PTOLEMY DIAGRAMS AND TORSION PAIRS IN THE CLUSTER CATEGORIES OF DYNKIN TYPE D

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Abstract. We give a complete classification of torsion pairs in the cluster category of Dynkin type $D_n$, via a bijection to new combinatorial objects called Ptolemy diagrams of type $D$. For the latter we give along the way different combinatorial descriptions. One of these allows us to count the number of torsion pairs in the cluster category of type $D_n$ by providing their generating function explicitly.

1. Introduction

Torsion theory is a classic subject in algebra. The fundamental example of a torsion pair appears for abelian groups with the class of torsion abelian groups and the class of torsion-free abelian groups as the two entries. For arbitrary abelian categories the concept of torsion pairs goes back to a paper by Dickson [6] from the mid 1960’s. Since then torsion theory appeared naturally in various contexts, in the representation theory of finite dimensional algebras most notably in the framework of tilting theory. In recent years the focus of several modern developments in representation theory has been on derived categories and related triangulated categories, e.g. stable module categories or cluster categories. A notion of torsion pairs in triangulated categories has been introduced by Iyama and Yoshino [14].

In this paper we will study and classify combinatorially torsion pairs in cluster categories of Dynkin type $D_n$.

Cluster categories have been introduced by Buan, Marsh, Reineke, Reiten and Todorov [5] as a categorical model for Fomin and Zelevinsky’s cluster algebras. Roughly speaking, the indecomposable objects in the cluster category correspond to the cluster variables and certain direct sums of indecomposable objects, called cluster tilting objects, then correspond to the clusters in the cluster algebras. Most importantly, the fundamental mutation operation on clusters in cluster algebras is reflected by exchanging summands in cluster tilting objects in the cluster category. This categorification approach to cluster algebras via cluster categories has been and still is highly successful in that numerous important results on cluster algebras have been proven by using cluster categories.

In representation theory, the advent of cluster categories has created an entirely new research area, namely cluster tilting theory; one of the important aspects of this

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new theory is that it provides a long-awaited generalization of the classic Bernstein-
Bernstein-Gelfand reflection functors and of the more general APR-tilting.

Due to the importance of cluster categories in the theory of cluster algebras a lot of
research goes into understanding the structure of cluster categories. From the point
of view of torsion theory a systematic study of torsion pairs in cluster categories
has only started recently.

In her thesis, Ng [18] classified torsion pairs in the cluster categories of type $A_\infty$.
These categories have been studied in detail in [10]; they are generated by a spher-
érical object and hence fit into the work of Keller, Yang and Zhou [16] where it is
in particular shown that this category is uniquely determined up to triangulated
equivalence.

Ng’s classification of torsion pairs in the cluster category of type $A_\infty$ is combinato-
rial in the sense that she uses a combinatorial model, namely arcs of the infinity-gon
(see [10]), to give an explicit bijection between torsion pairs and certain configura-
tions of arcs.

Similar configurations, then called Ptolemy diagrams, appeared later also in our
classification of torsion pairs in cluster categories of Dynkin type $A_n$ [11]. More-
over, we enumerated these torsion pairs and gave an explicit closed formula for the
number of torsion pairs in the cluster category of type $A_n$.

Classifications and enumerations of torsion pairs have also been achieved for cluster
categories coming from tubes [12] (see related work of Baur, Buan and Marsh [1]
on torsion pairs in the abelian tube categories).

In the present paper we will provide a complete classification and enumeration of
torsion pairs in the cluster category of Dynkin type $D_n$. The situation is more
complicated than in Dynkin type $A_n$, caused by the exceptional vertices appearing
in a Dynkin diagram of type $D$, and hence the corresponding exceptional objects
in the cluster category.

Our work in this paper will be based on a combinatorial model for the cluster
category of type $D_n$ which first appeared in a paper by Fomin and Zelevinsky
[7]. There the indecomposable objects are parametrized by pairs of rotationally
symmetric arcs and by diameters in two colours in a regular 2n-gon. For a precise
description of this model we refer to Section 2 below.

Torsion pairs in the cluster category $D_n$ of Dynkin type $D_n$ are pairs $(X, X^\perp)$ of
subcategories of $D_n$ closed under direct sums and direct summands which satisfy
the condition $X = \perp(X^\perp)$; see [14]. Here, the perpendicular subcategories are taken
with respect to $\text{Hom}$. See Section 3 for more details.

Since the subcategories appearing in a torsion pair are closed under direct sums
and direct summands they are uniquely determined by the set of indecomposable
objects they contain. For a subcategory $X$ let $\mathcal{X}$ be the collection of arcs of the
$2n$-gon corresponding to $X$ in the Fomin-Zelevinsky model.

Our first main result gives a combinatorial characterization for those collections $\mathcal{X}$
corresponding to a subcategory $X$ appearing as the first half of a torsion pair in
$D_n$. To this end we introduce the new notion of a $Ptolemy diagram of type D$ in
Definition 4.1.

Then we go on to show the following main result in Section 4.
Theorem 1.1. Let $\mathcal{X}$ be a collection of arcs of the $2n$-gon which is invariant under rotation by 180 degrees, and let $\mathcal{X}$ be the corresponding subcategory of the cluster category $\mathcal{D}_n$. Then the following conditions are equivalent.

(a) $(\mathcal{X}, \mathcal{X}^\perp)$ is a torsion pair in $\mathcal{D}_n$.
(b) $\mathcal{X}$ is a Ptolemy diagram of type $D$.

As an application of this combinatorial classification we then deal in Section 5 with the enumeration of torsion pairs in $\mathcal{D}_n$. We first give an alternative description of Ptolemy diagrams of type $D$ which is closely linked to the Ptolemy diagrams of Dynkin type $A$ studied in our earlier paper [11].

This alternative description then allows us to work out the generating function for Ptolemy diagrams of type $D$, as an explicit expression involving the corresponding generating function for torsion pairs in type $A$. The main result of Section 5 can be summarized as follows (for unexplained notation we refer to Section 5).

Theorem 1.2. For $n \geq 4$ let $\mathcal{D}_n$ be the cluster category of Dynkin type $D_n$. Then the number of torsion pairs in $\mathcal{D}_n$ is given by the generating function

$$
P_D(y) := \sum_{n \geq 1} \# \{ \text{Ptolemy diagrams of type } D \text{ of the } 2n\text{-gon} \} y^n$$

$$= y \mathcal{P}_A'(y) \frac{1 + 12 \mathcal{P}_A(y) - \mathcal{P}_A^2(y) - 2 \mathcal{P}_A^3(y)}{1 - 2 \mathcal{P}_A(y) - \mathcal{P}_A^2(y)}$$

$$= y + 16y^2 + 82y^3 + 500y^4 + 3084y^5 + 19400y^6 + \ldots$$

where $\mathcal{P}_A(y)$ is the generating function for Ptolemy diagrams of Dynkin type $A$, as studied in [11].

The paper is organized as follows. In Section 2 we recall in some detail Fomin and Zelevinsky’s combinatorial model for a cluster category of Dynkin type $D_n$ from [7]. In Section 3 we first briefly review the fundamentals on torsion theory in triangulated categories, as introduced by Iyama and Yoshino [14]. Then we apply and make explicit this concept for the cluster categories $\mathcal{D}_n$. In particular we show in Proposition 3.5 that the following holds: $(\mathcal{X}, \mathcal{X}^\perp)$ is a torsion pair in $\mathcal{D}_n$ if and only if the corresponding collection $\mathcal{X}$ of arcs satisfies $\mathcal{X} = \text{nc nc } \mathcal{X}$ (where $\text{nc } \mathcal{X}$ consists of the arcs not crossing any arc from $\mathcal{X}$). Section 4 then constitutes one of the two main parts of the paper. Namely, we first introduce Ptolemy diagrams of type $D$ by imposing explicit combinatorial conditions on collections of arcs (see Definition 4.1) and then make a detailed analysis to show that these new Ptolemy diagrams of type $D$ are precisely the collections of arcs satisfying $\mathcal{X} = \text{nc nc } \mathcal{X}$, i.e. the ones corresponding to torsion pairs in $\mathcal{D}_n$. The second main part of the paper is Section 5 in which we enumerate torsion pairs in the cluster category of Dynkin type $D_n$. To this end we establish an alternative description of Ptolemy diagrams of type $D$ via different types of central regions to which Ptolemy diagrams of type $A$ are glued.

2. A geometric model for cluster categories of Dynkin type $D$

In this section we will briefly recall the definition and describe the structure of the cluster category of Dynkin type $D_n$. Moreover, we recall in detail a geometric
model, introduced by Fomin and Zelevinsky [7], for this cluster category on which we will build throughout the paper.

For a quiver $Q$ without oriented cycles the cluster category (over a field $k$) has been introduced by Buan, Marsh, Reineke, Reiten and Todorov [5] as the orbit category

$$\mathcal{C}_Q := \text{D}^b(kQ)/((\tau^{-1} \circ \Sigma)$$

where $\tau$ and $\Sigma$ are the Auslander-Reiten translation and the suspension on the bounded derived category of the path algebra $kQ$. It has been shown by Keller [15] that $\mathcal{C}_Q$ is a triangulated category.

Now let $Q$ be a Dynkin quiver of type $D_n$ for an integer $n \geq 4$. Since the path algebras for different orientations are known to be derived equivalent, we can assume that $Q$ has the orientation as given in Figure 1. The vertices $(n-1)^\pm$ are called exceptional, the others non-exceptional. By a result of Happel [8, Corollary 4.5(i)],
the Auslander-Reiten quiver of the derived category $\mathcal{D}^b(kQ)$ is the repetitive quiver $\mathbb{Z}D_n$ shown in Figure 2.

