Chapter 6

Describing Random Sequences

6.1 Introduction

Signals that follow a periodic pattern and have precisely defined sample values are generally referred to as *deterministic*. By contrast the individual values of *random signals* are not defined, nor can their future be predicted with certainty. In general, deterministic signals are useful for testing systems, such as finding the frequency response of a filter for example, whilst random signals are found wherever there is the transmission of information. In fact it is the unpredictability of the signal that fundamentally provides its information bearing capability. In the field of DSP, the knowledge of random functions is essential in understanding both the information content of digital signals, and the properties of digital noise. The underlying principles are those of probability and statistics, which can be used to define useful average measures of random sample sequences.

6.2 Mean, Mean-Square and Variance

The *mean* is found by multiplying each allowed value of the sequence by the probability with which it occurs, followed by summation. It is sometimes referred to as the *dc value* and is represented by Equation 6.1, where E denotes mathematical expectation.

$$\overline{x[n]} = E\{x[n]\} = \sum_{c=-\infty}^{\infty} c \cdot P_{x[n]=c} \tag{6.1}$$

where c is the range of real numbers. Another practical measure is the *mean-square* and it is sometimes called the *average power* and is defined by Equation 6.2 below. It is calculated by squaring each sequence value, multiplying by the relevant probability, and summing over all possible values.

$$\overline{x^2[n]} = E\{x^2[n]\} = \sum_{c=-\infty}^{\infty} c^2 \cdot P_{x[n]=c} \tag{6.2}$$

The third important measure of a signal is the *variance*, which refers to fluctuations around the mean. The variance is defined by Equation 6. 5 below, it is similar to the mean-square but with the mean removed.

$$\sigma^2 = E\{(x[n] - m_x)^2\} \tag{6.3}$$

As the mean-square is equivalent to the total average power of a random variable, and the square of the mean represents the power of its *dc* component, it follows that the variance is a measure of the average power in all of its other frequency components. We may therefore like to think of the variance as a measure of the *ac power*.

6.3 Ensemble Averages and Time Averages

So far we have implied that the probabilities associated with the discrete values of a random digital sequence are known in advance, or can be calculated in some way. Furthermore, the probabilities have also been used to deduce the mean and variance of the sequence. This raises a number of points that need explanation and justification. In particular, we should clarify the distinction between what are known as *ensemble averages* and *time averages*.

A random (or *stochastic*) digital process is one giving rise to an infinite set of random variables, of which a particular sample sequence $x[n]$, $-\infty < n < \infty$, is just one realisation. The set of all sequences that could be generated by the process is known as an infinite ensemble, and it is this to which the probability functions and expected values defined by Equations 6.1 – 6.5 strictly refer. Such measures are therefore known as ensemble averages.

In DSP we prefer to deal with individual sequences rather than ensembles. Each sample represents a single value of one of the variables described by the underlying stochastic process. We must therefore make a connection between ensemble properties and those of an individual sequence existing over a period of time. This is achieved by *time* *averages* and we define such measures as the time-averaged mean and variance by Equation 6.4 and Equation 6.5 respectively.

$$\overline{x[n]} = E\{x[n]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n] \tag{6.4}$$

$$\overline{x^2[n]} = E\{(x[n])\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n])^2 \tag{6.5}$$

$$\sigma^2 = E\left\{(x[n] - \overline{x[n]})\right\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n] - \overline{x[n]})^2 \tag{6.6}$$

The limits used in these equations only exist if the sequence $x[n]$ has a finite mean value and is *stationary*. A process is said to be stationary in the *strict sense* if all of its statistics are independent of time origin. However, in the case of linear DSP, it is accepted that the less stringent conditions of *wide sense* stationary may be used instead. This requires that only the mean, and the correlation functions are independent of the time origin. Given wide sense stationary, we also require that the digital sequence we are dealing with obey the *ergodic* hypothesis. A process is ergodic if the time averages equal the ensemble expected averages.

