannot be maximal.
2 The geometric setting: orthogonal polarities commuting with a unitary polarity

In $PG(n-1, q^2)$ a non–singular Hermitian variety is defined to be the set of all absolute points of a non–degenerate unitary polarity $\mathcal{U}$, and is denoted by $\mathcal{H}(n-1, q^2)$. The number of points on $\mathcal{H}(n-1, q^2)$ is given in [19]:

$$[q^n + (-1)^{n-1}][q^{n-1} - (-1)^{n-1}]/(q^2 - 1).$$

We write $\Sigma$ for $PG(n-1, q^2)$ and $\mathcal{H}$ for $\mathcal{H}(n-1, q^2)$ when the context is clear.

Let $\mathcal{B}$ be an orthogonal polarity commuting with the unitary polarity $\mathcal{U}$ associated with $\mathcal{H}$. Set $\mathcal{V} = \mathcal{B}\mathcal{U} = \mathcal{U}\mathcal{B}$. Then $\mathcal{V}$ is a non–linear collineation and from [19], the fixed points of $\mathcal{V}$ on $\mathcal{H}$ form a non–degenerate quadric $Q$. Moreover, the complete set of points of $\Sigma$ fixed by $\mathcal{V}$ forms a subgeometry $\Sigma_0$ isomorphic to $PG(n-1, q^2)$ such that $Q = \Sigma_0 \cap \mathcal{H}$. Notice that the points of $\Sigma$ fixed under $\mathcal{V}$ are those admitting the same tangent or polar space with respect to both the unitary polarity and the orthogonal polarity. If $\Pi$ is a subspace of $\Sigma$ such that $\mathcal{V}(\Pi) = \Pi$, then $\Pi$ is a subspace of $\Sigma$ generated by a subspace $\Pi_0$ of $\Sigma_0$: we also say that $\Pi$ is an extension of $\Pi_0$ to a subspace of $\Sigma$. If $P$ is a point of $\Sigma \setminus \Sigma_0$, then $P + \mathcal{V}(P)$ is a line of $\Sigma$ extending a line of $\Sigma_0$. Moreover if $L$ is a line of $\Sigma$ extending a line $L_0$ of $\Sigma_0$ and if $P \in L \setminus L_0$, then $P + \mathcal{V}(P) = L$. Note that we use the terms secant line and external line to refer to lines of $\Sigma_0$ that are secant and external with respect to $Q$.

In terms of forms, let us assume that $(V, H)$ is an $n$–dimensional unitary space over $K = GF(q^2)$; $H$ is a non-degenerate hermitian form on $V$ with $H(\alpha u, \beta v) = \alpha \beta^n H(u, v)$ for all $\alpha, \beta \in K$ and all $u, v \in V$. Let $K_0$ be the subfield $GF(q)$ of $K$. Choose a basis $v_1, \ldots, v_n$ of $V$ such that $H(v_i, v_j) \in K_0$ for all $i, j$ and let $W$ denote the $K_0$–span of these vectors. It turns out that the restriction $H_0$ of $H$ to $W$ is a non–degenerate symmetric bilinear form. If the basis is an orthonormal basis, then the discriminant of $H_0$ is a square. Replacing $v_1$ by $\xi v_1$, where $\xi$ is a generator of $GF(q^2)^*$ (the multiplicative group of $GF(q^2)$), the discriminant of $H_0$ is a non–square. Therefore, when $n$ is even we obtain embeddings $O_n(q^\epsilon) < U_n(q^2)$ for both $\epsilon = +$ and $\epsilon = -$.

As above, let $\xi$ be a generator for $GF(q^2)^*$. Then $\lambda = \xi^{q+1}$ is a generator for $GF(q)^*$ (the multiplicative group of $GF(q)$) and is necessarily a non-square.
in $GF(q)$. Let $\mathcal{G} = \langle GO_n(q), \xi I_n \rangle$ ($\xi I_n$ being the centre of $GU_n(q^2)$), let $\mathcal{G}_1 = \mathcal{G} \cap U_n(q^2)$, let $\mathcal{G}_2 = \mathcal{G} \cap SU_n(q^3)$ and let $\mathcal{G}_3 = \langle SO_n(q), \xi I_n \rangle \cap SU_n(q^2)$. We denote by $G$ and $S$ the stabilizers of $Q$ in $SU(n, q^2)$ and $U(n, q^2)$ and by $\bar{G}$ the stabilizer of $Q$ in $PSU(n, q^2)$. It is of some interest to know the structure of $G$ and related groups.

**Proposition 2.1.** $S = \mathcal{G}_1$ and $G = \mathcal{G}_2$ with $\bar{G}$ the image of $G$ in $PSU(n, q^2)$.

Proof. Suppose that $h \in S$. Observe that if $L$ is a line of $\Sigma$ extending a secant line $L_0$ of $\Sigma_0$ and if $\{X, Y\} = Q \cap L$, then $hX, hY$ are non-orthogonal points in $Q$ and so $hL$ also extends a secant line of $\Sigma_0$. Moreover if there is a third point $R$ of $L_0$ such that $hR \in \Sigma_0$, then $hP \in \Sigma_0$ for all $P \in L_0$. Suppose that $n \geq 4$. For any point $Z$ of $\Sigma_0 \setminus Q$, the polar space $Z^\perp$ is spanned by points of $Q$ and the same is true of $h(Z^\perp)$. Then $h(Z^\perp)$ extends a subspace of $\Sigma_0$ to one of $\Sigma$ and therefore the same is true of $hZ$, i.e., $hZ \in \Sigma_0$. If $n = 3$, then the same argument works for one class of points of $\Sigma_0 \setminus Q$. If $Z$ lies in the second class, then $Z^\perp$ is spanned by points of the first class and the same argument can be applied again. Hence $h$ fixes $\Sigma_0$ and $h$ must be the product of an element of $GL_n(q)$ and an element in the centre of $U(n, q^2)$. It is well known [13] that the stabilizer of $Q$ in $GL_n(q)$ is $GO_n(q)$. Therefore $h \in \mathcal{G}_1$. The elements of $\mathcal{G}_1$ stabilize $Q$ so $S = \mathcal{G}_1$ and the stabilizer, $G$, of $Q$ in $SU_n(q^2)$ is $\mathcal{G}_2$. The stabilizer, $\bar{G}$, of $Q$ in $PSU_n(q^2)$ is then the image of $G$ in $PSU_n(q^2)$.

$\square$

**Proposition 2.2.** The stabilizer $\bar{G}$ of $Q$ in $PSU_n(q^2)$ is isomorphic to $PSO_n(q)$. 2 when $n$ is even and $PSO_n(q)$ when $n$ is odd.

Proof. We identify the elements of $\mathcal{G}$ and $GO_n(q)$ that lie in $U_n(q^2)$ and can then consider $\bar{G}$ as $PGO_n(q) \cap PSU_n(q^2)$. Recall that $\lambda$ is a generator for $GF(q)^*$.