In the sequel we denote the cluster category $\mathcal{C}_{D_n}$ just by $D_n$. In this cluster category, objects of the derived category are identified modulo the action of the functor $\tau^{-1} \circ \Sigma$. So let us describe this action explicitly.

We will frequently use in this paper the coordinate system on $\mathbb{Z}D_n$ (first appearing in a paper by Iyama [13, Section 4]) as given in Figure 3.

The inverse Auslander-Reiten translation $\tau^{-1}$ acts by shifting one unit to the right. More precisely, in the coordinate system this means that $\tau^{-1}(i, j) = (i+1, j+1)$ for the non-exceptional vertices, and $\tau^{-1}(i, i+n)^\pm = (i+1, i+n+1)^\pm$ for the exceptional vertices.

The action of the suspension functor is more subtle and depends on the parity of $n$; it can be deduced from a paper by Miyachi and Yekutieli [17, table p. 359].

If $n$ is even, then the suspension $\Sigma$ acts by shifting $n-1$ units to the right. Expressed with the coordinate system we thus have $\Sigma(i, j) = (i+n-1, j+n-1)$ for the non-exceptional vertices and $\Sigma(i, i+n)^\pm = (i+n-1, i+2n-1)^\pm$ for the exceptional vertices.

If $n$ is odd, then $\Sigma$ acts by shifting $n-1$ units to the right and switching each pair of exceptional vertices, i.e. $\Sigma(i, j) = (i+n-1, j+n-1)$ for the non-exceptional vertices, but for the exceptional vertices we have $\Sigma(i, i+n)^\pm = (i+n-1, i+2n-1)^\mp$.

Therefore, for obtaining the Auslander-Reiten quiver of the cluster category $D_n$ one has to make the following identifications of vertices from the Auslander-Reiten quiver of the derived category. For the non-exceptional vertices we have $(\tau^{-1} \circ \Sigma)(i, j) = (i+n, j+n)$, i.e. indices just have to be taken modulo $n$. For the exceptional vertices we have $(\tau^{-1} \circ \Sigma)(i, i+n)^\pm = (i+n, i+2n)^\pm$ if $n$ is even, and $(\tau^{-1} \circ \Sigma)(i, i+n)^\pm = (i+n, i+2n)^\mp$ if $n$ is odd.
Accordingly, the Auslander-Reiten quiver of the cluster category $\mathcal{D}_n$ has the shape of a cylinder of circumference $n$; in particular $\mathcal{D}_n$ has precisely $n^2$ indecomposable objects.

There are different variations of combinatorial models for the cluster category of Dynkin type $D_n$. The first one appeared in work of Fomin and Zelevinsky [7]; it works in a regular $2n$-gon and uses pairs of arcs which are obtained by 180 degree rotation and diameters in two colours (corresponding to the two exceptional vertices in type $D_n$).

Later there has been a variation of this model by Schiffler [20] using a punctured disc and homotopy classes of paths between vertices. This more recent model is often used because it fits well into the framework of triangulations of surfaces.

However, in this paper we will work with the Fomin-Zelevinsky model. For us, it has two main advantages; namely it is combinatorially very simple to describe and to work with and secondly it is analogous to the standard combinatorial model which we used in our earlier classification of torsion pairs in the cluster categories of Dynkin type $A_n$.

Let us now recall the Fomin-Zelevinsky model for cluster categories of Dynkin type $D_n$ [7, Section 3].

For any $n \geq 1$ we consider a regular $2n$-gon $P$ (although the Dynkin diagrams $D_n$ only appear for $n \geq 4$, on the combinatorics side it makes sense to include the small values of $n$).

We label the vertices of $P$ counterclockwise by $0, 1, \ldots, 2n-1$ consecutively. In our arguments below vertices will also be numbered by some $r \in \mathbb{N}$ which might not be in the range $0 \leq r \leq 2n-1$; in this case the numbering of vertices always has to be taken modulo $2n$.

An arc in $P$ is a set $\{i, j\}$ of vertices of $P$ with $j \not\in \{i-1, i, i+1\}$, i.e. $i$ and $j$ are different and non-neighboring vertices. The arcs connecting two opposite vertices $i$ and $i+n$ are called diameters. We need two different copies of each of these diameters and denote them by $\{i, i+n\}_g$ and $\{i, i+n\}_r$, where $0 \leq i \leq n-1$. The indices should indicate that these diameters are coloured in the colours green and red, which is a convenient way to think about and to visualize the diameters. By a slight abuse of notation, we sometimes omit the indices and just write $\{i, i+n\}$ for diameters, to avoid cumbersome definitions or statements.

Any arc in $P$ which is not a diameter is of the form $\{i, j\}$ where $j \in [i+2, i+n-1]$; here $[i+2, i+n-1]$ stands for the set of vertices of the $2n$-gon $P$ which are met when going counterclockwise from $i+2$ to $i+n-1$ on the boundary of $P$.

Such an arc has a partner arc $\{i+n, j+n\}$ which is obtained from $\{i, j\}$ by a rotation by 180 degrees. We denote the pair of arcs $\{\{i, j\}, \{i+n, j+n\}\}$ by $\{i, j\}$ throughout this paper.

The model of Fomin and Zelevinsky [7] (see also [20]) for the cluster category $\mathcal{D}_n$ of Dynkin type $D_n$ builds on the following crucial fact parametrizing indecomposable objects in $\mathcal{D}_n$ by certain objects coming from the regular $2n$-gon $P$. Namely, the indecomposable objects in $\mathcal{D}_n$ are in bijection with the union of the set of pairs $\{i, j\}$ of non-diameter arcs and the set of diameters $\{i, i+n\}_g$ and $\{i, i+n\}_r$ in two different colours.
The above parametrization of indecomposable objects of \( D_n \) via the \( 2n \)-gon can be made explicit by looking at the structure of the Auslander-Reiten quiver of the cluster category \( D_n \) and the coordinate system in Figure 3.

For the pairs of non-diameter arcs \( \{i, j\} \) the corresponding indecomposable object has coordinates \((i, j)\); note that the coordinates are only determined modulo \( n \) so both arcs \( \{i, j\} \) and \( \{i + n, j + n\} \) in the pair \( \{i, j\} \) yield the same coordinate in the Auslander-Reiten quiver. The diameters \( \{i, i + n\}_g \) and \( \{i, i + n\}_r \) correspond to the exceptional vertices \((i, i + n)^+\) and \((i, i + n)^-\) in the Auslander-Reiten quiver. For specifying precisely which coloured diameter corresponds to which of the two exceptional vertices one has to make a choice.

We will use the following bijection between exceptional vertices and coloured diameters; the motivation for this particular choice will become clear in Section 3 below; see in particular the proof of Proposition 3.2.

We start by pairing the exceptional vertex \((0, n)^+\) with the green diameter \( \{0, n\}_g \) and \((0, n)^-\) with the red diameter \( \{0, n\}_r \). Then we continue in an alternating manner. We assign green diameters to the exceptional vertices \((1, n + 1)^-, (2, n + 2)^+, (3, n + 3)^-\) etc., and red diameters to the exceptional vertices \((1, n + 1)^+, (2, n + 2)^-, (3, n + 3)^+\) etc.

It is a crucial observation that this assignment is compatible with the identification of vertices in the cluster category \( D_n \). In fact, if \( n \) is even then \((i, i + n)^\pm\) get assigned to diameters of the same colour as the vertex \((\tau^{-1} \circ \Sigma)(i, i + n)^\pm = (i + n, i + 2n)^\pm\) obtained after shifting \( n \) steps to the right.

However, if \( n \) is odd, then the functor \( \tau^{-1} \circ \Sigma \) shifts \( n \) units to the right but also flips the exceptional vertices. Therefore, also in this case any exceptional vertex \((i, i + n)^\pm\) gets assigned to a diameter of the same colour as the vertex \((\tau^{-1} \circ \Sigma)(i, i + n)^\pm = (i + n, i + 2n)^\mp\) obtained after identification.

3. Torsion theory in triangulated categories

In this section we summarize the fundamental definitions and properties on torsion pairs in triangulated categories from the seminal paper by Iyama and Yoshino [14], and then apply this abstract concept to the cluster category of Dynkin type \( D_n \).

A torsion pair \((X, Y)\) in a triangulated category \( \mathcal{C} \) with suspension functor \( \Sigma \) is a pair \((X, Y)\) of full subcategories closed under direct sums and direct summands such that

(i) the morphism space \( \text{Hom}_\mathcal{C}(x, y) \) is zero for \( x \in X, y \in Y \),

(ii) each object \( c \in \mathcal{C} \) appears in a distinguished triangle \( x \to c \to y \to \Sigma x \) with \( x \in X, y \in Y \).

This definition in particular includes t-structures, as introduced by Beilinson, Bernstein, and Deligne [2] (with the additional condition \( \Sigma X \subseteq X \)) and the co-t-structures of Bondarko and Pauksztello [4], [19] (with the additional condition \( \Sigma^{-1} X \subseteq X \)).

Any torsion pair \((X, Y)\) is determined by one of its entries, namely we have

\[ Y = X^\perp := \{ c \in \mathcal{C} | \text{Hom}_\mathcal{C}(x, c) = 0 \text{ for each } x \in X \}, \]

and \( X = Y^\perp := \{ c \in \mathcal{C} | \text{Hom}_\mathcal{C}(c, y) = 0 \text{ for each } y \in Y \} \).

