EXT VANISHING AND INFINITE AUSLANDER-BUCHSBAUM

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Abstract. A vanishing theorem is proved for Ext groups over non-commutative graded algebras. Along the way, an “infinite” version is proved of the non-commutative Auslander-Buchsbaum theorem.

0. Introduction

Let $R$ be a noetherian local commutative ring, and let $X$ be a finitely generated $R$-module of finite projective dimension. The classical Auslander-Buchsbaum theorem states

$$\text{pd } X = \text{depth } R - \text{depth } X.$$  

This can also be phrased as an Ext vanishing theorem, namely, if $M$ is any $R$-module, then

$$\text{Ext}^i_R(X, M) = 0 \text{ for } i > \text{depth } R - \text{depth } X. \quad (1)$$

In [1] is proved a surprising variation of this: Suppose that $R$ is complete in the $m$-adic topology. Then equation (1) remains true if $X$ is any $R$-module of finite projective dimension, provided $M$ is finitely generated. In other words, the condition of being finitely generated is shifted from $X$ to $M$.

In theorem 2.3 below, this result will be generalized to the situation of a non-commutative noetherian N-graded connected algebra.

The route goes through an “infinite” version of the non-commutative Auslander-Buchsbaum theorem, given in theorem 1.4. This result is a substantial improvement of the original non-commutative Auslander-Buchsbaum theorem, as given in [3, thm. 3.2], in that the condition of dealing only with finitely generated modules is dropped.

The notation of this paper is standard, and is already on record in several places such as [2] or [3]. So I will not say much, except that throughout, $k$ is a field, and $A$ is a noetherian N-graded connected

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$k$-algebra. However, let me give one important word of caution: Everything in sight is graded. So for instance, $D(A)$ stands for $D(\text{Gr} A)$, the derived category of the abelian category $\text{Gr}(A)$ of $\mathbb{Z}$-graded $A$-left-modules and graded homomorphisms of degree zero.

1. Auslander-Buchsbaum

**Definition 1.1.** For $X$ in $D(A)$, define the $k$-flat dimension by

$$k.fdl \ X = - \inf k \overset{L}{\otimes}_A X.$$  

**Remark 1.2.** Using a minimal free resolution, it is easy to see that if the cohomology of $X$ is bounded and finitely generated, then

$$k.fdl \ X = \text{fd} X = \text{pd} X,$$

where fd stands for flat dimension and pd stands for projective dimension.

In the following lemma is used $D^b(A)$, the full subcategory of $D(A)$ consisting of complexes with bounded cohomology, and $D^+(A^{op})$, the full subcategory of $D(A^{op})$ consisting of complexes whose cohomology vanishes in low degrees.

**Lemma 1.3.** Let $X$ in $D^b(A)$ have $\text{fd} X < \infty$, and let $T$ in $D^+(A^{op})$ be so that $\text{H}^i(T)$ is a graded torsion module for each $i$. Then

$$\inf T \overset{L}{\otimes}_A X = \inf T - k.fdl \ X.$$  

**Proof.** Observe that $\text{fd} X < \infty$ implies

$$\inf k \overset{L}{\otimes}_A X > -\infty,$$  

hence

$$k.fdl \ X < \infty.$$  

If $T$ is zero then $\text{inf} T = \text{inf} T \overset{L}{\otimes}_A X = \infty$, and then the inequality (3) implies that the lemma’s equation trivially reads $\infty = \infty$. So for the rest of the proof I can assume that $T$ is non-zero and hence $\text{inf} T < \infty$. Note that $T$ is in $D^+(A^{op})$, so $\text{inf} T > -\infty$, so $\text{inf} T$ is a finite number.

First, the special case where $T$ is concentrated in degree zero. Here $T$ is just a non-zero graded torsion $A$-right-module, and the lemma claims

$$\inf T \overset{L}{\otimes}_A X = -k.fdl \ X = \inf k \overset{L}{\otimes}_A X.$$  

Let me start by showing more modestly

$$\inf T \overset{L}{\otimes}_A X \geq \inf k \overset{L}{\otimes}_A X.$$  

(5)
If $F$ is a flat resolution of $X$, then this amounts to
\[ \inf T \otimes_A F \geq \inf k \otimes_A F. \] (6)

To prove this, note that as $T$ is a graded torsion module, it is the colimit of the system
\[ T'(1) \subseteq T'(2) \subseteq \cdots \]
where
\[ T'(j) = \{ t \in T \mid A_{\geq j}t = 0 \}. \]