Assume that $n$ is even with $n = 2m$. We begin by constructing an element $g_\lambda \in GO_n(q)$ such that $GO_n(q) = \langle O_n(q), g_\lambda \rangle$. The detailed structure of $g_\lambda$ depends on the Witt index of $H_0$ and on the value of $q$. We take a basis $x_1, x_2, \ldots, x_{m-1}, y_1, \ldots, y_{m-1}, x_m, y_m$ for $V$ such that $H_0(x_i, x_j) = H_0(y_i, y_j) = H_0(x_i, x_m) = H_0(x_i, y_m) = H_0(y_i, x_m) = H_0(y_i, y_m) = 0$ and $H_0(x_i, y_j) = \delta_{ij}$.
for $i, j \leq m - 1$. If the Witt index of $H_0$ is $m$ (i.e., we are in the hyperbolic case), then we take $H_0(x_m, y_m) = H_0(y_m, x_m) = 0$ and $H_0(x_m, y_m) = 1$. If the Witt index is $m - 1$ (i.e., we are in the elliptic case), then we take $H_0(x_m, y_m) = 1, H_0(y_m, y_m) = \lambda, H_0(x_m, y_m) = 0$ when $q \equiv 1 \pmod{4}$ and $H_0(x_m, y_m) = 1, H_0(y_m, y_m) = 1, H_0(x_m, y_m) = 0$ when $q \equiv 3 \pmod{4}$. Let $g_\lambda$ be the block-diagonal matrix $(\lambda I_{m-1}, I_{m-1}, X)$, where $X$ is a $2 \times 2$ matrix given in the three cases above by: 

\[
\begin{bmatrix}
\lambda & 0 \\
0 & 1
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\
-\lambda & 0
\end{bmatrix}
\] and $X = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (where $a^2 + b^2 = \lambda$) respectively. The element $g_\lambda$ so constructed has multiplier $\lambda$ (i.e., $H_0(g_\lambda(x), g_\lambda(y)) = \lambda H_0(x, y)$). Any element of $GO_n(q)$ may be written $hg_\lambda^s$ for some $0 \leq s \leq q - 2$ and some $h \in O_n(q)$ (as $g_\lambda^s \in O_n(q)$) and any element in $\mathcal{G}$ can be written $hg_\lambda^s \xi^t I_n$. Note that $\xi I_n$ has multiplier $\lambda$ so $k_\lambda = g_\lambda \xi^{-1} I_n$ has multiplier $1$, i.e., $k_\lambda \in U_n(q^2)$. If we examine the elements in $S$ we see that they may be written $hg_\lambda^s \xi^t I_n = hk_\lambda^s \xi^{r+t} I_n$ with $\xi^{(q+1)(r+t)} = 1$, so that $\xi^{r+t} = \xi^{(q-1)s}$ for some $0 \leq s \leq q$. In other words elements in $S$ may be written $hk_\lambda^s l^t$ where $l = \xi^{q-1} I_n$ and $0 \leq s \leq q$. In fact $g_\lambda^2$ has the same multiplier as $\lambda I_n$ so $g_\lambda^2 = h_1 \lambda I_n$ (for some $h_1 \in O_n(q)$) and therefore $k_\lambda^2 = g_\lambda^2 \xi^{-2} I_n = h_1 \lambda I_n \xi^{-2} I_n = h_1 l$, so elements in $S$ may be written $hk_\lambda^s l^t$ with $r = 0$ or $1$. We can characterize the elements of $G$ as lying in exactly one of the following classes:

I. $\mathcal{G}_3$

II. $\{hl^s : h \in O_n(q); \det h = -1; \det l^s = -1\}$

III. $\{hk_\lambda l^s : h \in SO_n(q); \det k_\lambda l^s = 1\}$

IV. $\{hk_\lambda l^s : h \in O_n(q); \det h = -1; \det k_\lambda l^s = -1\}$

We can characterize the classes II, III and IV as follows:

- Class II: $\xi^{(q-1)s}n = -1$ for some $0 \leq s \leq q$, i.e., $(q + 1)$ divides $sn - (q + 1)/2$.

- Class III: $\xi^{(q-1)s}n \xi^{(q-1)n/2} = 1$ for some $0 \leq s \leq q$, i.e., $(q + 1)$ divides $sn + n/2$.

- Class IV: $\xi^{(q-1)s}n \xi^{(q-1)n/2} = -1$ for some $0 \leq s \leq q$, i.e., $(q + 1)$ divides $sn + n/2 - (q + 1)/2$. 

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Notice that two of II, III, IV cannot occur simultaneously. Let \( d = \text{hcf}(n, q + 1) \). We can consider three possibilities, exactly one of which occurs.

First suppose that \((q + 1)/d\) is even but that \( n/d \) is odd. Let \( s = (q + 1)/(2d) \), then \( sn - (q + 1)/2 = (q + 1)[(n/d) - 1]/2 \) so case II occurs.

Second suppose that \((q + 1)/d\) is odd but that \( n/d \) is even. Let \( s = [[(q + 1)/d] - 1]/2 \), then \( sn + n/2 = (q + 1)[(n/d) - 1]/2 \) so case III occurs.

Finally suppose that \((q + 1)/d\) and \( n/d \) are both odd. Let \( s = [[(q + 1)/d] - 1]/2 \), then \( sn + n/2 - (q + 1)/2 = (q + 1)[(n/d) - 1]/2 \) and so case IV occurs.

Hence exactly one of cases II, III, IV occurs, depending on the values of \( n \) and \( q \). Moreover two elements of a given class differ by an element of \( G_3 \), so \( |G : G_3| = 2 \). Factoring out by scalars we arrive at \( \bar{G} = \text{PSO}_n(q^2) \).

Proposition 2.3. \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \) and \( \bar{G} \) is the normalizer of \( \text{PSO}_n(q) \) in \( \text{PSU}_n(q^2) \).

Proof. Certainly the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \) contains \( G \). One orbit of \( SO_n(q) \) on \( \Sigma \) is \( Q \) so if \( h \in SU_n(q^2) \) normalizes \( G \), then \( hQ \) is also an orbit of \( SO_n(q) \). We show that \( hQ \) must be \( Q \) so that \( h \in G \).

There are four types of line (three in small cases) in \( \Sigma_0 \): external, secant, tangent to \( Q \), on \( Q \). Suppose that \( L_0 \) is a line of \( \Sigma_0 \) and \( L \) is its extension to a line of \( \Sigma \). If \( L_0 \) is an external line, then \( L \) is non-isotropic with \( q + 1 \) points on \( H \setminus Q \) and these lie in a single orbit under \( O_2^- \) (the action on \( L \) of the stabilizer in \( SO_n(q) \) of \( L_0 \)). If \( L_0 \) is a secant line, then \( L \) is non-isotropic with two points on \( Q \), and \( q - 1 \) singular points on \( H \setminus Q \) in a single orbit under \( O_2^+ \) (again, the action on \( L \) of the stabilizer in \( \text{PSO}_n(q) \) of \( L_0 \)). If \( L_0 \) is a tangent line, then \( L \) contains exactly one point of \( H \) and that is a point of \( Q \). If \( L_0 \) is a line on \( Q \), then \( L \) contains \( q + 1 \) points of \( Q \) and \( q^2 - q \) of \( H \) that lie off \( Q \) but lie in a single orbit under the stabilizer in \( SO_n(q) \) of \( L_0 \).
acting as $PGL_2(q)$ on $L$.

If $n \geq 5$, then $SO_n(q)$ has a single orbit of each of the three types of line: external, secant, and on $Q$. If $n = 4$ and $Q$ is an elliptic quadric or if $n = 3$, then there are no lines on $Q$ but $SO_n(q)$ has a single orbit of external lines and a single orbit of secant lines. If $n = 4$ and $Q$ is a hyperbolic quadric, then $SO_n(q)$ has a single orbit of external lines and a single orbit of secant lines, but has two (equally sized) orbits of lines on $Q$. Moreover $SO_n(q)$ is transitive on the points of $Q$ for all $n \geq 3$. Thus $SO_n(q)$ has 3, 4 or 5 orbits of points on $H$, depending on $n$ and the nature of $Q$.

The numbers of points and lines in $\Sigma_0$ of various types are stated in [11, pp 23, 25]. In each case we list the relevant number for (i) the elliptic quadric ($n = 2m$) followed by (ii) the hyperbolic quadric ($n = 2m$), and (iii) the parabolic quadric ($n = 2m + 1$).

The number, $N_0$, of points on $Q$ is:

(i) $[(q^m + 1)(q^{m-1} - 1)]/[q - 1]$
(ii) $[q^m - 1)(q^{m-1} + 1)]/[q - 1]$
(iii) $[(q^m - 1)(q^m + 1)]/[q - 1]$.

The number, $N_1$, of lines on $Q$ is:

(i) $[(q^m + 1)(q^{m-1} + 1)(q^{m-1} - 1)(q^{m-2} - 1)]/[(q - 1)(q^2 - 1)]$
(ii) $[(q^m - 1)(q^{m-1} - 1)(q^{m-1} + 1)(q^{m-2} + 1)]/[(q - 1)(q^2 - 1)]$
(iii) $[(q^m - 1)(q^{m-1} - 1)(q^{m-1} + 1)(q^{m-1} + 1)]/[(q - 1)(q^2 - 1)]$.