If the triangulated category \( \mathcal{C} \) is \( \text{Hom} \)-finite over a field and \( \text{Krull-Schmidt} \) (conditions which are satisfied for the categories considered in this paper) we have the
following characterisation, see [14, Prop. 2.3]. Let $X$ be a contravariantly finite full subcategory of $C$ which is closed under direct sums and direct summands. Then $(X, X^\perp)$ is a torsion pair if and only if $X = \perp(X^\perp)$.

We want to apply this general concept to the cluster category $D_n$ of Dynkin type $D_n$. First note that we can ignore the condition on contravariant finiteness since the cluster category of type $D_n$ has only finitely many indecomposable objects and hence every subcategory is contravariantly (and also covariantly) finite. Moreover, any subcategory of $D_n$ closed under direct sums and direct summands is completely determined by the set of indecomposable objects it contains.

For understanding the perpendicular subspaces one needs to understand which homomorphism spaces between indecomposable objects are non-zero. This can be explicitly described by certain regions in the Auslander-Reiten quiver, as first observed by Iyama [13, Section 4].

Consider an indecomposable object $x$ in $D_n$ with coordinates $(i, j)$ in the Auslander-Reiten quiver, cf. Figure 3. To such an indecomposable object we consider the regions (including boundaries) shown in Figures 4 and 5, respectively. If $x = (i, j)$ is a non-exceptional vertex, then all exceptional vertices in these regions belong to $H^+(x)$ and $H^-(x)$, respectively.
If \( x = (i, i + n)^\pm \) is an exceptional vertex then only half of the exceptional vertices belong to the regions \( H^+(x) \) and \( H^-(x) \). To be precise, if \( x = (i, i + n)^\pm \) then \( H^+(x) \) contains \((i + 1, i + n + 1)^\pm, (i + 2, i + n + 2)^\pm, (i + 3, i + n + 3)^\pm\) etc., but does not contain \((i + 1, i + n + 1)^\pm, (i + 2, i + n + 2)^\pm, (i + 3, i + n + 3)^\pm\) etc.

Similarly for the region \( H^-(x) \).

The crucial observation in [13, Section 4] (which follows from the mesh relations on the Auslander-Reiten quiver) is the following. Let \( x \) be an indecomposable object in the cluster category \( D_n \). Then the indecomposable objects \( c \in D_n \) with \( \text{Hom}_{D_n}(x, c) \neq 0 \) are precisely those which lie in the region \( H^+(x) \). Similarly, the indecomposable objects \( c \in D_n \) with \( \text{Hom}_{D_n}(c, x) \neq 0 \) are precisely those which lie in the region \( H^-(x) \).

Let us connect this crucial observation with the combinatorial model of the cluster category \( D_n \) given by pairs \( \{i, j\} \) of non-diameter arcs and by green and red diameters \( \{i, i + n\}_g, \{i, i + n\}_r \) in a regular 2n-gon. For this we shall need the following notion of crossings of arcs. For the non-diameter arcs this crossing will exactly reflect the geometric intuition of when two arcs cross. For the diameters one has to be careful with the different colours.

**Definition 3.1.**

(a) We say that two non-diameter arcs \( \{i, j\} \) and \( \{k, \ell\} \) cross precisely if the elements \( i, j, k, \ell \) are all distinct and come in the order \( i, k, j, \ell \) when moving around the 2n-gon \( P \) in one direction or the other (i.e. counterclockwise or clockwise). In particular, the two arcs in \( \{i, j\} \) do not cross.

Similarly, in the case \( j = i + n \), the above condition defines when a diameter \( \{i, i + n\}_g \) (or \( \{i, i + n\}_r \)) crosses the non-diameter arc \( \{k, \ell\} \).

(b) We say that two pairs \( \{i, j\} \) and \( \{k, \ell\} \) of non-diameter arcs cross if there exist two arcs in these two pairs which cross in the sense of part (a). (Note that then necessarily the other two rotated arcs also cross.)

Similarly, the diameter \( \{i, i + n\}_g \) (or \( \{i, i + n\}_r \)) crosses the pair \( \{k, \ell\} \) of non-diameter arcs if it crosses one of the arcs in \( \{k, \ell\} \). (Note that it then necessarily crosses both arcs in \( \{k, \ell\} \).

(c) Two diameters \( \{i, i + n\}_g \) and \( \{j, j + n\}_r \) of different colour cross if \( j \notin \{i, i + n\}_g \), i.e. if they have different endpoints. But \( \{i, i + n\}_g \) and \( \{i, i + n\}_r \) do not cross. Moreover, any diameters of the same colour do not cross.

Then the vertices in the above regions \( H^+(x) \) and \( H^-(x) \) of the Auslander-Reiten quiver can be expressed as follows in terms of arcs.

**Proposition 3.2.** Let \( x \) be a vertex in the Auslander-Reiten quiver of \( D_n \) with coordinates \( (i, j) \) (where in the case of exceptional vertices this means \((i, i + n)^+ \) or \((i, i + n)^- \)).

(a) The vertices \( y = (k, \ell) \) in the region \( H^+(x) \) are precisely those for which the corresponding arc \( \{k, \ell\} \) of the 2n-gon crosses (at least) one of the arcs \( \{i - 1, j - 1\} \) and \( \{i + n - 1, j + n - 1\} \).

(b) The vertices \( y = (k, \ell) \) in the region \( H^-(x) \) are precisely those for which the corresponding arc \( \{k, \ell\} \) of the 2n-gon crosses (at least) one of the arcs \( \{i + 1, j + 1\} \) and \( \{i + n + 1, j + n + 1\} \).
Proof. This follows by direct inspection immediately from the definition of the regions $H^\pm(x)$ and from Definition 3.1. We leave the details to the reader.

For the exceptional vertices note that the alternating membership of the other exceptional vertices to $H^\pm(x)$ directly corresponds to the alternating assignment of green and red colours as described at the end of Section 2.

Now we start describing torsion pairs combinatorially in terms of the arc model. Recall the fact from [14] mentioned above that a pair $(X, X^\perp)$ of subcategories (closed under direct sums and direct summands) of $\mathcal{D}_n$ is a torsion pair if and only if $X = ^\perp(\overline{X})$.

The subcategory $X$ has to be closed under direct sums and direct summands hence is determined by the set of indecomposable objects of $\mathcal{D}_n$ it contains. Let $\mathcal{X}$ be the collection of non-diameter arcs and coloured diameters of the $2n$-gon corresponding to the indecomposable objects of $X$; note that the non-diameter arcs in this collection $\mathcal{X}$ come in pairs obtained by 180 degree rotation, i.e. the collection $\mathcal{X}$ is invariant under rotation by 180 degrees.

The following definition will be crucial.

**Definition 3.3.** For $n \geq 1$ let $P$ be a regular $2n$-gon. If $\mathcal{X}$ is a set of arcs in $P$, then we set

$$n\mathcal{X} = \{ \alpha = \{i, j\} \text{ is an arc in } P \mid \alpha \text{ crosses no arc in } \mathcal{X} \}.$$  

(Note that $\alpha$ can be a diameter $\{i, i+n\}$ or $\{i, i+n\}$, here, we avoid the indices for simplicity.)

**Example 3.4.** Let us consider two examples for $n = 5$, i.e. collections of arcs of a regular 10-gon which are invariant under 180 degree rotation. For better visibility we draw the red diameters in a wavelike form and the green ones as straight lines.

\[
\begin{align*}
\mathcal{X}_1 &= \quad & n\mathcal{X}_1 &= \quad \\
\mathcal{X}_2 &= \quad & n\mathcal{X}_2 &= \quad 
\end{align*}
\]

Then torsion pairs in $\mathcal{D}_n$ can be characterized combinatorially as follows.

**Proposition 3.5.** Let $X$ be a subcategory of the cluster category $\mathcal{D}_n$, closed under direct sums and direct summands, and let $\mathcal{X}$ be the corresponding collection of arcs of the regular $2n$-gon.

Then $(X, X^\perp)$ is a torsion pair in $\mathcal{D}_n$ if and only if $X = n\mathcal{X}$.
Proof. Let $x$ and $y$ be indecomposable objects in $\mathcal{D}_n$, given by their coordinates $(i,j)$ and $(k,\ell)$ in the Auslander-Reiten quiver. Then we have that

$$\text{Ext}^1_{\mathcal{D}_n}(x,y) = \text{Hom}_{\mathcal{D}_n}(x,\Sigma y) = \text{Hom}_{\mathcal{D}_n}(\Sigma^{-1}x,y) \neq 0$$

if and only if the corresponding arcs $\{i,j\}$ and $\{k,\ell\}$ cross. In fact, by Iyama’s observation from [13], $(k,\ell)$ has to be in the region $H^+(\Sigma^{-1}x)$ which equals $H^+(\tau^{-1}x)$ since $\tau = \Sigma$ in the cluster category $\mathcal{D}_n$. But $\tau^{-1}x$ has coordinates $(i+1,j+1)$ and then by Proposition 3.2(a) the arc corresponding to $y$ crosses the arc corresponding to $x$.

For the perpendicular subcategory we therefore have

$$\mathcal{X}^\perp = \{ c \in \mathcal{D}_n \mid \text{Hom}_{\mathcal{D}_n}(x,c) = 0 \text{ for each } x \in \mathcal{X} \}$$

$$= \{ c \in \mathcal{D}_n \mid \text{Ext}^1_{\mathcal{D}_n}(x,\Sigma^{-1}c) = 0 \text{ for each } x \in \mathcal{X} \}$$

$$= \{ c \in \mathcal{D}_n \mid \text{Ext}^1_{\mathcal{D}_n}(\Sigma x,c) = 0 \text{ for each } x \in \mathcal{X} \},$$

which corresponds to the set $\text{nc} \Sigma \mathcal{X}$ of arcs, by definition of the nc operator.