6.4 Autocorrelation

The amplitude distribution of a random sequence gives a complete statistical description of its amplitude fluctuation and tells us about the probability of finding an individual sample value at various levels. Unfortunately it does not tell us about whether successive sample values are related to one another. However, it is possible to characterise the time-domain structure of a sequence, using the *autocorrelation function*. The autocorrelation function (ACF), as defined by Equation 6.6, is the average product of the sequence $x[n]$ with a time shifted m, version of itself. Autocorrelation is a valuable measure of the statistical dependence between values of $x[n]$ at different times, and summarises its time-domain structure. A close relation of the autocorreletion function is the autocovariance function and is defined by Equation 6.8 below.

$$\phi_{xx}[m] = E\{x[n]x[n+m]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]x[n+m] \tag{6.7}$$

$$\gamma_{xx}[m] = E\left\{(x[n] - \overline{x[n]})(x[n+m] - \overline{x[n]})\right\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n] - \overline{x[n]})(x[n+m] - \overline{x[n]}) \tag{6.8}$$

Close inspection of these two functions reveals that the autocorrelation and autocovariance functions are identical, except that the autocovariance has the mean removed. In the case of a random variable having zero mean, the autocorrelation and autocovariance functions are identical. Since Equations 6.6 and 6.7 have been derived on the assumption of *wide sense stationarity*, it follows that $\phi_{xx}[m]$ and $\gamma_{xx}[m]$ are both symmetrical about m=0.

The autocorrelation functions of five examples are illustrated in Figure 6.1. On the left-hand side of the diagram, the digital sequences are shown, and on the right-hand side their computed autocorrelation functions. Parts (a) and (b) of the Figure show two deterministic sequences: a sinusoid; and a cosinusoid with a dc offset. The autocorrelation functions of these sequences are clearly visible. As Equation 6.6 indicates, we multiply a long portion of the sequence by a *time shifted* version of itself, and then average the result. This process is again repeated for different values of the shift parameter m. It is understandable that when the shift is zero, positive peaks of the sinusoids align with positive peaks, and negative peaks with negative peaks, giving a large positive product. Therefore we expect the autocorrelation function to be large and positive for m = 0, as depicted in the diagram. Now as we introduce time shift (sliding one version of the sequence beneath the other, cross-multiplying and summing), the positive and negative peaks begin to loose their alignment. By the time all positive peaks are aligned with negative ones and visa-versa, the autocorrelation function will have a large negative value. As the sliding process continues the ACF traces out a repetitive function, whose period in the time shift variable m equals the period of the sequence itself in the time variable n. We can conclude that a repetitive sequence $x[n]$ has an autocorrelation function which is also repetitive, with the same period. In the special case of the sinusoid, it turns out the ACF takes the form of a cosine.

We now consider the autocorrelation function of three random functions. Figure 6.1 (c) shows a sequence of uniformly distributed random numbers in the range –1 to 1. Where successive values may be assumed as being independent. When such a sequence is autocorrelated, we get a large central peak at m = 0, with small non-zero values either side. At zero time shift, positive and negative values align with themselves giving a large positive value. But at any other value of time shift a given sample is just as likely to align with one of the opposite signs, so the cross product averages out to zero. In other words there is no correlation between adjacent samples. This type of sequence is more commonly referred to as *white noise*.