The number, $N_2$, of secant lines to $Q$ is:

(i) $[(q^m + 1)(q^{m-1} - 1)q^{2m-2}]/[2(q - 1)]$
(ii) $[(q^{m-1} + 1)(q^m - 1)q^{2m-2}]/[2(q - 1)]$
(iii) $[(q^m + 1)(q^m - 1)q^{2m-1}]/[2(q - 1)]$.

The number, $N_3$, of external lines to $Q$ is:
The number of points of \( \mathcal{H} \) arising from lines on \( \mathcal{Q} \), secant lines to \( \mathcal{Q} \) and external lines to \( \mathcal{Q} \) is 
\((q^2 - q)N_1, (q - 1)N_2 \) and \((q + 1)N_3 \) respectively, these numbers being greater than \( 2\mathcal{N}_0, \mathcal{N}_0 \) and \( \mathcal{N}_0 \) respectively (excluding the cases where \( \mathcal{N}_1 = 0 \)). Thus \( \mathcal{Q} \) represents the smallest orbit of \( \bar{G} \) and \( h\mathcal{Q} \) being an orbit leads to \( h\mathcal{Q} = \mathcal{Q} \), i.e., \( h \in G \). Hence \( \bar{G} \) is the normalizer of \( PSO_n(q) \) in \( PSU_n(q^2) \).

3 The maximality

In this Section we prove that \( G \) is maximal in \( PSU_n(q^2) \) (with a small number of exceptions). In fact we prove something a little stronger: if \( F \) is a subgroup of \( SU_n(q^2) \) containing \( SO_n(q) \) but not contained in \( G \), then \( F = SU_n(q^2) \).

To this aim we use a dimension reduction argument. We use elements of \( SO_n(q) \) known as hyperbolic rotations. These are discussed in [6, p26]: a hyperbolic rotation is an element of \( SO_n(q) \) that fixes both points of \( \mathcal{Q} \) on a secant line \( L_0 \) of \( \Sigma_0 \) and acts as the identity on \( L_0^\perp \); in matrix terms it has the form diag(\( \alpha, \alpha^{-1}, 1, \ldots, 1 \)) with respect to an appropriate basis, where \( 0 \neq \alpha \in GF(q) \). It is shown in [6] that \( SO_n(q) \) is generated by such elements.

More generally there are such elements in \( SU_n(q^2) \) fixing non-orthogonal points \( X, Y \) of \( \mathcal{H} \) and acting as the identity on \( (X + Y)^\perp \); we continue to take \( \alpha \in GF(q) \), and when \( \alpha = \lambda \), a generator for the multiplicative group \( GF(q)^* \) we denote the element \( h_{XY} \). If \( x, y \) are vectors representing \( X, Y \), chosen such that \( H(x, y) = 1 \), then
\[
h_{XY}(v) = v + (\lambda - 1)H(v, y)x + (\lambda^{-1} - 1)H(v, x)y.
\]

Notice that if \( g \in SU_n(q^2) \), then \( gh_{XY}g^{-1} = h_{XY'} \) where \( X' = gX \) and \( Y' = gY \). The subgroup \( H_{XY} = H_{YX} \) generated by \( h_{XY} \) has order \( q - 1 \), fixes \( X, Y \) and each point in \( (X + Y)^\perp \), has orbits of length \( (q - 1)/2 \) on the remaining points of \( X + Y \) and orbits of length \( q - 1 \) on points of \( \Sigma \) not in \( X + Y \) or \( (X + Y)^\perp \). In our reduction argument we show that usually \( F \) must
contain a hyperbolic rotation \( h_{XY} \) not in \( G \) such that \((X + Y)^\perp \) contains a point \( Z \in \Sigma_0 \setminus Q \). If we write \( F_{n-1} \) for the subgroup of \( SU_{n-1}(q^2) \) so that \( 1 \times F_{n-1} = F \cap (1 \times SU_{n-1}(q^2)) \) (written with respect to the decomposition \( Z \oplus Z^\perp \)), then we may regard \( h_{XY} \) as an element of \( F_{n-1} \) that does not lie in \( SO_{n-1}(q) \) and we can apply induction to conclude that \( F_{n-1} = SU_{n-1}(q^2) \).

The initial case, \( n = 3 \), was established many years ago by Mitchell in [18], although exceptions when \( q = 3 \) or 5 necessitate careful argument in those cases.

The first result largely characterizes the hyperbolic rotations that lie in \( G \). If \( X, Y \in Q \) (with \( Y /\not\in X^\perp \)), then \( h_{XY} \in SO_n(q) \). Otherwise \( h_{XY} \) does not usually lie in \( G \).

**Proposition 3.1.** Suppose that \( n \geq 4 \).

(i) If \( X \in Q \) and \( Y \in H \setminus Q \) (with \( Y \notin X^\perp \)), then \( h_{XY} \notin G \) except when \( q = 3 \) and \( X \in Y + \mathcal{V}(Y) \).

(ii) If \( X, Y \in H \setminus Q \) (with \( Y \notin X^\perp \)) but with \( X + \mathcal{V}(X) = Y + \mathcal{V}(Y) \), then \( h_{XY} \notin G \) except when \( q = 3 \).

(iii) If \( X, Y \in H \setminus Q \) (with \( Y \notin X^\perp \)) with \( L = X + \mathcal{V}(X) \neq N = Y + \mathcal{V}(Y) \), then \( h_{XY} \notin G \) except (conceivably) when \( n = 4 \), \( L^\perp = L \) and \( N^\perp = N \).

Proof.

(i) Let \( L = Y + \mathcal{V}(Y) \), then \( L \) is a line of \( \Sigma \) fixed by \( \mathcal{V} \) and is thus the extension to \( \Sigma \) of a line \( L_0 \) of \( \Sigma_0 \). If \( X \in L \), then \( L_0 \) is a secant line of \( \Sigma_0 \) and \( H_{XY} \) fixes two points of \( L_0 \) with all other points lying in orbits of length \((q - 1)/2 \). Thus, assuming that \( q \neq 3 \), the second point of \( Q \) lying on \( L \) is mapped by \( h_{XY} \) to a point off \( Q \). Now assume that \( X \) does not lie on \( L \), write \( \Pi_0 \) for the plane \( X + L_0 \) in \( \Sigma_0 \) and \( \Pi \) for its extension to a plane of \( \Sigma \) fixed by \( H_{XY} \). Then \( \Pi_0 \cap X^\perp \) is a line \( N_0 \) of \( \Pi_0 \) whose extension to \( \Sigma \) is a line \( N \) fixed by \( H_{XY} \). The remaining \( q^2 \) points of \( \Pi_0 \) lie neither in \( X + Y \) nor \((X + Y)^\perp \) and hence lie in orbits of length \( q - 1 \) under \( H_{XY} \). It follows that for some point \( P \in \Pi_0 \setminus N_0 \), we have \( h_{XY}(P) \notin \Sigma_0 \). Thus \( h_{XY} \notin G \).

(ii) This time \( H_{XY} \) fixes two points of \( L = Y + \mathcal{V}(Y) \) (and the two points do not lie on \( Q \)) with all other points lying in orbits of length \((q - 1)/2 \).
Assuming that $q \neq 3$, $H_{XY}$ has exactly two fixed points on $L$, namely $X$ and $Y$. Suppose that $L_0$ is the line of $\Sigma_0$ underlying $L$. If $L_0$ is a secant line of $\Sigma_0$, then the subgroup of $G$ that fixes $L$ and acts as the identity on $L^\perp$ must act on $L$ as $SO_2^+(q)$, having order $q - 1$ and fixing exactly two points of $L$, both in $L_0$: hence $h_{XY} \notin G$. If $L_0$ is an external line, then the subgroup of $G$ that fixes $L$ and acts as the identity on $L$ must act on $L$ as $SO_2^+(q)$, having order $q + 1$ and therefore not containing $H_{XY}$.