Similarly, the left perpendicular subcategory $\mathcal{X}^\perp X$ corresponds to $\text{nc} \Sigma^{-1} \mathcal{X}$.

Thus, $(\mathcal{X},\mathcal{X}^\perp)$ is a torsion pair in $\mathcal{D}_n$ if and only if $\mathcal{X} = \text{nc} \Sigma^{-1}(\text{nc} \Sigma \mathcal{X}) = \text{nc} \text{nc} \mathcal{X}$.

For the last equation note that $\Sigma^\pm$, when interpreted in the arc model, just induces a rotation and hence commutes with the nc operator. ∎

4. Ptolemy diagrams of type $D_n$

We have seen in Proposition 3.5 that torsion pairs in the cluster category $\mathcal{D}_n$ can be characterized via their corresponding sets of arcs by the condition $\mathcal{X} = \text{nc} \text{nc} \mathcal{X}$.

The aim of this section is to characterize combinatorially those collections $\mathcal{X}$ of arcs of the $2n$-gon which are invariant under 180 degree rotation (i.e. correspond to a collection of indecomposable objects of $\mathcal{D}_n$) and satisfy $\mathcal{X} = \text{nc} \text{nc} \mathcal{X}$. It will turn out below, in the main result Theorem 4.2, that the following properties are the crucial ones.

The notion Ptolemy diagram is used because of the analogy to the Ptolemy diagrams in Dynkin type $A$ [11] whose visualization is very reminiscent of Ptolemy’s theorem about the relation between the lengths of the sides and the diagonals in a cyclic quadrilateral, see Figure 11.

Definition 4.1. (a) Let $\mathcal{X}$ be a collection of arcs of the $2n$-gon, $n > 1$, which is invariant under rotation of 180 degrees. Then $\mathcal{X}$ is called a Ptolemy diagram of type $D_n$ if it satisfies the following conditions. Let $\alpha = \{i,j\}$ and $\beta = \{k,\ell\}$ be crossing arcs in $\mathcal{X}$ (in the sense of Definition 3.1).

(Pt1) If $\alpha$ and $\beta$ are not diameters, then those of $\{i,k\}$, $\{i,\ell\}$, $\{j,k\}$, $\{j,\ell\}$ which are arcs in $P$ are also in $\mathcal{X}$. In particular, if two of the vertices $i,j,k,\ell$ are opposite vertices (i.e. one of $k$ and $\ell$ is equal to $i+n$ or $j+n$), then both the green and the red diameter connecting them are also in $\mathcal{X}$.

(Pt2) If both $\alpha$ and $\beta$ are diameters (necessarily of different colour by Definition 3.1(c)) then those of $\{i,k\}$, $\{i,k+n\}$, $\{i+n,k\}$, $\{i+n,k+n\}$ which are arcs of $P$ are also in $\mathcal{X}$.

(Pt3) If $\alpha$ is a diameter while $\beta$ is not a diameter, then those of $\{i,k\}$, $\{i,\ell\}$, $\{j,k\}$, $\{j,\ell\}$ which are arcs and do not cross the arc $\{k+n,\ell+n\}$ are
also in $X$. Additionally, the diameters $\{k, k + n\}$ and $\{\ell, \ell + n\}$ of the same colour as $\alpha$ are also in $X$.

(b) For the 2-gon, there is precisely one Ptolemy diagram of type $D_1$ containing only the two diameters.

(c) A collection $X$ of arcs is called a Ptolemy diagram of type $D_1$ if it is a Ptolemy diagram of type $D_n$ for some $n \geq 1$.

These conditions are illustrated in Figure 6, where dashed lines indicate non-diameter arcs and diameters forced by the crossing of $\alpha = \{i, j\}$ and $\beta = \{k, \ell\}$. Note that in Definition 4.1 the collection of arcs $X$ is supposed to be invariant under rotation by 180 degrees. Conditions (Pt1) and (Pt3) are only formulated for the one crossing of $\alpha$ and $\beta$, but the rotated arcs are also crossing arcs in $X$. Therefore, (Pt1) and (Pt3) also guarantee that the rotated arcs appearing in the pictures in Figure 6 are also in $X$, although they are not explicitly mentioned in Definition 4.1. Note that in Example 3.4, the collection $X_1$ is not a Ptolemy diagram (conditions (Pt1) and (Pt2) are violated), whereas the collection $X_2$ is a Ptolemy diagram.

We are now in the position to state the main result of this section.

**Theorem 4.2.** Let $X$ be a collection of arcs of the $2n$-gon, $n \geq 1$, which is invariant under rotation of 180 degrees. Then the following conditions are equivalent:

(a) $X = \text{nc nc } X$.

(b) $X$ is a Ptolemy diagram of type $D$.

Before embarking on the proof of Theorem 4.2 let us draw a few consequences. Note that these are not obvious from the combinatorial definition of Ptolemy diagrams of type $D$ in Definition 4.1.

**Corollary 4.3.**

(a) If $X$ is a Ptolemy diagram of type $D$, then $\text{nc nc } X$ is also a Ptolemy diagram of type $D$.

(b) For any $n \geq 1$, the operator $\text{nc}$ induces a bijection on the Ptolemy diagrams of type $D$ of the $2n$-gon.

**Proof.** (a) By assumption on $X$ and Theorem 4.2 we have that $X = \text{nc nc } X$. But then $\text{nc } X = \text{nc(nc nc } X) = \text{nc nc(nc } X)$, and hence $\text{nc } X$ is again a Ptolemy diagram of type $D$ by Theorem 4.2.

(b) The map induced by the operator $\text{nc}$ on Ptolemy diagrams of type $D$ is injective by Theorem 4.2. In fact if $X$ and $Y$ are Ptolemy diagrams of type $D_n$ and if $\text{nc } X = \text{nc } Y$ then applying $\text{nc}$ again we get $\text{nc } \text{nc } X = \text{nc } \text{nc } Y$ from which $X = Y$ follows by using Theorem 4.2. Clearly there are only finitely many Ptolemy diagrams of type $D_n$, so the map induced by $\text{nc}$ is also surjective. 

As a preparation for the proof of Theorem 4.2 we shall first state and prove a few useful lemmas.

The following notation will be useful in the sequel: for any vertices $i$ and $j$ of the $2n$-gon, we denote by $[i, j]$ the set of all vertices of the $2n$-gon which are met when going counterclockwise from $i$ to $j$ on the boundary of $P$ (including $i$ and $j$ themselves). Recall that our numbering of vertices of $P$ was also counterclockwise so that $[i, j]$ can be thought of as the interval between $i$ and $j$. Also note that the order now matters, $[i, j]$ and $[j, i]$ are different sets of vertices.
(Pt1) The first Ptolemy condition in Dynkin type $D$:

![Diagram of the first Ptolemy condition](image1)

(Pt2) The second Ptolemy condition in Dynkin type $D$:

![Diagram of the second Ptolemy condition](image2)

(Pt3) The third Ptolemy condition in Dynkin type $D$:

![Diagram of the third Ptolemy condition](image3)

Figure 6. The Ptolemy conditions in Dynkin type $D$.

Lemma 4.4. Let $\mathcal{X}$ be a Ptolemy diagram of type $D_n$, $n \geq 1$. Suppose that we had a pair $\{i, j\}$ of non-diameter arcs of $P$ which is in nc.nc.\mathcal{X} but not in $\mathcal{X}$.

(a) If the diameters $\{i, i + n\}_g$ and $\{i, i + n\}_r$ are not in $\mathcal{X}$ then there exists an arc $\{i, t\}$ in $\mathcal{X}$ where $t \in [j + 1, i + n - 1]$ if $j \in [i, i + n]$ and where $t \in [i + n + 1, j - 1]$ if $j \in [i + n, i]$. 

For one of the colours, the diameters attached to $i$ and $j$ of the same colour are in $X$ (i.e. $\{i,i+n\}_g$ and $\{j,j+n\}_g$ are in $X$ or $\{i,i+n\}_r$ and $\{j,j+n\}_r$ are in $X$).

Proof. We will only consider the case that $j \in [i,i+n]$, i.e. that the consecutive counterclockwise order of the vertices is $i,j,i+n,j+n$. The other case $j \in [j+n,i]$ in which the vertices appear in counterclockwise order as $i,j,i+n,j$ is completely symmetric.

(a) Consider the arc $\{i-1,i+1\}$. It crosses $\{i,j\} \in \text{ncnc} X$, hence $\{i-1,i+1\}$ must be crossed by an element from $X$. By assumption, the diameters attached to $i$ are not in $X$. So there exists a non-diameter arc $\{i,t\} \in X$ where $t \neq i+n$ and $t \not\in \{i-1,i,i+1\}$ (otherwise $\{i,t\}$ was not an arc). Moreover, by assumption the arc $\{i,j\} \not\in X$, thus also $t \neq j$.

If $t \in [j+1,i+n-1]$ then the claim in (a) is shown and we are done.

So we are left with two possibilities, namely $t \in [i+2,j-1]$ or $t \in [i+n+1,i-2]$.

Case 1: Let $t \in [i+2,j-1]$. W.l.o.g. we can suppose that $\{i,t\}$ has maximal length among the arcs $\{i,u\} \in X$ with $u \in [i+2,j-1]$ (i.e. the arcs $\{i,u\}$ with $u \in [t+1,j-1]$ are not in $X$).