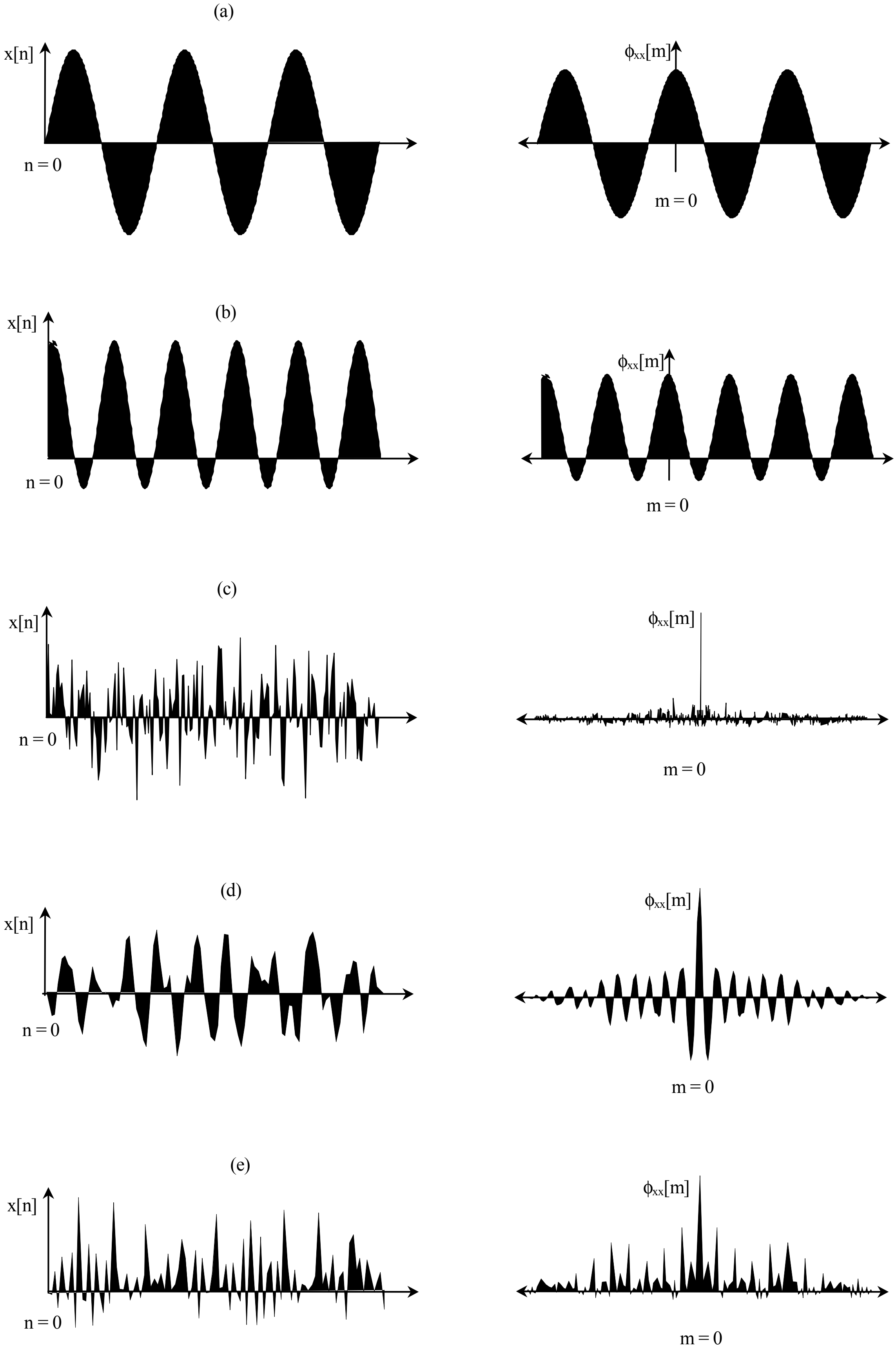

Figure 6.1: Five digital sequences and their corresponding autocorrelation functions.

In Figure 6.1 (d) it shows a similar sequence after band-pass filtering. The autocorrelation function is now spread considerably to either side of $m = 0$, reflecting correlation between adjacent sample values. In Figure (e), white noise is passed through a low pass filter and a repetitive impulse train with a period of 10 sampling intervals was added to the signal. Notice how the impulse train is not recognisable in the signal $x[n]$, but the autocorrelation brings it out quite clearly. Also notice how the filtered noise only contributes to the autocorrelation function around $m = 0$, whereas the pulse train, being strictly repetitive contributes over the complete range of time shift. This suggests an important practical application for autocorrelation – the detection of a repetitive signal in the presence of unwanted noise.