(iii) Given that $L \neq N$ it follows that $L^\perp \neq N^\perp$ and therefore $L^\perp$ is spanned by points $P$ of $\Sigma_0$ such that $P \notin N^\perp$. Suppose that $h = h_{XY}$ fixes $Q$ and that $P \in L^\perp \cap \Sigma_0$ with $P \notin N^\perp$, then $hP \neq P$ and $hP \in P + X$ so that $X \in P + hP$. Thus $L = P + hP$ for all such $P$. This can only happen if $L^\perp = L$ in which case $n = 4$. The same argument applies to $N$.

The following two propositions establish conditions sufficient for the reduction to lower dimension when $n \geq 5$.

**Proposition 3.2.** Suppose that $X, Y \in \mathcal{H}$ with $Y \notin X^\perp$. If either $n \geq 5$ and $X \in Q$ or $n \geq 7$, then there is a point $Z$ of $\Sigma_0 \setminus Q$ lying in $(X + Y)^\perp$. Moreover, if $h_{XY} \in F \setminus G$, then $h_{XY}$ does not fix $Z^\perp \cap Q$.

Proof. We write $M$ for $X + \mathcal{V}(X) + Y + \mathcal{V}(Y)$. Then $M$ is fixed by $\mathcal{V}$ so extends a subspace $M_0$ of $\Sigma_0$ and the projective dimension of $M_0$ can be no greater than 3.

If $n \geq 8$, then $M_0$ cannot contain $M_0^\perp \cap \Sigma_0$ so $M_0^\perp \cap \Sigma_0$ cannot lie on $Q$ (if $n = 8$, then the fact that $Y \notin X^\perp$ is pertinent). Hence there is a point of $\Sigma_0 \setminus Q$ lying in $M^\perp$.

If $n = 7$, then there is the possibility that $M_0^\perp \cap \Sigma_0$ lies on $Q$, but it can only happen if $M^\perp$ is a plane and $M$ a solid and in this case $M^\perp$ contains all points of $M$ lying on $\mathcal{H}$, including $X$ and $Y$, contrary to $Y \notin X^\perp$.

Now suppose that $n \geq 5$ with $X \in Q$. Then the projective dimension of $M_0$ can be no greater than 2 and $M_0^\perp \cap \Sigma_0$ cannot lie on $Q$ unless $n = 5$, $M$ is a
plane and $M^\perp$ is a line containing all the points of $M$ lying on $\mathcal{H}$, including $X$ and $Y$, contrary to $Y \notin X^\perp$.

In all cases here there is a point $Z$ of $\Sigma_0 \setminus Q$ lying in $M^\perp$ and such a point necessarily lies in $(X + Y)^\perp$. Finally, $h_{XY}$ acts as an element of $1 \times SU_{n-1}(q^2)$ with respect to the decomposition $Z \oplus Z^\perp$ and it follows from Proposition 3.1 that $h_{XY}$ cannot fix $Z^\perp \cap Q$. □

**Proposition 3.3.** Suppose that $n = 5$ or 6 and that $F$ contains $h_{XY}$ not in $G$ with $X, Y \notin Q$. Then either there is a point $Z$ of $\Sigma_0 \setminus Q$ lying in $(X + Y)^\perp$ such that $h_{XY}$ does not fix $Z^\perp \cap Q$ or there are $X' \in Q$, $Y' \in \mathcal{H}$ such that $h_{XY'} \in F \setminus G$.

Proof. Unless $M$ is a solid and either $n = 5$ with $M^\perp$ a point of $Q$ or $n = 6$ and $M^\perp$ a line extending a line on $Q$, we can argue as in Proposition 3.2 that there is a point $Z$ of $\Sigma_0 \setminus Q$ lying in $(X + Y)^\perp$ and $h_{XY}$ acts as an element of $1 \times SU_{n-1}(q^2)$ with respect to the decomposition $Z \oplus Z^\perp$. If $n = 6$, then it follows from Proposition 3.1 that $h_{XY}$ cannot fix $Z^\perp \cap Q$. Suppose that $n = 5$ and that $h_{XY}$ fixes $Z^\perp \cap Q$. Then $h = h_{XY}$ cannot lie in $1 \times SO_4(q)$. Let $B$ be a point of $Q$ not fixed by $h$. Then $B$ does not lie in $Z^\perp$ or $A^\perp$. Let $A$ be a point $Z^\perp \cap Q$ not in $B^\perp$. Then $hA \in Q$ but $hB \notin Q$. Moreover there is a point $C$ of $\Sigma_0 \cap Z^\perp$ such that $B$ lies on the line $Z + C$ and $h(Z + C)$ extends a line of $\Sigma_0$. Hence $hB + \mathcal{V}(hB) = h(Z + C)$ and this line cannot contain $hA$. Writing $X' = hA, Y' = hB$, it follows from Proposition 3.1 that $h_{XY'} \in F \setminus G$.

Suppose now that $M$ is a solid and that either $n = 5$ with $M^\perp$ a point of $Q$ or $n = 6$ and $M^\perp$ a line extending a line on $Q$. Then $X, \mathcal{V}(X), Y, \mathcal{V}(Y)$ form a simplex for $M$. We choose a point $X'$ in $\Sigma_0$: if $n = 5$ we choose $X' = M^\perp$ and if $n = 6$ we choose $X'$ to be any point of $M^\perp$ on $Q$. Note that if $B \in Q$, then $B \in X^\perp$ if and only if $B \in \mathcal{V}(X)^\perp$ with the corresponding statement for $Y$ also true. Thus if $B \in Q \setminus X'^\perp$, then $B \notin M$ and so $B \notin X^\perp \cap Y^\perp$. In consequence $Y' = h_{XY}(B) \notin Q$. Now $h_{XY} \in G$ and by Proposition 3.1 $h_{XY'} = h_{XY}h_{X'B}h_{X'_Y}^{-1} \in F \setminus G$, except possibly when $q = 3$.

Suppose that $q = 3$. If $n = 5$, then $X' = M^\perp$ (as chosen above) cannot lie in both $X + \mathcal{V}(X)$ and $Y + \mathcal{V}(Y)$ (as the two lines have no points in common): without loss of generality we may assume that $X' \notin L = Y + \mathcal{V}(Y)$. We write $L_0$ for the line of $\Sigma_0$ that is extended to give $L$, it must be a secant
or external line of $\Sigma_0$ and $L_0^\perp \cap \Sigma_0$ is a non-isotropic plane containing $X'$.
Thus we may choose $B \in (Q \cap L^\perp) \setminus X'^\perp$, so that $Y' = h_{XY}(B)$ lies on $(B + Y) \setminus Q$ and then $Y' + \mathcal{Y}(Y')$ lies in $B + L$. Thus $Y' + \mathcal{Y}(Y')$ cannot contain $X'$: by Proposition 3.1, $h_{XY'} \notin G$. If $n = 5$, then one possibility is that $L = Y + \mathcal{V}(Y)$ meets $M^\perp$ in a point of $M^\perp \cap Q$, in which case we can choose $X'$ to be any other point of $M^\perp$ on $Q$ and choose $B \in (Q \cap L^\perp) \setminus X'^\perp$ (so necessarily not in $M$) with $X' \notin B + L$. The alternative is that $L$ does not meet $M^\perp$ and then any choice of $X' \in M^\perp$ and $B \in L^\perp$ such that $B \in Q \setminus X'^\perp$ leads to $X' \notin B + L$: again by Proposition 3.1, $h_{XY'} \notin G$. In both cases ($n = 5$ and $n = 6$), the hyperbolic rotation $h_{XB}$ lies in $G$ so $F$ contains $h_{XY'} = h_{XY}h_{XB}h_{XY}'^{-1}$.

$\square$

The next four propositions are necessary for the reduction argument when $n = 4$.