Now consider the arc $\{i-1,t\}$. It crosses $\{i,j\} \in \text{ncnc} X$, hence $\{i-1,t\}$ must be crossed by an element from $X$.

But $\{i-1\}$ can not be crossed by a diameter from $X$. In fact, the diameters $\{i,i+n\}_g$ and $\{i,i+n\}_r$ are not in $X$ by assumption; if a diameter $\{i',i'+n\}_g$ (or $\{i',i'+n\}_r$) with $i' \in [i+1,t-1]$ was in $X$ then condition (Pt3) implied that $\{i,i+n\}_g \in X$ (or $\{i,i+n\}_r \in X$), contradicting the assumption.

Therefore $\{i-1,t\}$ must be crossed by a non-diameter arc $\{u,s\} \in X$ where $u \in [i,t-1]$ (and $s \in [t+1,i-2]$).

If $s \in [t+1,j]$ then condition (Pt1) provides an arc $\{i,s\}$ in $X$ which contradicts the maximality of $t$ (if $s \neq j$) or the assumption $\{i,j\} \not\in X$ (if $s = j$).

If $s \in [j+1,i+n-1]$ then condition (Pt1) implies the existence of an arc as claimed in (a) and we are done.

If $s = i+n$ then condition (Pt1) implies that the diameters attached to $i$ are in $X$, contradicting the assumption.

So it remains to deal with the case $s \in [i+n+1,i-2]$. By condition (Pt1) then also the arc $\{i,s\} \in X$. W.l.o.g. choose $s \in [i+n+1,i-2]$ so that $\{i,s\} \not\in X$ but $\{i,s'\} \in X$ for all $s' \in [i+n+1,s-1]$. See Figure 7 for an illustration.

Now consider the arc (possibly a diameter) $\{t,s\}$. It crosses $\{i,j\} \in \text{ncnc} X$, so it must be crossed by an element in $X$.

If $\{t,s\}$ is crossed by an arc in $X$ attached at $i$, then the other endpoint of this arc must be in $[j+1,i+n-1]$ and we are done; this follows from the choice of $t$ and $s$ and from the assumptions that the diameters attached at $i$ are not in $X$ and that $\{i,j\} \not\in X$.

So we can suppose that $\{t,s\}$ is crossed by an arc (possibly a diameter) in $X$ which also crosses one of $\{i,t\} \in X$ and $\{i,s\} \in X$. 

Now by arguments analogous to the above ones one uses the choice of \( t \) and \( s \) (as endpoints of arcs of maximal length) to conclude from the Ptolemy conditions (Pt1) and (Pt3) that there must be an arc \( \{i, v\} \) with \( v \in [j + 1, i + n - 1] \), as claimed.

**Case 2:** Let \( t \in [i + n + 1, i - 2] \).

This case is completely analogous (by symmetry) to Case 1; just interchange the roles of \( t \) and \( s \).

This completes the proof of part (a).

(b) Again it suffices by symmetry to deal with the case where \( j \in [i, i + n] \), i.e. where \( i, j, i + n, j + n \) is the counterclockwise order of these vertices.

Suppose first that all four diameters attached at \( i \) and \( j \) were not in \( \mathcal{X} \). Then we can apply part (a) to both \( i \) and \( j \). This gives arcs \( \{i, t\} \in \mathcal{X} \) with \( t \in [j + 1, i + n - 1] \) and \( \{j, s\} \in \mathcal{X} \) with \( s \in [j + n + 1, i - 1] \). These two arcs clearly cross and condition (Pt1) implied that \( \{i, j\} \in \mathcal{X} \), contradicting the assumption.

Secondly, suppose that for one of \( i \) and \( j \), say for \( i \), a diameter attached at \( i \) is in \( \mathcal{X} \), but for the other vertex \( j \), no diameter attached at \( j \) is in \( \mathcal{X} \). Then part (a), applied to \( j \) yields an arc \( \{j, s\} \in \mathcal{X} \) with \( s \in [j + n + 1, i - 1] \). This arc crosses the diameter attached at \( i \) which is supposed to be in \( \mathcal{X} \). But then condition (Pt2) implies that \( \{i, j\} \in \mathcal{X} \), a contradiction.

Finally, if for both \( i \) and \( j \) at least one diameter attached to each of them is in \( \mathcal{X} \) then these two diameters must have the same colour, as claimed in (b). In fact, if the two diameters had different colours they would cross and condition (Pt2) implied that \( \{i, j\} \in \mathcal{X} \), contradicting the assumption.

**Lemma 4.5.** Let \( \mathcal{X} \) be a Ptolemy diagram of type \( D_n \), \( n \geq 1 \). Suppose that there was a diameter, say \( \{i, i + n\}_r \), which is in \( \text{nc nc} \mathcal{X} \) but not in \( \mathcal{X} \).

Then the diameter \( \{i, i + n\}_g \) of the other colour must be in \( \mathcal{X} \).

**Proof.** We consider the non-diameter arc \( \{i - 1, i + 1\} \). It crosses the diameter \( \{i, i + n\}_r \) which is in \( \text{nc nc} \mathcal{X} \) by assumption. Hence \( \{i - 1, i + 1\} \) must be crossed by an element from \( \mathcal{X} \).

If it is crossed by a diameter in \( \mathcal{X} \) then the only possibility is that \( \{i, i + n\}_g \in \mathcal{X} \) (since \( \{i, i + n\}_r \notin \mathcal{X} \) by assumption), and we are done.
So from now on we can assume that \( \{i - 1, i + 1\} \) is crossed by a non-diameter arc \( \{i, u\} \in \mathcal{X} \). W.l.o.g. we can assume that \( u \in [i + 2, i + n - 1] \) (the other case \( u \in [i + n + 1, i - 2] \) is completely symmetric), and that \( u \) is ‘maximal’ in the sense that there are no arcs \( \{i, u'\} \in \mathcal{X} \) with \( u' \in [u + 1, i + n - 1] \).

Now consider the arc \( \{i - 1, u\} \). It crosses the diameter \( \{i, i + n\} \in \mathcal{Y} \) which is in \( \mathcal{X} \) since \( \mathcal{X} \) is invariant under 180 degree rotation. Then condition (Pt1) implies that the diameter \( \{i, i + n\} \in \mathcal{Y} \), and we are done.

Therefore we are left with the case that \( \{i, t\} \in \mathcal{X} \) and \( t \in [u + n, i - 2] \). W.l.o.g. we can choose such \( t \) whose arc has maximal length, i.e. none of the arcs \( \{i, t'\} \) with \( t' \in [u + n, t - 1] \) is in \( \mathcal{X} \).

Now we consider the arc (possibly a diameter) \( \{u, t\} \). It crosses the diameter \( \{i, i + n\} \) which is in \( \text{ncnc} \mathcal{X} \), so it must be crossed by an element from \( \mathcal{X} \).

If \( \{u, t\} \) is crossed by an arc from \( \mathcal{X} \) attached at the vertex \( i \) then by the choice of \( u \) and \( t \) (maximality of the arcs \( \{i, u\} \) and \( \{i, t\} \)) it can only be crossed by the green diameter \( \{i, i + n\} \in \mathcal{X} \), and we are done.
So we can assume that \( \{u, t\} \) is crossed by an arc from \( \mathcal{X} \) not attached at \( i \). Any such arc necessarily also crosses \( \{i, u\} \) or \( \{i, t\} \). But then we are in one of the situations already dealt with above where we have in each case concluded that the desired green arc \( \{i, i + n\}_g \) is in \( \mathcal{X} \) or obtained a contradiction.

This completes the proof of the lemma.

After these two preparatory results we now come to the proof of the main result of this section.

**Proof of Theorem 4.2.** The direction \( '(a) \implies (b)' \) is fairly straightforward. Let \( \mathcal{X} = \text{nc nc} \mathcal{X} \). In each of the conditions (Pt1), (Pt2), (Pt3) as visualized in Figure 6 we have to confirm that the dashed arcs must be in \( \text{nc nc} \mathcal{X} = \mathcal{X} \). Slightly reformulated this means that for each of the dashed arcs the following holds: if they are crossed by an arc (possibly diameter) \( \alpha \) of the \( 2n \)-gon \( P \) then \( \alpha \) must also cross an element in \( \mathcal{X} \). But this condition is indeed easily verified from looking at the figures provided in Figure 6 (where the solid arcs are in \( \mathcal{X} \) by assumption).

The direction \( '(b) \implies (a)' \) is much more involved.

Let \( \mathcal{X} \) be a Ptolemy diagram of type \( D \), so \( \mathcal{X} \) satisfies conditions (Pt1), (Pt2) and (Pt3). We have to show that \( \mathcal{X} = \text{nc nc} \mathcal{X} \). Note that the inclusion \( \mathcal{X} \subseteq \text{nc nc} \mathcal{X} \) always holds (by definition of the operator \text{nc}).

Thus we have to show that the conditions (Pt1), (Pt2) and (Pt3) imply that \( \text{nc nc} \mathcal{X} \subseteq \mathcal{X} \). This is where we shall make use of the preceding lemmas.

We have to consider the cases of pairs of non-diameters and of diameters separately.

First, suppose (for a contradiction) there was a pair \( \{i, j\} \) of non-diameter arcs in \( \text{nc nc} \mathcal{X} \setminus \mathcal{X} \). By Lemma 4.4 (b), two diameters of the same colour attached at \( i \) and \( j \) are then in \( \mathcal{X} \), say \( \{i, i + n\}_g \in \mathcal{X} \) and \( \{j, j + n\}_g \in \mathcal{X} \).