The central value of an autocorrelation function equals the *mean square value* of the sequence, and is therefore a measure of its *total power*. The central value is always the maximum value. It may be equalled at other values of time shift but it can never be exceeded. Thus by setting $m = 0$ in Equation 6.6 we obtain the following measure.

$$\phi_{xx}[0] = E\{(x[n]^2\} \tag{6.9}$$

Similarly, by setting $m = 0$ in Equation 6.7, the central value of an autocovariance function equals the *variance* of the corresponding sequence – equivalent to its *ac power*.

6.4.1 Signals in noise

One of the most important topics in digital signal processing concerns the extraction of wanted signals from unwanted noise. When a signal, contaminated by noise, is to be recovered or detected, a useful way of detecting it is by using autocorrelation. An expression for the autocorrelation function is derived below.

$$y[n] = s[n] + q[n]$$

$$\phi_{yy}[m] = E\{[s[n] + q[n]][s[n+m] + q[n+m]]\}$$

$$= E\{s[n] \cdot s[n+m]\} + E\{s[n] \cdot q[n+m]\} + E\{q[n] \cdot s[n+m]\} + E\{q[n] \cdot q[n+m]\}$$

$$= \phi_{ss}[m] + \phi_{qq}[m] + 2E\{s[n] \cdot q[n+m]\}$$

The periodic signal $s[n]$ and noise $q[n]$ are completely uncorrelated to each other. Another way of stating this is that the probability of any particular value of $q[n]$ is totally independent of the value of $s[n]$ (i.e. $P[q/s] = P[q]$, or in words, the probability of q given s is identically equal to the probability of s). Under these circumstances the last term in the equation above is determined by the following:

$$E\{s[n] \cdot q[n+m]\} = E\{s[n]\} \cdot E\{q[n+m]\} = 0$$

It is identically equal to zero because $E\{q[n+m]\}$ is the mean of the noise which is by definition itself zero. Hence the auto-correlation function of a signal with white noise of zero mean is found by replacing the last term with zero, hence

$$\phi_{yy}[m] = \phi_{ss}[m] + \phi_{qq}[m]$$

This is the Principle of Superposition that states the ACF is composed of the individual ACF's of both the signal and noise, *providing that signal and noise are uncorrelated*. This is an extremely important relationship, which is often used to detect the signal from the unwanted noise.

6.5 Wiener-Khintchine Power Theorem

The Wiener-Khintchine theorem states that the power spectrum of a signal can be expressed as the Fourier transform of its *ACF*. The power spectrum of a signal can be defined by:

$$P_{xx}(\Omega) = |X(\Omega)|^2 = X(\Omega) \cdot X^*(\Omega) = \left\{\sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}\right\}\left\{\sum_{n=-\infty}^{\infty} x[n]e^{j\Omega n}\right\} \tag{6.10}$$

Where $X(\Omega)$ and $X^*(\Omega)$ are the Fourier transform of $x[n]$ and its complex conjugate respectively. However, the complex conjugate of the Fourier transform of $x[n]$ is identical to the Fourier transform of $x[-n]$, as we can see below by a simple replacement of n by $-n$ in the second summation:

$$P_{xx}(\Omega) = \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right\} \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{j\Omega n} \right\} = \left\{ \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n} \right\} \left\{ \sum_{n=-\infty}^{\infty} x[-n] e^{-j\Omega n} \right\} \tag{6.11}$$

Since Equation 6.11 describes the product of the Fourier Transforms of $x[n]$ and $x[-n]$, it can also be expressed as the Fourier Transform of the convolution of $x[n]$ and $x[-n]$.