Each quotient $T'(j)/T'(j-1)$ is annihilated by $A_{\geq 1}$ so has the form $\prod_{\alpha} k(\ell_{\alpha})$, so there are short exact sequences of the form
\[ 0 \rightarrow T'(j - 1) \rightarrow T'(j) \rightarrow \prod_{\alpha} k(\ell_{\alpha}) \rightarrow 0. \]

Tensoring such a sequence with $F$ gives a short exact sequence of complexes because $F$ consists of graded flat modules. The corresponding cohomology long exact sequence consists of pieces
\[ h^i(T'(j - 1) \otimes_A F) \rightarrow h^i(T'(j) \otimes_A F) \rightarrow \prod_{\alpha} h^i(k(\otimes_A F)(\ell_{\alpha}). \]

Induction on $j$ now makes clear that
\[ h^i(k \otimes_A F) = 0 \]
implies
\[ h^i(T'(j) \otimes_A F) = 0 \text{ for each } j, \]
and this further gives
\[ h^i(T \otimes_A F) \cong h^i(\text{colim } T'(j) \otimes_A F) \cong \text{colim } h^i(T'(j) \otimes_A F) = 0, \]
so the inequality (6) follows, and hence, so does the inequality (5).

Note that the proof even works for $\inf k \otimes_A X = \infty$.

Let me now step this up to show equation (4). Note that if
\[ \inf k \otimes_A X = \infty \]
then the inequality (5) forces
\[ \inf T \otimes_A X = \infty, \]
and so equation (4) holds.

So I can assume
\[ \inf k \otimes_A X < \infty. \]
Because of the inequality (2), it follows that \( \inf k \otimes_A X \) is a finite number. By the inequality (5), equation (4) will follow if I can prove

\[
h^{\inf k \otimes_A X}(T \otimes_A X) \neq 0. \tag{7}
\]

But since \( T \) is non-zero and graded torsion, there is a short exact sequence \( 0 \rightarrow k(\ell) \rightarrow T \rightarrow \hat{T} \rightarrow 0 \) of graded \( A \)-right-modules. This gives a distinguished triangle \( k(\ell) \rightarrow T \rightarrow \hat{T} \rightarrow \) in \( D(A^{op}) \), and tensoring with \( X \) and taking the cohomology long exact sequence gives a sequence consisting of pieces

\[
h^i(k(\ell) \otimes_A X) \rightarrow h^i(T \otimes_A X) \rightarrow h^i(\hat{T} \otimes_A X).
\]

As \( T \) is graded torsion, so is \( \hat{T} \). The inequality (5) applied to \( \hat{T} \) gives

\[
h^i(\hat{T} \otimes_A X) = 0 \text{ for } i < \inf k \otimes_A X.
\]

Hence there is a piece of the long exact sequence which reads

\[
0 \rightarrow h^{\inf k \otimes_A X}(k(\ell) \otimes_A X) \rightarrow h^{\inf k \otimes_A X}(T \otimes_A X),
\]

proving equation (7) and hence equation (4).

Secondly, the general case where \( T \) is not necessarily concentrated in degree zero. There is a spectral sequence

\[
E_2^{pq} = h^p(h^q(T) \otimes_A X) \Rightarrow h^{p+q}(T \otimes_A X)
\]

which can be obtained as the second usual spectral sequence of the double complex defined by \( M^{pq} = T^p \otimes_A F^q \), where \( F \) is a flat resolution of \( X \); cf. [4, thm. 11.19]. The spectral sequence converges because \( \text{fd} \ X < \infty \) implies that it is first quadrant up to shift. Now, \( h^q(T) \) is graded torsion for each \( q \), so if \( h^q(T) \) is non-zero, then the special case of the lemma dealt with above applies to \( T \otimes_A X \) and shows

\[
\inf h^q(T) \otimes_A X = -\text{k.fd} \ X. \tag{8}
\]

There are now two cases. The first case is

\[
\text{k.fd} \ X = -\infty. \tag{9}
\]

Here equation (8) gives that if \( h^q(T) \) is non-zero, then \( \inf h^q(T) \otimes_A X = \infty \), that is, \( h^p(h^q(T) \otimes_A X) \) is zero for each \( p \). Of course this also holds for \( h^q(T) \) equal to zero, and so in the spectral sequence, \( E_2^{pq} \).
is identically zero. Therefore the limit $h^{p+q}(T \otimes_A X)$ of the spectral sequence is also zero, so $T \otimes_A X$ is zero, so

$$\inf T \otimes_A X = \infty.$$  \hfill (10)

But $\inf T$ is a finite number, and combining this with equations (9) and (10) says that the lemma’s equation reads

$$\infty = (\text{a finite number}) - (-\infty)$$

which is true.