We can w.l.o.g. (by symmetry) assume that \( j \in [i, i + n] \), i.e. \( i, j, i + n, j + n \) come in this order when going counterclockwise around around the boundary of \( P \).

We then consider the vertices \( s \in [i + 1, j - 1] \), see Figure 9. We call such a vertex free if there is no arc \( \{u, v\} \in \mathcal{X} \) with endpoints \( u \in [i, s - 1] \) and \( v \in [s + 1, j] \). Such a free vertex must exist. In fact, if \( i + 1 \) is free, we are done. Otherwise, there exists an arc \( \{i, s_1\}_g \in \mathcal{X} \) with \( s_1 \in [i + 2, j - 1] \) (\( s_1 \neq j \) since \( \{i, j\} \notin \mathcal{X} \) by assumption).

Choose a longest such arc; then the vertex \( s_1 \) is free since otherwise condition (Pt1) would produce a longer arc than \( \{i, s_1\} \) in \( \mathcal{X} \) or implies that \( \{i, j\} \in \mathcal{X} \), in both cases a contradiction.

Now take a free vertex \( s \in [i + 1, j - 1] \) and consider the green diameter \( \{s, s + n\}_g \).

It crosses \( \{i, j\} \in \text{nc nc} \mathcal{X} \), so it must be crossed by an element from \( \mathcal{X} \).

We claim that \( \{s, s + n\}_g \) can not be crossed by a non-diameter arc in \( \mathcal{X} \). In fact, with \( s \) being a free vertex the diameter \( \{s, s + n\}_g \) can not be crossed by an arc from \( \mathcal{X} \) having both endpoints in \( [i, j] \). If \( \{s, s + n\}_g \) was crossed by an arc having one endpoint in \( [i, j] \) and the other outside \( [i, j] \) then such an arc would cross one of the green diameters \( \{i, i + n\}_g \in \mathcal{X} \) and \( \{j, j + n\}_g \in \mathcal{X} \) and condition (Pt3) would produce an arc in \( \mathcal{X} \) contradicting the freeness of \( s \) (use the rotational symmetry of \( \mathcal{X} \) if initially the freeness of \( s + n \) is violated). By rotational symmetry all arguments apply equally well to the polygon bounded by \( \{i + n, j + n\}_g \) and the edges of \( P \) along the interval \( [i + n, j + n] \), and the free vertex \( s + n \) therein. Thus the only possibility left is that \( \{s, s + n\}_g \) was crossed by a non-diameter arc \( \{u, v\} \in \mathcal{X} \) with
$u \in [j + 1, i + n - 1]$ and $v \in [j + n + 1, i - 1]$, see Figure 9. Then $\{u, v\}$ crossed both green diameters $\{i, i + n\}_g \in \mathcal{X}$ and $\{j, j + n\}_g \in \mathcal{X}$ and condition (Pt3) implies that $\{i, u\} \in \mathcal{X}$. But the latter crosses $\{j, j + n\}_g \in \mathcal{X}$ and another application of (Pt3) shows that $\{i, j\} \in \mathcal{X}$, a contradiction.

This completes the proof of the claim that the diameter $\{s, s + n\}_g$ can not be crossed by any non-diameter arc in $\mathcal{X}$.

Therefore, $\{s, s + n\}_g$ must be crossed by a diameter in $\mathcal{X}$ which is necessarily red, say by $\{s', s' + n\}_r$. If $s' \not\in \{i, i + n, j, j + n\}$ then condition (Pt2) implies the existence of a non-diameter arc in $\mathcal{X}$ crossing $\{s, s + n\}_g$. But this has just been excluded by the preceding claim. Finally, if $s' \in \{i, j, i + n, j + n\}$ then $\{i, i + n\}_r \in \mathcal{X}$ or $\{j, j + n\}_r \in \mathcal{X}$; but then condition (Pt2) implies that $\{i, j\} \in \mathcal{X}$, contradicting the assumption.

This completes the proof that there are no non-diameter arcs in $(\text{nc nc} \mathcal{X}) \setminus \mathcal{X}$.

Secondly, suppose (for a contradiction) that there was a (w.l.o.g. red) diameter $\{i, i + n\}_r$ in $(\text{nc nc} \mathcal{X}) \setminus \mathcal{X}$. By Lemma 4.5 we have that the green diameter $\{i, i + n\}_g$ is in $\mathcal{X}$.

We consider the green diameter $\{i + 1, i + n + 1\}_g$. It crosses the red diameter $\{i, i + n\}_r \in \text{nc nc} \mathcal{X}$, so $\{i + 1, i + n + 1\}_g$ must be crossed by an element from $\mathcal{X}$. If $\{i + 1, i + n + 1\}_g$ is crossed by a (necessarily red) diameter $\{s, s + n\}_r \in \mathcal{X}$, then $s \not\in \{i, i + n\}$ by assumption. But then $\{s, s + n\}_r$ crosses $\{i, i + n\}_g \in \mathcal{X}$ and condition (Pt2) implies that $\{i + 1, i + n + 1\}_g$ is also crossed by a non-diameter arc $\{i, s\} \in \mathcal{X}$.

So from now on we can assume that $\{i + 1, i + n + 1\}_g$ is crossed by a non-diameter arc from $\mathcal{X}$. Using condition (Pt3) we can even assume that $\{i + 1, i + n + 1\}_g$ is crossed by a non-diameter arc of the form $\{i, u\} \in \mathcal{X}$ with $u \in [i + 2, i + n - 1]$. (In fact, any non-diameter arc not attached at $i$ crossing $\{i + 1, i + n + 1\}_g$ also crosses $\{i, i + n\}_g \in \mathcal{X}$ and (Pt3) can be applied.) W.l.o.g. we choose $u$ maximal with this property, i.e. $\{i, r\} \not\in \mathcal{X}$ for all $r \in [u + 1, i + n - 1]$.

Now consider the green diameter $\{u, u + n\}_g$, see Figure 10.
Figure 10.

It crosses the red diameter \( \{i, i+n\}_r \in \text{nc nc } \mathcal{X} \), so \( \{u, u+n\}_g \) must be crossed by an element from \( \mathcal{X} \).

Suppose first that \( \{u, u+n\}_g \) is crossed by a (red) diameter \( \{v, v+n\}_r \in \mathcal{X} \). Then one of the endpoints, say \( v \), must be in \([i+1, i+n-1]\) \((v = i \) is impossible by the assumption that \( \{i, i+n\}_r \not\in \mathcal{X} \)). If \( v \in [i+1, u - 1] \) then \( \{v, v+n\}_r \in \mathcal{X} \) crosses \( \{i, u\} \in \mathcal{X} \) and condition (Pt3) implies that \( \{i, i+n\}_r \in \mathcal{X} \), contradicting the assumption. If \( v \in [u+1, i+n-1] \) then \( \{v, v+n\}_r \) crosses \( \{i, i+n\}_g \in \mathcal{X} \) and condition (Pt2) yields an arc \( \{i, v\} \in \mathcal{X} \) contradicting the maximality of \( u \).

So we are left with the case that \( \{u, u+n\}_g \) is crossed by a non-diameter arc \( \{v, w\} \in \mathcal{X} \) (and hence also by the rotated arc \( \{v+n, w+n\} \in \mathcal{X} \)). If none of the arcs in the pair \( \{v, w\} \) crosses the green diameter \( \{i, i+n\}_r \) then one of the arcs, say \( \{v, w\} \) crosses \( \{i, u\} \in \mathcal{X} \) and condition (Pt1) yields an arc contradicting the maximality of \( u \) or the assumption that \( \{i, i+n\}_r \not\in \mathcal{X} \).

So we can assume from now on that the non-diameter arc \( \{v, w\} \in \mathcal{X} \) crossing \( \{u, u+n\}_g \) also crosses the green diameter \( \{i, i+n\}_g \in \mathcal{X} \).

Then one of the endpoints, say \( v \), must be in the interval \([i+1, i+n-1]\), but \( v \neq u \) (otherwise the arc can not cross \( \{u, u+n\}_g \)).

If \( v \in [i+1, u - 1] \) then \( w \in [i+n+1, u+n-1] \). But then \( \{v, w\} \in \mathcal{X} \) crosses \( \{i+n, u+n\} \in \mathcal{X} \) and condition (Pt1) implies that \( \{v, i+n\} \in \mathcal{X} \); but \( \{v, i+n\} \) crosses \( \{i, u\} \in \mathcal{X} \) and hence condition (Pt1) yields that the red diameter \( \{i, i+n\}_r \in \mathcal{X} \), contradicting the assumption.

Finally, if \( v \in [u+1, i+n-1] \) then \( w \in [u+n+1, i-1] \). But then condition (Pt3) would imply that \( \{i, v\} \in \mathcal{X} \), contradicting the maximality of \( u \).

This completes the proof that there are no diameters in \( (\text{nc nc } \mathcal{X}) \setminus \mathcal{X} \).

Together with the earlier proof that there are no non-diameter arcs in \( (\text{nc nc } \mathcal{X}) \setminus \mathcal{X} \) we have shown that conditions (Pt1), (Pt2) and (Pt3) imply that \( \text{nc nc } \mathcal{X} = \mathcal{X} \).

This completes the proof of Theorem 4.2. \(\square\)
In this section our aim is to count the torsion pairs in the cluster category of Dynkin type $D_n$. As a main result we will give the generating function for the number of torsion pairs explicitly. This will be achieved by first providing an alternative description of Ptolemy diagrams of type $D$. In this description we will build on Ptolemy diagrams of Dynkin type $A$, as introduced in [11]. We briefly recall the definition and the facts needed for our purposes.