$$P_{xx}(\Omega) = X(\Omega) \cdot X^*(\Omega) = \sum_{n=-\infty}^{\infty} \{x[n] * x[-n]\} \cdot e^{-j\Omega n} \tag{6.12}$$

Furthermore, the convolution of $x[n]$ with $x[-n]$ is identical to the correlation of $x[n]$ and $x[n]$ (i.e. the ACF ϕ_{xx}). Therefore the power spectrum can be obtained by correlating the signal $x[n]$ with $x[-n]$ and then taking the Fourier transform.

$$P_{xx}(\Omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}[m] \cdot e^{-j\Omega m} \tag{6.13}$$

There also exists the inverse Fourier transform expressing ϕ_{xx} as a function of $P_{xx}(\Omega)$:

$$\phi_{xx}[m] = \frac{1}{2\pi} \int_{2\pi} P_{xx}(\Omega) \cdot e^{j\Omega m} d\Omega = \int_{-0.5}^{0.5} P_{xx}(F) \cdot e^{j2\pi F m} dF \tag{6.14}$$

When we calculate the autocorrelation function, it is customary to work with a finite portion of the sequence and estimate the ACF for a limited set of time shift values. Figure 6.2 illustrates the estimated autocorrelation functions and power spectra for a band-pass filtered noise sequence. The ACF has been estimated by cross-multiplying 512 values of an infinite sequence with time shift values in the range $-128 \le m \le 128$, and the corresponding estimated power spectrum was calculated using the Fourier transform.

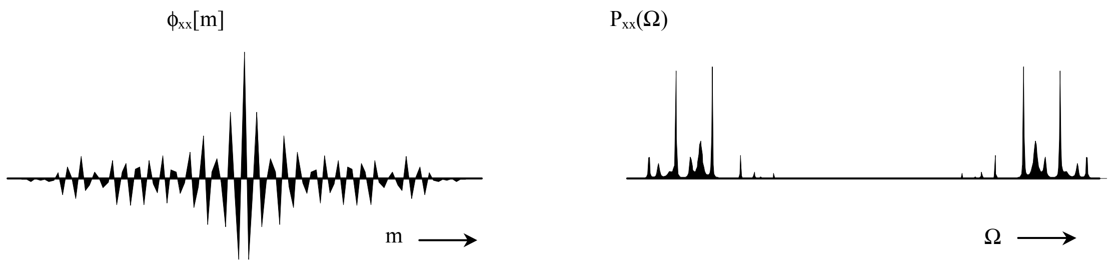

Figure 6.2: Autocorrelation and power spectral estimates for a band-pass filtered noise sequence.

If we assume the ACF to be zero outside the estimated range, we are effectively truncating it with a rectangular window. This tends to produce substantial sidelobes in the power spectrum, owing to the Gibb's phenomenon (directly comparable to the *spectral leakage* problem described in the chapter on the DFT).

6.6 Cross-Correlation

The autocorrelation function can be used to characterise a sequence's time domain structure. Cross-correlation is essentially the same process but instead of comparing a sequence with a time shifted version of itself, it compares two different sequences. The *cross-correlation function* (CCF) of two sequences $x[n]$ and $y[n]$, and the *cross-covariance* function are defined in terms of time averages by Equation 6.14 and Equation 6.15 respectively.

$$\phi_{xy}[m] = E\{x[n]y[n+m]\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]y[n+m] \tag{6.15}$$

$$\gamma_{xy}[m] = E\{(x[n] - \overline{x[n]})(y[n+m] - \overline{y[n]})\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} (x[n] - \overline{x[n]})(y[n+m] - \overline{y[n]}) \tag{6.16}$$

Both of these functions are second-order measures, with the CCF providing a statistical comparison of two sequences as a function of the time-shift between them. Cross-covariance is the same as the CCF, except that the mean values of the two sequences are removed. The CCF reflects the various frequency components *held* in *common* between the two sequences $x[n]$ and $y[n]$. In addition, it also holds vital information about the relative phases of shared frequency components. Unfortunately, when the cross-correlation of two sequences is performed, sometimes the fine detail of the shared frequency components is hard to interpret. If a detailed spectral analysis of the signals is required then it is better to use the cross-spectrum approach. However from a practical point of view there is one situation where the CCF is useful – namely when there are *timing differences* between two sequences. For example, suppose that $x[n]$ and $y[n]$ are identical white noise sequences which differ only in the time origin. Their CCF will then be zero for all values of m, except the one which corresponds to the timing difference.