**Proposition 3.4.** Suppose that $n = 4$. Then either there exists $k \in F \setminus G$ such that $Q \cap kQ$ is non-empty or there exist non-orthogonal points $X, Y \in Q$ and $k \in F$ such that $k(X + Y) = X + Y$ but $k(X), k(Y) \notin Q$.

**Proof.** Suppose that for all $k \in F \setminus G$ and for all $X \in Q$ we have $kX \notin Q$. Suppose that for some $X \in Q$, the line $N = kX + \mathcal{V}(kX)$ extends a line $N_0$ of $\Sigma_0$ that lies on $Q$. Then $Q$ is hyperbolic and, as there are only $2(q + 1)$ lines on $Q$ there is a choice of $X$ such that $N$ contains a second point $kA$ of $kQ$. It follows that all $q + 1$ points of $X + A$ in $\Sigma_0$ are mapped by $k$ into $N$ and none of the images lie on $Q$. The stabilizer of $N_0$ in $SO_4^+(q)$ acts as $GL_2(q)$ on $N$ with order $(q^2 - 1)(q^2 - q)$, there being two orbits on $N$: the $q + 1$ points of $N_0$ and the remaining $q^2 - q$ points. The stabilizer of one of the points of $N \setminus N_0$ has fixed two points of $N \setminus N_0$ and has orbits of length $q + 1$ on the remainder. It follows that there is a point $P \in kQ \cap N$ and a $g \in \text{Stab}_{SO_4^+(q)}(N_0)$ such that $gkX = kX$ but $gP \notin kQ$. Hence $k^{-1}gk \in F \setminus G$ with $k^{-1}gkQ \cap Q$ non-empty.

Suppose that $Q$ is hyperbolic and that for some $k \in F \setminus G$ and some $X \in Q$, the line $N = kX + \mathcal{V}(kX)$ extends an external line $N_0$ of $\Sigma_0$. Then $G$ has a subgroup $E$ acting as $SO_3^-(q)$ on $N^\perp$ and as the identity on $N$; the order of $E$ is $q + 1$. The orbits of $E$ acting on the points of $\mathcal{H}$ not in $N$ or $N^\perp$ have length $q + 1$. The number of points of $kQ$ not in $N$ or $N^\perp$ lies between $q^2 + 2q - 3$ and $q^2 + 2q$ so these points cannot be represented as a union of
orbits under $E$ and for some $g \in E$ we have $k^{-1}gk \in F \setminus G$ with $k^{-1}gkQ \cap Q$ non-empty.

Suppose that $Q$ is elliptic. Then there are $q^2 + 1$ points on $Q$ and $(q^2+1)(q+1)/2$ points of $H$ lying on extensions of external lines of $\Sigma_0$. However $SU_q(q^2)$ has order $q^6(q^2-1)(q^3+1)(q^4-1)$, not divisible by $q^2+1+(q^2+1)q^2(q+1)/2$, so that these points cannot form an orbit under $F$. Hence there must be some $k \in F \setminus G$ and some $X \in Q$ such that the line $N = kX + \mathcal{V}(kX)$ extends a secant line $N_0$ of $\Sigma_0$.

Finally suppose that for some $k \in F \setminus G$ and some $X \in Q$, the line $N = kX + \mathcal{V}(kX)$ extends a secant line $N_0$ of $\Sigma_0$. This time $G$ has a subgroup $E$ acting as $SO_q^+(q)$ (respectively $SO_q^-(q)$) on $N^\perp$ if $Q$ is hyperbolic (respectively elliptic) and as the identity on $N$; the order of $E$ is $q - 1$ (respectively $q + 1$). The orbits of $E$ acting on the points of $H$ not in $N$ or $N^\perp$ have length $q - 1$ (respectively $q + 1$). The number of points of $kQ$ not in $N$ or $N^\perp$ lies between $q^2 + 2q - 3$ and $q^2 + 2q$ in the hyperbolic case and between $q^2 - 1$ and $q^2$ in the elliptic case. Either these points cannot be represented as a union of orbits under $E$, in which case for some $g \in E$ we have $k^{-1}gk \in F \setminus G$ with $k^{-1}gkQ \cap Q$ non-empty (as above), or $N$ contains two points of $kQ$. In this latter case there is a $Y \in Q$ and a $g \in G$ such that $N = k(X + Y)$ and $gk(X + Y) = X + Y$ with neither $U = gkX$ nor $W = gkY$ in $Q$.

□

**Proposition 3.5.** Suppose that $n = 4$ and $q \neq 3$. If there exists $k \in F \setminus G$ with $Q \cap kQ$ non-empty, then there exists $h_{X,Y'} \in F \setminus G$ with $X' \in Q$. If no such $k$ exists, then there exists $h_{X,Y'} \in F \setminus G$ with $X', Y' \notin Q$ but $X' + Y'$ extending a secant line of $\Sigma_0$.

Proof. Suppose that there exists $k \in F \setminus G$ with $Q \cap kQ$ non-empty, then (as $G$ is transitive on $Q$) we may assume that $kA = A$ for some $A \in Q$. If there is a $B \in Q \setminus A^\perp$ such that $kB \notin Q$, then $h_{AB} \in G$ and we can take $X' = A, Y' = kB$ so that $h_{X,Y'} = kh_{AB}k^{-1}$ has the required properties. Otherwise there is a $C \in Q \cap A^\perp$ such that $kC \notin Q$. There must exist $B \in Q$ such that $B \notin A^\perp$ and $B \notin C^\perp$; now $h_{BC} \in G$ and we can take $X' = kB, Y' = kC$ with $h_{X,Y'} = kh_{BC}k^{-1}$ having the required properties.

Otherwise, by Proposition 3.4, there exist non-orthogonal points $X, Y \in Q$ and $k \in F$ such that $k(X + Y) = X + Y$ but $k(X), k(Y) \notin Q$. Let $X' = kX,$
\[ Y' = kY. \] Then \( h_{XY'} = kh_{XY}k^{-1} \in F \setminus G \) by Proposition 3.1, so it has the required properties.

\[ \Box \]

**Proposition 3.6.** Suppose that \( n = 4, q \neq 3 \) and that \( Q \) is elliptic. Suppose that \( F \) contains \( h_{XY} \notin G \) with \( X \in Q \) but \( Y \notin Q \) (and \( Y \notin X^\perp \)). Then there is a point \( Z \) of \( \Sigma_0 \setminus Q \) lying in \( (X + Y)^\perp \) and moreover \( h_{XY} \) does not fix \( Z^\perp \cap Q \).

**Proof.** The argument is similar to that for \( n = 5 \) and makes use of Proposition 3.1. Either \( M = X + Y + V(Y) \) extends a secant line of \( \Sigma_0 \) and any point \( Z \) of \( \Sigma_0 \setminus Q \) lying on \( M^\perp \) has the required properties. Or \( M \) is a plane with \( Z = M^\perp \) a point of \( \Sigma_0 \), necessarily not on \( Q \) since \( Q \) is elliptic, and \( Z \) has the required properties.

\[ \Box \]

**Proposition 3.7.** Suppose that \( n = 4 \) and that \( Q \) is hyperbolic. Suppose that \( F \) contains \( h_{XY} \notin G \) with \( X \in Q \). If \( q \neq 3 \), then there is a point \( Z \in \Sigma_0 \setminus Q \) such that \( F \) contains an \( h_{XY'} \notin G \) with \( X', Y' \in Z^\perp \) and with \( X' \in Q \) and such that \( h_{XY'} \) does not fix \( Z^\perp \cap Q \). If \( q = 3 \), then either there exists \( h_{XY'} \) with these same properties or there exists a line \( L \) of \( \Sigma \) extending a secant line of \( \Sigma_0 \) and an element of \( F \) fixing \( L \) and fixing exactly one point of \( L \cap Q \).

**Proof.** Let \( M = X + Y + V(Y) \). If \( M^\perp \) is not a point of \( Q \) (i.e., \( M^\perp \) is either a line or is a point of \( \Sigma_0 \setminus Q \)), then \( M^\perp \) contains a point \( Z \in \Sigma_0 \setminus Q \) and we may take \( X' = X, Y' = Y \).