For any $n \geq 1$, let $P$ be a regular $(n + 3)$-gon with a distinguished oriented edge which we refer to as the distinguished base edge. An edge of $P$ is a set of two neighbouring vertices of the polygon. As before, an arc is a set of two non-neighbouring vertices of $P$.

Two arcs $\{i,j\}$ and $\{k,\ell\}$ cross if their end points are all distinct and come in the order $i,k,j,\ell$ when moving around the polygon $P$ in one direction or the other. This corresponds to an obvious notion of geometrical crossing. Note that an arc does not cross itself and that two arcs sharing an end point do not cross.

Let $X$ be a set of arcs in $P$. Then $X$ is a Ptolemy diagram of type $A$ if it has the following property: when $\{i,j\}$ and $\{k,\ell\}$ are crossing arcs from $X$, then those of $\{i,k\}, \{i,\ell\}, \{j,k\}, \{j,\ell\}$ which are arcs are also in $X$, see Figure 11.

In [11] we have classified and enumerated the Ptolemy diagrams of type $A$. In particular, we have shown in [11, Section 3] that the generating function for Ptolemy diagrams of type $A$, 

$$P_A(y) := \sum_{N \geq 1} \# \{\text{Ptolemy diagrams of type } A \text{ of the } (N+1)\text{-gon}\} y^N. \quad (1)$$

satisfies 

$$P_A(y) = y + \frac{P_A(y)^2}{1 - P_A(y)} + \frac{P_A(y)^3}{1 - P_A(y)}.$$

We now turn back to Ptolemy diagrams of type $D$. In this section we shall determine the generating function for Ptolemy diagrams of type $D$, 

$$P_D(y) := \sum_{N \geq 1} \# \{\text{Ptolemy diagrams of type } D \text{ of the } 2N\text{-gon}\} y^N.$$
Roughly speaking we will decompose a Ptolemy diagram of type $D$ into a ‘central region’ (again with $180^\circ$ rotational symmetry) containing all the diameters, bounded by a polygon and a circular arrangement of Ptolemy diagrams of type $A$ ‘glued’ to the edges of this central polygon as sketched in Figure 12.

Before we can define the ‘central region’ of a Ptolemy diagram of type $D$ precisely, we need the following fact about the structure of Ptolemy diagrams:

**Lemma 5.1.** Let $X$ be a Ptolemy diagram of type $D$ in the $2n$-gon $P$. Suppose that $X$ does not contain a diameter. Then there exists a diameter $\{i, i+n\}$ (green or red) which is not crossed by any arc in $X$.

**Proof.** Let $\{i,j\} \in X$ (with $j \in [i,i+n]$) such that the set of vertices $[i,j]$ is maximal. We show that $\{i,i+n\}$ is not crossed by any arc in $X$.

Suppose that an arc $\{k,\ell\} \in X$ (with $\ell \in [k,k+n]$) crosses $\{i,i+n\}$. There are two cases to distinguish: if $k, i, j, \ell$ come in this order when going counterclockwise then $[k,\ell] \supset [i,j]$, contradicting the maximality assumption. On the other hand, if the order is $k, i, \ell, j$, then $\{k,\ell\}$ and $\{i,j\}$ cross and condition (Pt1) implies that $\{k,j\} \in X$. Now, if $j \in [k,k+n]$ then $[k,j] \supset [i,j]$ contradicts the maximality assumption. On the other hand, if $j + n \in [k,k+n]$ we have that $\{k,j\}$ and $\{j+n,i+n\}$ cross. Condition (Pt1) then forces $\{j,j+n\} \in X$, contradicting the hypothesis of the lemma. \qed

**Definition 5.2.** Let $X$ be a Ptolemy diagram of type $D$ in the $2n$-gon $P$. Then the **central region** is a set of vertices and arcs of $X$ constructed as follows:

Suppose that $X$ does not contain any diameter. Then, by Lemma 5.1 there is a diameter $\{i,i+n\}$ (green or red) which is not crossed by any arc in $X$. Let $\mathcal{V}$ be the shortest sequence of vertices ($i = i_0, i_1, \ldots, i_k = i + n$) such that $i_j$ and $i_{j+1}$ are connected by an edge of $P$ or an arc in $X$ for $0 \leq j \leq k-1$.

If $X$ does contain a diameter $\{i,i+n\}$ (green or red) then let $\mathcal{V}$ be the shortest sequence of vertices ($i = i_0, i_1, \ldots, i_k = i + n$) such that $i_j$ and $i_{j+1}$ are connected
by an edge of \( P \) or an arc in \( \mathcal{X} \) for \( 0 \leq j \leq k - 1 \) and that contains one end point of every diameter from \( \mathcal{X} \).

The central region of \( \mathcal{X} \) is the polygon containing the vertices in \( \mathfrak{V} \), their opposite vertices and the edges of \( P \) and arcs in \( \mathcal{X} \) connecting vertices in \( \mathfrak{V} \). We say that the edges and arcs \( \{i_j, i_{j+1}\} \) and \( \{i_j + n, i_{j+1} + n\} \) for \( 0 \leq j \leq k - 1 \) bound the central region.

**Lemma 5.3.** In a Ptolemy diagram of type \( D \) there is no arc crossing one of the edges or arcs bounding the central region. Consequently, the diagrams attached to the central region are Ptolemy diagrams of type \( A \).

**Proof.** Let \( \mathfrak{V} \) be the set of vertices on the boundary of the central region. Let \( i \) and \( j \) be two vertices in \( \mathfrak{V} \) and let \( \{i, j\} \in \mathcal{X} \) (with \( j \in [i, i + n] \)). Suppose that the arc \( \{i, j\} \) is crossed by an arc \( \{k, \ell\} \in \mathcal{X} \) (with \( \ell \in [k, k + n] \)). Let us assume without loss of generality that \( k \in [i, j] \). We show that this forces that a diameter \( \{k, k + n\} \in \mathcal{X} \), so the arc \( \{i, j\} \) cannot be on the boundary of the central region.

If \( \{k, \ell\} \) is a diameter there is nothing to show, so we suppose that this is not the case. If \( \{k, \ell\} \) is crossed by a diameter then condition (Pt3) implies that \( \{k, k + n\} \) of \( \mathcal{X} \) and we are done.

Suppose now that \( \{k, \ell\} \) is not crossed by a diameter. In particular, \( \{i, j\} \) is not a diameter either. Condition (Pt1) then implies that \( \{i, \ell\} \in \mathcal{X} \). It follows that \( \ell \notin \mathfrak{V} \) because otherwise the sequence \( \mathfrak{V} \) is not minimal: replacing the vertices in \( [i, \ell] \) by just \( i \) and \( \ell \) yields a shorter sequence (at least the vertex \( j \) does not appear).

However, if \( \ell \notin \mathfrak{V} \), there must be an arc \( \{r, s\} \in \mathcal{X} \) with \( r, s \in \mathfrak{V} \) that crosses \( \{k, \ell\} \) and such that \( i, j, r, s \) appear in this order when going around the polygon counterclockwise. Since this arc cannot be a diameter by assumption, condition (Pt1) implies that \( \{i, s\} \in \mathcal{X} \), which again violates the minimality of \( \mathfrak{V} \).

The fact that the diagrams attached to the central region are Ptolemy diagrams of type \( A \) follows, because such a component cannot contain any diameters and condition (Pt1) coincides with the Ptolemy condition in type \( A \). \( \square \)

**Proposition 5.4.** Let

\[
\mathcal{C}(y) = \sum_{k \geq 0} \# \{\text{central regions with } 2k + 2 \text{ bounding edges}\} y^k
\]

be the generating function for central regions. Then the generating function for Ptolemy diagrams of type \( D \) equals

\[
\mathcal{P}_D(y) = y \mathcal{P}_A(y) \mathcal{C}(\mathcal{P}_A(y)).
\]

**Proof.** Suppose we are given a central region with \( 2k + 2 \) bounding edges. We can then construct every Ptolemy diagram of type \( D \) with this central region in a unique way from a list \( (\mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_k) \) of Ptolemy diagrams of type \( A \) together with an additional distinguished edge \( \delta \) in \( \mathcal{X}_0 \). As we will see we have to insist that \( \delta \) is different from the distinguished base edge of \( \mathcal{X}_0 \) except if \( \mathcal{X}_0 \) is the degenerate Ptolemy diagram of type \( A \).

Namely, to construct a diagram from the given data we glue in clockwise order \( \mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_k, \mathcal{X}_0, \mathcal{X}_1, \ldots, \mathcal{X}_k \) onto the bounding edges of the central region along their respective distinguished base edges. Finally, we declare the edge in the resulting diagram that corresponds to \( \delta \) in one copy of \( \mathcal{X}_0 \) in \( \mathfrak{V} \) to be the distinguished edge.
Figure 13. The central regions of type (I) with at most 6 bounding edges.

base edge of $\mathcal{X}$. Because of the symmetry of the resulting diagram it does not matter which of the two copies of $\mathcal{X}_0$ we select.

We now notice that the generating function for Ptolemy diagrams of type $A$ with an additional distinguished edge satisfying the condition mentioned above is $yP'_A(y)$, where the prime denotes the derivative. Thus, the generating function for Ptolemy diagrams of type $D$ with a central region of size $2k+2$ equals

$$\sum_{k=0}^{\infty} \#\{\text{central regions with } 2k+2 \text{ bounding edges}\} y^k P'_A(y) P_A(y)^{k}.$$ 

Summing over all $k$ we obtain the claim.