Now let us suppose that the two signals $x[n]$ and $y[n]$, are completely *uncorrelated* with each other. From Equation 6.14, it can be shown that their CCF is a product of the expectation of each signal, as illustrated below.

$$\phi_{xy}[m] = E\{x[n]\} \cdot E\{y[n+m]\}$$

6.7 Cross-Correlation Coefficient

Sometimes it is preferable to express the cross correlation of two signals in terms of the *cross-correlation coefficient*. It is calculated by normalising the cross-correlation of the two signals with the power of the two signals i.e. by setting m = 0, as illustrated in Equation 6.16 below. The cross-correlation coefficient lies between -1 and +1, with zero indicating no correlation between the two signals.

$$\ell_{xy}[m] = \frac{\phi_{xy}[m]}{\left[\phi_{xx}[0] \cdot \phi_{yy}[0]\right]^{\frac{1}{2}}} \tag{6.17}$$

6.8 Cross Spectrum

The frequency domain counterpart of a cross-correlation function relating two sequences is known as the *cross-spectral density* or the *cross-spectrum*. It is an indication of the frequencies held in common between $x[n]$ and $y[n]$. If the two sequences have no shared frequencies, or frequency ranges, then their cross-spectrum (like their CCF) is zero. These types of sequences are said to be *linearly independent*, or *orthogonal* to each other. By definition the cross-correlation function and cross-spectrum of a digital sequence are related as a Fourier transform pair, given by Equation 6.17 and Equation 6.18 respectively.

$$\phi_{xy}[m] = \frac{1}{2\pi} \int_{2\pi} P_{xy}(\Omega) \cdot e^{j\Omega m} d\Omega \tag{6.18}$$

$$P_{xy}(\Omega) = \sum_{m=-\infty}^{\infty} \phi_{xy}[m] \cdot e^{-j\Omega m} \tag{6.19}$$

In section 6.5, we described the difficulty of obtaining a reliable estimate for the power spectrum of a time-limited random sequence. The same problems also apply herein the cross-spectrum estimation.

6.9 Examples

It is shown below how to calculate the auto correlation functions for a number of signals directly from their power spectra. This can be achieved by using the relationship between the power spectrum and the auto-correlation function given by Equation 6.13 and Equation 6.14. Essentially they are both related by a Fourier transform pair, so the auto-correlation function can be found by integrating the power spectrum.

$$\phi_{xx}[m] = \int_{-0.5}^{0.5} P_{xx}(F) \cdot e^{j2\pi Fm} dF$$

1.) Auto-correlation of a DC signal.

The power spectrum of a DC signal is given by the following: $P_{xx}(F) = A^2\delta(F)$

$$\phi_{xx}[m] = A^2 \int_{-0.5}^{0.5} \delta(F) \cdot e^{j2\pi Fm} dF \tag{6.20}$$

$$\phi_{xx}[m] = A^2 \tag{6.21}$$

2.) Auto-correlation of a sine wave signal

A sine wave function $x[n]$ is depicted in Figure 6.1 (a), with its auto-correlation function depicted on the right hand side. However, an alternative mathematical proof can be shown by using the Fourier transform relationship.