Suppose that \( M^\perp \) is a point \( U \) of \( Q \). Let \( T \) be a point of \( Q \cap X^\perp \) such that \( T \notin U^\perp \) and let \( W \) be the second point of \( Q \) on \( (U + T)^\perp \). Let \( u, t, x, w \) be vectors representing \( U, T, X, W \), chosen so that points of \( \Sigma_0 \) are represented by \( GF(q) \)-linear combinations of \( u, t, x, w \) and so that \( H(u, t) = H(x, w) = 1 \). Then \( h_{XY} \) takes \( u \) to \( u \), \( x \) to \( \lambda x \) and \( t \) to \( t + \alpha x \) for some \( \alpha \in GF(q^2) \).

Suppose that \( \alpha \in GF(q) \). Then \( SO_4(q) \) contains an element \( g \) such that \( g(u) = u, g(x) = \lambda^{-1} x \), and \( g(t + \alpha x) = t \). Thus \( f = gh_{XY} \in F \setminus G \) and fixes each of \( u, x, t \). If \( q = 3 \), then \( f \) has the required property with \( L = (U + T)^\perp \).

If \( q \neq 3 \), then \( f(w) = w + \beta x \) for some \( \beta \notin GF(q) \) and \( fW \notin Q \). We may
take $X' = X$, $Y' = fW$, since $h_{XW} \in G$ and $h_{X'Y'} = fh_{XW}f^{-1} \in F \setminus G$ with $Z$ any point of $(U + T) \cap \Sigma_0$ but not in $Q$.

Suppose that $\alpha \notin GF(q)$. Let $r = t + \alpha x$ and let $R$ be the corresponding point of $\Sigma$. Then $G$ contains $h_{UT}$ and $F$ contains $h_{UR} = h_{XY}h_{UT}^{-1}$. Let $k = h_{UT}^{-1}h_{UR}$. Then $k(u) = u$, $k(x) = x$, $k(t) = t + (\lambda^{-1} - 1)\alpha x$ and $k(w) = w + (\lambda^{-1} - 1)\bar{\alpha}u$ so $k(t + w) = t + w + (\lambda^{-1} - 1)(\alpha x - \bar{\alpha}u)$ (here $\bar{\alpha}$ denotes $\alpha^q$). Let us write $A, B$ and $C$ for the points of $\Sigma$ represented by $t + w$, $\alpha x - \bar{\alpha}u$ and $x - u$. Observe that $kA \in A + B$, $kA + C$ contains exactly one point of $\Sigma_0$ (namely $C$), and $\mathcal{V}(kA) \in A + \mathcal{V}(B)$. Therefore $kA + \mathcal{V}(kA)$ is a line of $\Sigma$ extending a line of $\Sigma_0$ (so it cannot contain $C$) and it meets $X + U$ at a point $X' \neq C$. Note that $C = (X + U) \cap A^\perp$ so $X' \notin A^\perp$. Let $Y' = kA$. Then $h_{XY}A \in G$ and $kh_{XY}A^{-1} = h_{X'Y'} \in F$ with $Y' \notin Q$ but $X' \in Y' + \mathcal{V}(Y')$. If $q \neq 3$, then $h_{X'Y'} \notin G$ and any point $Z$ of $(X' + Y')^\perp$ lying in $\Sigma_0$ but not in $Q$ has the required properties. If $q = 3$, then $h_{X'Y'} \in G$. However $X' + A$ and $k(X' + A)$ are both lines of $\Sigma$ extending secant lines of $\Sigma_0$, so there exists a $g \in G$ such that $gk(X' + A) = X' + A$ and $gkX' = X'$. Let $L = X' + A$, then $gkA = gY' \notin Q$, so $f = gk \in F$ and fixes exactly one point of $L \cap Q$.

We now turn to the main theorem. The following results represent initial cases in an induction argument. They state that our Theorem is true for $n = 3$ and also that it is true in the elliptic case when $n = 4$ and $q = 3$.

**Result 3.8.** [2] For $q = 3$, any subgroup of $PSU_3(q^2)$ containing $PSO_5^- (q)$ either lies in $PSO_5^- (q)$ or is isomorphic to to $PSO_3(4)$; in particular $PSO_5^- (q)$ is maximal in $PSU_3(q^2)$.

Notice that in [10, Theorem 19.3.18], the group $PSL_3(4)$ is identified as the subgroup of $PGU_4(9)$ fixing a hemisystem whose dual structure in $PG(5, 3)$ is the Hill’s cap, studied in [9] and [5]. In [5] the stabilizer in $\Omega^+_5(3)$ of Hill’s cap is shown to be maximal.

**Result 3.9.** [18] For odd $q$, $PSO_3(q)$ is maximal in $PSU_3(q^2)$ except when $q = 3, 5$. When $q = 3$, any proper overgroup of $PSO_3(q)$ properly contained in $PSU_3(q^2)$ is isomorphic to $PSL_2(7)$. When $q = 5$, any proper overgroup of $PSO_3(q)$ properly contained in $PSU_3(q^2)$ is isomorphic to $A_7$. In all cases, $PSO_3(q)$ is self-normalizing in $PSU_3(q^2)$.
We noted earlier that some care is necessary when \( q = 3 \) and when \( q = 5 \). When \( n = 5 \) and \( q = 3 \) we have to ensure that our reduction leads to an elliptic quadric and that the overgroup of \( SO_4^-(q) \) corresponding to \( PSL_3(4) \) is avoided. When \( n = 4 \) and \( q = 5 \) we have to ensure that our reduction to \( n = 3 \) avoids the overgroup of \( SO_3(q) \) corresponding to \( A_7 \).

**Corollary 3.10.** Suppose that \( q = 3 \) or \( 5 \), that \( PSO_3^-(q) < \bar{F} \leq PSU_3(q^2) \) and that for some \( P \in Q \), \( Stab_{\bar{F}} P \neq Stab_{PSO_3(q)} P \). Then \( \bar{F} = PSU_3(q^2) \).

**Proof.** Suppose first that \( q = 3 \). There are 28 points of \( H \) and 4 of \( Q \). The remaining points of \( H \) fall into two orbits under \( PSO_3(q) \), each of size 12. The order of \( PSL_2(7) \) is 168, not divisible by 16, so such a subgroup of \( PSU_3(q^2) \) is transitive on \( H \) and the stabilizer in such a group of a point of \( Q \) is the same as the stabilizer in \( PSO_3(q) \). Hence an overgroup \( \bar{F} \), as given, cannot be isomorphic to \( PSL_2(7) \) and must therefore be \( PSU_3(q^2) \).

Now suppose that \( q = 5 \). Essentially the same arguments work except that now there are 126 points of \( H \) and 6 of \( Q \) with the remaining points of \( H \) falling into two orbits of size 60 under \( PSO_3(q) \), and the order of \( A_7 \) is 2520.

**Corollary 3.11.** Suppose that \( SO_3(q) \leq F \leq SU_3(q^2) \) with \( F \) not a subgroup of \( G \). Suppose further that if \( q = 3 \) or \( 5 \), then for any \( P \in Q \), \( Stab_{F} P \neq Stab_{G} P \). Then \( F = SU_3(q^2) \).

**Proof.** Let \( \bar{F} \) denote the image of \( F \) in \( PSU_3(q^2) \). Then by Result 3.9 and Corollary 3.10 above, \( \bar{F} = PSU_3(q^2) \). Hence \( F.C = SU_3(q^2) \) where \( C \) is the centre of \( SU_3(q^2) \). Let \( \sigma \) be any transvection in \( SU_3(q^2) \), then \( \sigma.\nu I_3 \in F \) for some \( \nu \in GF(q^2)^* \), the orders of \( \sigma \) and \( \nu I_3 \) are coprime so \( \sigma \in F \). Now \( SU_3(q^2) \) is generated by its transvections so \( F = SU_3(q^2) \).