We remark that we defined the degenerate Ptolemy diagram of type $D$ with two vertices such as to make this decomposition agree with Condition (Pt1).

We now distinguish three kinds of central regions occurring in Ptolemy diagrams of type $D$. Although not imperative for the rest of this article these are chosen in a way such that the set of diagrams of each kind is closed under the operator $nc$ and under rotation. Note that the central region of a Ptolemy diagram of type $D$ is itself a Ptolemy diagram of type $D$. Thus we actually define three kinds of Ptolemy diagrams, according to the kind of their central regions:

**Definition 5.5.** Let $\mathcal{X}$ be a Ptolemy diagram of type $D$. We say that two diameters in $\mathcal{X}$ are paired if they connect the same two vertices (and are thus of different colour).

Every Ptolemy diagram $\mathcal{X}$ of type $D$ is of one of the following three kinds:

(I) All diameters (if any) in $\mathcal{X}$ are paired.

(II) All diameters in $\mathcal{X}$ are of the same colour and there are at least two diameters in $\mathcal{X}$.

(III) Not all diameters in $\mathcal{X}$ are paired and if there are several diameters, both colours occur.

We remark that according to the construction in the proof of Proposition 5.4 the type of a Ptolemy diagram is determined by its central region. In Figures 13–15 the central regions with few bounding edges are listed.

In the remainder of this section we describe each of the three kinds of central regions precisely and determine in each case the corresponding generating function.

5.1. **Ptolemy diagrams of the first kind.** Let $\mathcal{X}$ be a central region with $2k+2$ bounding edges. If all diameters in $\mathcal{X}$ are paired then by Condition (Pt2) all arcs connecting end points of the diameters in $\mathcal{X}$ are also in $\mathcal{X}$. In particular, since $\mathcal{X}$ is
a central region, if $X$ contains a diameter then all $2k + 2$ possible diameters ($k + 1$ of each colour) must be present.

Finally, again because $X$ is a central region, if $X$ does not contain a diameter then $X$ is a polygon without any (internal) arcs. In this case $k$ must be greater than zero since the central region with only two vertices contains both diameters.

Thus, for $k = 0$ there is one central region of the first kind while for $k \geq 1$ there are two. In other words, the generating function for central regions of the first kind equals

$$C_{D,1}(y) = \frac{1+y}{1-y} = 1 + 2y + 2y^2 + 2y^3 + 2y^4 + 2y^5 + \ldots,$$

Clearly, for $k \geq 1$ the operator nc maps the central region of the first kind without diameters to the central region of the first kind with all diameters and vice versa. The degenerate central region with two vertices is a fixed point under the operation of nc.

5.2. **Ptolemy diagrams of the second kind.** We now consider the case that there are at least two diameters in $X$ and all of them are of the same colour. Suppose that there is a non-diameter arc $\{i, j\}$ crossing a diameter. Let us traverse the polygon starting at $i$ and ending at $j$, where the direction of travel is chosen in such a way that $j$ is encountered before $j + n$. Let $\mathcal{B}$ be the set of end points of diameters in $X$, including $i$ and $j$ encountered in this way. Then Condition (Pt1) and Condition (Pt3) imply that precisely the arcs connecting vertices in $\mathcal{B}$ are
present in $\mathcal{X}$. Furthermore, by Condition (Pt3) and Condition (Pt1) there are at least two diameters in $\mathcal{X}$ which are not crossed by another diagonal, and only arcs connecting end points of diameters in $\mathcal{X}$ can cross a diameter in $\mathcal{X}$.

It remains to derive the generating function of central regions of the second kind. Such a central region is determined by a selection of at least two diameters that are not crossed by any arcs and a subset of these diameters that are crossed by arcs. However, end points of diameters that are chosen to be crossed by arcs can only be neighboured by end points of other diameters in the selected set. Therefore, the central regions can be encoded by words from an alphabet with three letters $o$ (not selected), $l$ (diameter) and $x$ (crossed diameter) such that $o$ and $x$ are not consecutive, where we consider the first and the last letter of the word adjacent and $l$ occurs at least twice.

One way to perform the computation of the generating function

$$W(y) = \sum_{k \geq 2} \# \{\text{words of length } k\} y^k$$

of such words is as follows: let $W_o$ (respectively $W_x$) be the set of words that do not contain the sequence $ox$ or $xo$ and end in $o$ (respectively $x$). Furthermore, let $W'$ be the set of all words that do not contain the sequence $ox$ or $xo$. Then we have the following combinatorial equations, omitting parenthesis around singleton sets to improve readability:

$$W_o = o + W' \cdot l \cdot o + W_o \cdot o$$
$$W_x = x + W' \cdot l \cdot x + W_x \cdot x$$
$$W' = \emptyset + (\emptyset + W' \cdot l) \cdot (l + o + x) + W_o \cdot (l + o) + W_x \cdot (l + x)$$
$$W = l \cdot W' \cdot l \cdot (\emptyset + o^* + x^*)$$
$$+ o^* \cdot l \cdot W' \cdot l \cdot o^* + x^* \cdot l \cdot W' \cdot l \cdot x^*$$

In these equations $+$ denotes the union of sets, $\mathcal{F} \cdot \mathcal{G}$ is the set of all words obtained by appending a word from $\mathcal{G}$ to a word from $\mathcal{F}$, $\emptyset$ is the empty word, $a^*$ denotes the set of words composed of the letter $a$ only, including the empty word, and $a^+$ equals $a^*$ without the empty word.

Passing to generating functions (we assign every letter the weight $y$) and solving the system of equations we obtain $W'(y) = \frac{1+y}{1-2y-y^2}$ and $W(y) = \frac{y^2(1+y)(1+2y-y^2)}{(1-y)^2(1-2y-y^2)}$.

Since we have to choose one of two colours for the diameters we conclude that the
generating function for central regions of the second kind equals

\[
C_{D,\II} = 2 \frac{W(y)}{y} = 2 \frac{y(1 + y)(1 + 2y - y^2)}{(1 - y)^2(1 - 2y - y^2)}
\]

(II)

\[
= 2y + 14y^2 + 50y^3 + 142y^4 + 370y^5 + \ldots
\]

The action of the operator \( nc \) is most easily explained by its effect on the corresponding words: if \( X \) is a central region corresponding to a word \( w \) then the word corresponding \( ncX \) is obtained from \( w \) by interchanging the letters \( o \) and \( x \).

5.3. Ptolemy diagrams of the third kind. Suppose first that not all diameters in \( X \) are paired and both colours occur. In this case, if there is a paired diameter in \( X \) then all other diameters must be of the same colour.

Namely, let \( a = \{i, i + n\} \) be a paired diameter in \( X \), and let \( b = \{j, j + n\} \) and \( c \) be unpaired diameters of different colours, say \( b \) red and \( c \) green. Then Condition (Pt2) applied to \( a \) and \( b \) implies that one of \( \{i, j\} \) and \( \{i, j + n\} \) crosses \( c \). But then Condition (Pt3) applied to this crossing forces the presence of the green diameter pairing \( b \), a contradiction.

Furthermore, if there is a paired diameter in \( X \) and all other diameters are of the same colour then on both sides of the paired diameter the end points of all diameters are all connected by Condition (Pt2) and Condition (Pt1). However, the paired diameter itself is not crossed by any non-diameter arcs, since Condition (Pt3) would then force all diameters to be paired.

Otherwise, if there is no paired diameter then \( X \) contains at most two diameters. Suppose on the contrary that \( a = \{i, i + n\} \) is a red unpaired diameter and \( b = \{j, j + n\} \) and \( c \) are both unpaired green diameters. Then Condition (Pt2) applied to \( a \) and \( b \) forces the presence of an arc \( d \) crossing \( c \). In turn, Condition (Pt3) applied to \( c \) and \( d \) implies that \( a \) must be paired, contradicting our assumption.

As in Ptolemy diagrams of the first and second kind no other arcs can cross a diameter.

Thus, central regions of the third kind having one paired diameter or a single (unpaired) diameter can be constructed by choosing a colour for the unpaired diameter(s) and one diameter in a polygon with at least four edges. Finally, there are two additional central region with four vertices and two unpaired diameters of different colours:

\[
C_{D,\III} = 2y + 4 \sum_{k \geq 1} (k + 1)y^k = 2y + 4 \frac{2y - y^2}{(1 - y)^2}
\]

(III)

\[
= 10y + 12y^2 + 16y^3 + 20y^4 + 24y^5 + \ldots
\]

The operator \( nc \) maps the two central regions with two unpaired diameters onto each other. It maps the central region with a single unpaired diameter, say red, to the central region that has this diameter paired and all other diameters of red colour.
5.4. The grand total. Combining Equations (I)–(III) and applying Proposition 5.4 we obtain the generating function for Ptolemy diagrams of type $D$:

$$P_D(y) = yP'_A(y)\frac{1 + 12P_A(y) - P^2_A(y) - 2P^3_A(y)}{1 - 2P_A(y) - P^2_A(y)}$$

$$= y + 16y^2 + 82y^3 + 500y^4 + 3084y^5 + 19400y^6 + \ldots$$

It follows from the algebraicity of $P_A(y)$ that also $P_D(y)$ is an algebraic generating function. However, the equation is not particularly appealing:

$$(4y^2 - 47y^2 - 48y + 8)P^3(y) - 2(y - 2)(4y^3 - 47y^2 - 48y + 8)P^2(y) + (4y^5 - 99y^3 + 628y^3 - 246y^2 - 240y + 32)P(y) - 2y(20y^3 - 319y^2 + 152y + 16) = 0$$

References