The power spectrum of a sine wave is given by: $P_{xx}(F) = \frac{A^2}{4}[\delta(F + F_c) + \delta(F - F_c)]$

Hence the auto-correlation function is found by integration:

$$\phi_{xx}[m] = \frac{A^2}{4} \int_{-0.5}^{0.5} [\delta(F + F_c) + \delta(F - F_c)] \cdot e^{j2\pi Fm} dF$$

$$\phi_{xx}[m] = \frac{A^2}{4}\left[e^{j2\pi F_c m} + e^{-j2\pi F_c m}\right]$$

$$\phi_{xx}[m] = \frac{A^2}{2} \cos(2\pi F_c m) \tag{6.22}$$

Notice how the expression is the same as the diagram on the right side of Figure 6.1 (a).

3.) Auto-correlation of white noise

A white noise sequence $x[n]$ is depicted in Figure 6.1 (c), with its auto correlation function depicted on the right hand side. We shall now derive the auto correlation function directly from its power spectrum.

The power spectrum of white noise is given by the following: $P_{xx}(F) = S$

Hence the auto-correlation function is found by integration:

$$\phi_{xx}[m] = S \int_{-0.5}^{0.5} e^{j2\pi Fm} dF$$

$$\phi_{xx}[m] = \begin{cases} S & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases} \tag{6.23}$$

Notice how the expression above is the same as the diagram on the right hand side of Figure 6.1 (c).

4.) Auto-correlation of a low-pass filtered white noise sequence

A low-pass filtered white noise sequence was one of the components depicted Figure 6.2. Its power spectrum is:

$$P_{xx}(F) = \begin{cases} S & \text{for } -F_1 \leq F \leq F_1 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\phi_{xx}[m] = S \int_{-F_1}^{F_1} e^{j2\pi Fm} dF$$

$$= \frac{S}{j2\pi m}\left[e^{jm2\pi F_1} - e^{-jm2\pi F_1}\right]$$

$$= S\frac{\sin(m\Omega_1)}{\pi m}$$

$$= S \cdot 2F_1 \frac{\sin(m\Omega_1)}{(m\Omega_1)} \quad (6.24)$$

The expression Equation 6.23 is the well known *sinc* function. Note that the width of the main peak of the *sinc* (i.e. between the first two zero crossings) is equal to $1/F_1$ samples (i.e. first zero crossings at $m = \pm\pi/\Omega_1 = \pm 1/2F_1$).

5.) Auto-correlation of a band-pass filtered white noise sequence

A band-pass filtered white noise sequence $x[n]$ is depicted in Figure 6.1 (d), with its auto correlation function depicted on the right hand side. We shall now once again derive the auto correlation function directly from the power spectrum. The power spectrum of a band-pass filtered white noise sequence is:

$$P_{xx}(F) = \begin{cases} S & \text{for } F_1 \le |F| \le F_2 \\ 0 & \text{otherwise} \end{cases}$$

Hence:

$$\phi_{xx}[m] = S\int_{-F_2}^{-F_1} e^{j2\pi Fm}\,dF + S\int_{F_1}^{F_2} e^{j2\pi Fm}\,dF \quad (6.25)$$

$$= \frac{S}{j2\pi m}\left[e^{-jm2\pi F_1} - e^{-jm2\pi F_2} + e^{jm2\pi F_2} - e^{jm2\pi F_1}\right]$$

$$= \frac{S}{j2\pi m}\left[e^{jm2\pi F_2} - e^{-jm2\pi F_2}\right] - \frac{S}{j2\pi m}\left[e^{jm2\pi F_1} - e^{-jm2\pi F_1}\right]$$

$$= S\frac{\sin(m\Omega_2)}{\pi m} - S\frac{\sin(m\Omega_1)}{\pi m}$$

$$= S2F_2\frac{\sin(m\Omega_2)}{(m\Omega_2)} - S2F_1\frac{\sin(m\Omega_1)}{(m\Omega_1)} \quad (6.26)$$

The expression above is the sum of two *sinc* functions and is exactly the same as the diagram on the right hand side of Figure 6.1 (d).