**Proposition 3.12.** Suppose that \( n = 5 \) and \( q = 3 \) and that \( F \) contains \( h_{XY} \) not in \( G \) with \( Y \notin Q \). Then there is a point \( Z \in \Sigma_0 \setminus Q \) such that \( F \) contains \( 1 \times SU_4(9) \) (written with respect to the decomposition \( Z \oplus Z^\perp \)).

**Proof.** By Proposition 3.1, there is a point \( R \) of \( \Sigma_0 \setminus Q \) and an \( h_{XY'} \) in \( F \) such that \( X', Y' \in R^\perp \) and \( h_{XY'} \) does not fix \( R^\perp \cap Q \). Let \( F_4 \) be the subgroup of \( SU_4(9) \) determined by \( 1 \times F_4 = (1 \times SU_4(9)) \cap F \) (written with
respect to the decomposition $R \oplus R^\perp$. Then $h_{X,Y'} \in 1 \times F_1$ so $SO_4(3) \leq F_4$ but $F_4 \not\leq SO_4(3).2$. We show that there is a plane $\Pi$ extending a plane $\Pi_0$ of $\Sigma_0$ such that $\Pi_0$ meets $Q$ in a conic, such that the stabilizer in $F$ of $\Pi$ and a point of $\Pi \cap Q$ does not fix $\Pi \cap Q$. Consider the two possibilities: $R^\perp \cap Q$ is hyperbolic or elliptic.

Suppose first the case where $R^\perp \cap Q$ is elliptic. Then $SO_4^-(3) \leq F_4$ but $F_4 \not\leq SO_4^-(3).2$ so by Result 3.8, the image of $F_4$ in $PSU_4(9)$ is either $PSU_4(9)$ or $PSL_3(4)$; in either case the image of $F_4$ contains $PSL_3(4)$. The centre of $SU_4(9)$ is cyclic of order 4 and the centre of $SO_4^-(3)$ has order 2 so $F_4$ contains a subgroup with structure $2.PSL_3(4)$ or $4.PSL_3(4)$ and the order of $F_4$ is divisible by either 40320 or 80640. Let $T \in R^\perp \cap Q$. The number of points in $R^\perp \cap H$ is 280 so the stabilizer of $T$ in $1 \times F_4$ is at least 144 if $F_1 = 2.PSL_3(4)$ and at least 288 if $F_1 = 4.PSL_3(4)$. On the other hand there are 10 points in $R^\perp \cap Q$ and $SO_4^-(3)$ has structure $2.A_6$ and hence the stabilizer of $T$ in $1 \times SO_4^-(3)$ has order 72. It follows that $1 \times F_4$ contains an element $f = (1, f')$ with $f'$ not in the centre of $F_4$ such that $f' \notin SO_4^-(3).2$ and $f'T = T$. As $f'$ cannot fix $R^\perp \cap Q$ there must be a point $U \in R^\perp \cap Q$ such that $f'U \notin Q$. If $T \notin f'U + V(f'U)$, then $\Pi = T + f'U + V(f'U)$ is a plane of $\Sigma$ extending a plane of $\Sigma_0$, $\Pi^\perp \cap R^\perp$ is a point not on $Q$ (so $\Pi$ meets $Q$ in a conic) and $f'h_{T,U}.f'^{-1} = h_{T,f'U} \in F$ fixes $\Pi$ and $T$ but not $\Pi \cap Q$. If $T \in f'U + V(f'U)$, then $L = T + f'U + V(f'U)$ is a secant line of $\Sigma$ extending a secant line of $\Sigma_0$ so there exists a $g \in 1 \times SO_4^-(3)$ such that $gfT = T, gf(T + U) = T + U$ and $gfU \neq U$. This time let $\Pi = R + L$, then $\Pi$ has the required properties.

Suppose now that $R^\perp \cap Q$ is hyperbolic. By Proposition 3.7, either there is a point $W \in (R^\perp \cap \Sigma_0) \setminus Q$ such that $1 \times F_1$ contains an $h_{X,Y'} \not\in G$ with $X', Y' \in (R + W)^\perp$ and $X' \in Q$ and such that $h_{X,Y'}$ does not fix $(R + W)^\perp \cap Q$ or there exists a line $L$ of $R^\perp$ extending a secant line of $\Sigma_0$ and an element of $1 \times F_4$ fixing $L$ and fixing exactly one point of $L \cap Q$. In the first case we take $\Pi = (R + W)^\perp$ and in the second we take $\Pi = R + L$.

Consider the orthogonal decomposition $\Pi \oplus \Pi^\perp$ of $\Sigma$. The projection $\tilde{F}_3$ of $Stab_E(\Pi)$ onto $\Pi$ is a subgroup of $U_3(9)$ containing $SO_3(3)$ in which the stabilizer of a point of $\Pi \cap Q$ does not fix $\Pi \cap Q$. Noting that $PSU_3(9)$ is the whole of $PU_3(9)$, it follows from Corollary 3.10 that $\tilde{F}_3$ maps onto $PU_3(9)$. The subgroup $F_3$ of $SU_3(9)$ determined by $1 \times F_3 = (1 \times SU_3(9)) \cap F$ is
a normal subgroup of $\bar{F}$ and maps onto a normal subgroup of $PSU_3(9)$. By Corollary 3.11, $F_3 = SU_3(9)$. Choose a point $Z$ of $\pi^\perp \cap \Sigma_0$ such that $Z \notin Q$ and $Z^\perp \cap Q$ is elliptic. With respect to the decomposition $Z \oplus Z^\perp$, $F$ contains subgroups $1 \times SO^-_3(3)$ and $1 \times (1 \times SU_3(9))$. The latter group has order 6048 which does not divide 80640. Hence $F \cap (1 \times SU_4(9)) = 1 \times SU_4(9)$ and $F$ contains $1 \times SU_4(9)$ as required.

\[ \square \]

**Proposition 3.13.** Suppose that $n = 4$ and $q = 5$. Then there exists a point $Z$ of $\Sigma_0 \setminus Q$ such that $F$ contains $1 \times SU_3(q^2)$, written with respect to the decomposition $Z \oplus Z^\perp$.

Proof. We first show that there exists a point $Z$ of $\Sigma_0 \setminus Q$, a point $X \in Z^\perp \cap Q$ and an element of $F$ acting as an element of $1 \times SU_3(q^2)$ with respect to the decomposition $Z \oplus Z^\perp$, fixing $X$ but not fixing $Z^\perp \cap Q$.

By Proposition 3.5, either there exists $h_{X^\perp Y^\perp} \in F \setminus G$ with $X^\perp \in Q$ or there exists $h_{X^\perp Y^\perp} \in F \setminus G$ with $X^\prime, Y^\prime \notin Q$ but $X^\prime + Y^\prime$ extending a secant line of $\Sigma_0$. If the former possibility arises, then the existence of an element in $F$ with the required properties was established in Propositions 3.6 and 3.7. It thus remains to consider the latter possibility. Let $L$ be the line $X^\prime + Y^\prime$ and let $X, Y$ be the points of $Q$ on $L$. The elements of $F$ stabilizing $L$ act as elements of $U_2(q^2) \times U_2(q^2)$ with respect to the decomposition $\Sigma = L \oplus L^\perp$. Denote by $F_a$ the projection of $F$ onto the first copy of $U_2(q^2)$ and by $F_b$ the kernel of the projection of $F$ onto the second copy of $U_2(q^2)$. Thus $F_b \times 1 \leq Stab_F L \leq F_a \times U_2(q^2)$ and $F_b$ is normal in $F_a$. Let $G_a$ and $G_b$ denote the corresponding groups for $G$ and let $\bar{F}_a, \bar{F}_b, \bar{G}_a, \bar{G}_b$ denote the images of $F_a, F_b, G_a, G_b$ in $PU_2(q^2)$, with $PU_2(q^2)$ being isomorphic to $S_5$. Then $\bar{F}_b, \bar{G}_b \leq PSU_2(q^2)$, but $\bar{G}_a \notin PSU_2(q^2)$. We observe that $G$ contains $SO_2^+(q) \times 1$, the element $g_A$ described in Proposition 2.2 can be regarded as lying in $G$ and further $G$ contains an element $g$ of $O_2(q) \times O_2(q)$ that has determinant $-1$ in each component (such an element is a product of symmetries and switches $X$ and $Y$). All the elements of $G_a$ and $G_b$ are identified in this way: $|G_a| = 16, |G_b| = 4, |\bar{G}_a| = 8, |\bar{G}_b| = 2$. Now $F$ contains the element $h_{X^\perp Y^\perp}$ and this is an element of $F_b \times 1$. The possibilities for $(\bar{F}_a, \bar{F}_b)$, in terms of subgroups of $S_5$ are: $(S_5, A_5), (S_4, A_4), (S_4, V_4)$ and $(D_8, V_4)$. In each case $\bar{F}$ contains the four-group $V_4$ that is the normalizer of $\bar{G}_b$ in $A_5$. 18
Let \( f \in F_b \setminus G_b \) such that \( \langle f, G_b \rangle \) corresponds to \( V_4 \). Then \( f \) must switch \( X \) and \( Y \) and so \( gf \) fixes each of \( X \) and \( Y \). Moreover there are orthogonal points \( Z, W \) of \( (L^\perp \cap \Sigma_0) \setminus Q \) such that \( g \) fixes each of \( Z \) and \( W \) and \( g \) lies in \( 1 \times SU_3(q^2) \) with respect to the decomposition \( \Sigma = Z \oplus Z^\perp \). Finally \( gf \in 1 \times SU_3(q^2) \), \( gfX = X \) but \( gf \) cannot fix \( Z^\perp \cap Q \).

Now let \( F_3 \) be the subgroup of \( SU_3(q^2) \) determined by \( 1 \times F_3 = (1 \times SU_4(9)) \cap F \). The argument above shows that \( SO_3(q) \leq F_3 \) with the stabilizer in \( F_3 \) of a point of \( Z^\perp \cap Q \) not fixing \( Z^\perp \cap Q \). Then, by Corollary 3.11, \( F_3 = SU_3(q^2) \).

**Theorem 3.14.** Suppose that \( n \geq 3 \) and that \( F \) is a subgroup of \( SU_n(q^2) \) containing \( SO_n(q) \) but not contained in \( G \), then \( F = SU_n(q^2) \) except when \( n = 3 \) and \( q = 3 \) or 5 and when \( n = 4 \) and \( q = 3 \).

Proof. The case \( n = 3 \) (and \( q \neq 3, 5 \)) is proved in Corollary 3.11. We assume now that \( n \geq 4 \) if \( q \neq 3 \) and \( n \geq 5 \) if \( q = 3 \). The main part of the proof is concerned with identifying a point \( Z \) of \( \Sigma_0 \setminus Q \) such that, with respect to the decomposition \( Z \oplus Z^\perp \) of \( \Sigma \), \( F \) contains \( 1 \times SU_{n-1}(q^2) \). This is already established in Propositions 3.12 and 3.13 in the cases \( n = 4, q = 5 \) and \( n = 5, q = 3 \).

Suppose \( n \geq 4 \) when \( q > 5 \), that \( n \geq 5 \) when \( q = 5 \), that \( n \geq 6 \) when \( q = 3 \) and that the theorem holds for subgroups of \( SU_{n-1}(q^2) \) containing \( SO_{n-1}(q) \).

Consider first the situation when \( n \geq 5 \). We have noted that \( SO_n(q) \) is generated by hyperbolic rotations and we have proved that \( G \) is the normalizer of \( SO_n(q) \) in \( SU_n(q^2) \). It follows that for some \( A, B \in Q \) (with \( B \notin A^\perp \)) and some \( f \in F \), we have \( fh_{AB}f^{-1} \in F \setminus SO_n(q) \). Then \( X = fA \) and \( Y = fB \) cannot both be in \( Q \) and by Proposition 3.1, \( h_{XY} \in F \setminus G \). By Propositions 3.2 and 3.3 there is a point \( Z \in \Sigma_0 \setminus Q \) such that \( F \) contains a \( h_{XY'} \) with \( X', Y' \in Z^\perp \) and \( h_{XY'} \notin G \), moreover \( h_{XY'} \) does not fix \( Z^\perp \cap Q \).

Now consider the case where \( n = 4 \). By Propositions 3.5, 3.6, and 3.7, there is a point \( Z \) of \( \Sigma_0 \setminus Q \) such that \( F \) contains an \( h_{XY'} \notin G \) with \( X', Y' \in Z^\perp \), \( Y' \notin Q \) and \( h_{XY'} \) not fixing \( Z^\perp \cap Q \).
Consider the decomposition $\Sigma = Z \oplus Z^\perp$ with $Z$ as described in the paragraphs above and consider the subgroup $1 \times SU_{n-1}(q^2)$ of $SU_n(q^2)$ with respect to this decomposition. Denote by $F_{n-1}$ the subgroup of $SU_{n-1}(q^2)$ determined by $1 \times F_{n-1} = F \cap (1 \times SU_{n-1}(q^2))$. Denote also by $G_{n-1}$ the normalizer of $SO_{n-1}(q)$ in $SU_{n-1}(q^2)$ and note that $1 \times G_{n-1}$ is not usually a subgroup of $G$. In each of the cases above we have an element of $1 \times F_{n-1}$ not lying in $1 \times G_{n-1}$. By induction, $1 \times F_{n-1} = 1 \times SU_{n-1}(q^2)$.

Now suppose $n \geq 4$ when $q > 3$, that $n \geq 5$ when $q = 3$. Let $Z'$ be a non-isotropic point in $Z^\perp \cap \Sigma_0$ of the same class as $Z$. Then (conjugating $1 \times SU_{n-1}(q^2)$ by an element of $SO_n(q)$ taking $Z$ to $Z'$) we see that $F$ contains $SU_{n-1}(q^2) \times 1$ (relative to the decomposition of $\Sigma$ into $Z^\perp \oplus Z'$). Now any element of $SU_n(q^2)$ stabilizing $Z$ has the form $(\lambda \times \lambda^{-1} I_{n-2}) (1 \times h)$ for some $\lambda \in GF(q^2)$ with $\lambda^{q+1} = 1$ and some $h \in SU_{n-1}(q^2)$. Both $(\lambda \times \lambda^{-1} I_{n-2})$ and $(1 \times h)$ lie in $F$ so $\text{Stab}_{SU_n(q^2)}(Z) < F \leq SU_n(q^2)$. However this stabilizer is known to be maximal by [12], so $F = SU_n(q^2)$.

The following corollary follows immediately from Theorem 3.14 and Results 3.8 and 3.9 and establishes our main theorem.

**Corollary 3.15.** Suppose that $n \geq 3$. The normalizer of $SO_n(q)$ in $SU_n(q^2)$ is a maximal subgroup of $SU_n(q^2)$. The stabilizer of $Q$ in $PSU_n(q^2)$ is a maximal subgroup of $PSU_n(q^2)$. In both cases there are exceptions when $n = 3$ and $q = 3$ or 5 and when $n = 4$, $q = 3$ and $Q$ is hyperbolic.

### 4 The exceptional case: $n = 4$ and $q = 3$

We consider here the case where $n = 4$ and $q = 3$ and where $Q$ is a hyperbolic quadric. It is known that in this case $G$ is not maximal (for example, it does not appear as a maximal subgroup in [2]). We describe here, briefly, an overgroup of $G$ that fixes a geometric configuration.

There are 8 lines on $Q$, falling in two sets of four, each a regulus of $\Sigma_0$. There are 112 lines on $H$. It can be shown that $G$ has orbits of lengths 8, 16, 16 and 72 on this set. The orbits of size 16 are each complete spans of $H$. Given either of these spans, the stabilizer in $PSU_4(9)$ of the span is isomorphic to $2^4.A_6$ and contains $G$. From [2] it is apparent that the subgroup $2^4.A_6$ is
maximal in \( \text{PSU}_4(9) \). More information on spans of \( \mathcal{H}(n - 1, q^2) \) in general, and \( \mathcal{H}(3, 9) \) in particular, is given in [8].

References