The second case is

$$\kfd X > -\infty.$$  

Here equation (8) gives that if $h^q(T)$ is non-zero, then $h^p(h^q(T) \otimes_A X)$ is non-zero for $p = -\kfd X$, but zero for $p < -\kfd X$. And of course, if $h^q(T)$ is zero, then $h^p(h^q(T) \otimes_A X)$ is zero for each $p$. So in the spectral sequence, $E^{pq}_2$ is non-zero for $p = -\kfd X$ and $q = \inf T$, but zero for lower $p$ or $q$. Hence $E^{\kfd X, \inf T}_2$ can be used in a standard corner argument which shows that the lowest non-zero term in the limit $h^{p+q}(T \otimes_A X)$ of the spectral sequence has degree $p + q = -\kfd X + \inf T$. Hence

$$\inf T \otimes_A X = -\kfd X + \inf T,$$

proving the lemma’s equation. \hfill \Box

Observe that in the following theorem and the rest of the paper, depth $A$ stands for the depth of $A$ viewed as a left-module over itself.

**Theorem 1.4** (Infinite Auslander–Buchsbaum). Assume that $A$ satisfies that each $\mathrm{Ext}^i_A(k, A)$ is a graded torsion $A$-right-module. Let $X$ in $\mathcal{D}^b(A)$ have $\kfd X < \infty$. Then

$$\depth X = \depth A - \kfd X.$$  

*Proof.* I have

$$\mathrm{RHom}_A(k, X) \cong \mathrm{RHom}_A(k, A \otimes_A X) \cong \mathrm{RHom}_A(k, A) \otimes_A X,$$

where the second $\cong$ holds by [3, prop. 2.1] because $X$ is in $\mathcal{D}^b(A)$ and has $\kfd X < \infty$. 
Thus
\[
\text{depth } X = \inf \text{RHom}_A(k, X) \\
= \inf \text{RHom}_A(k, A) \otimes_A X \\
\overset{(a)}{=} \inf \text{RHom}_A(k, A) - \text{k.fd } X \\
= \text{depth } A - \text{k.fd } X,
\]
where (a) is by lemma 1.3. The lemma applies because \( \text{RHom}_A(k, A) \) is in \( D^+(A^{op}) \), and has \( h^i \text{RHom}_A(k, A) = \text{Ext}_A^i(k, A) \) a graded torsion \( A \)-right-module for each \( i \) by assumption.

\[\square\]

**Remark 1.5.** Theorem 1.4 even holds for depth \( A = \infty \), where the theorem states depth \( X = \infty \).

On the other hand, suppose depth \( A < \infty \). Then it is easy to see that it makes sense to rearrange the equation in theorem 1.4 as
\[
\text{k.fd } X = \text{depth } A - \text{depth } X.
\]

If the cohomology of \( X \) is bounded and finitely generated, then the equation of theorem 1.4 reads
\[
\text{depth } X = \text{depth } A - \text{pd } X
\]
by remark 1.2. This is the original non-commutative Auslander-Buchsbaum theorem, as proved in [3, thm. 3.2].

2. **EXT VANISHING**

**Lemma 2.1.** Assume that \( A \) has depth \( A < \infty \) and satisfies that each \( \text{Ext}_A^i(k, A) \) is a graded torsion \( A \)-right-module.

Let \( X \) in \( D^b(A) \) have \( \text{fd } X < \infty \), and let \( T \) in \( D^+(A^{op}) \) be so that \( h^i(T) \) is a graded torsion module for each \( i \). Then
\[
\inf T \otimes_A X = \inf T + \text{depth } X - \text{depth } A.
\]

**Proof.** Using lemma 1.3 and remark 1.5 gives
\[
\inf T \otimes_A X = \inf T - \text{k.fd } X = \inf T + \text{depth } X - \text{depth } A.
\]

\[\square\]

In the following theorem is used \( D^-_{fg}(A) \), the full subcategory of \( D(A) \) consisting of complexes whose cohomology vanishes in high degrees and consists of finitely generated graded modules.
Theorem 2.2. Assume that $A$ has depth $A < \infty$ and satisfies that each $\text{Ext}^i_A(k, A)$ is a graded torsion $A$-right-module.

Let $X$ in $D^b(A)$ have $\text{fd} X < \infty$, and let $M$ be in $D^+_A(A)$. Then

$$\sup \text{RHom}_A(X, M) = \sup M - \text{depth } X + \text{depth } A.$$ 

Proof. It is easy to see that since $M$ is in $D^+_A(A)$, the Matlis dual $M'$ is in $D^+(A^\text{op})$ and has $\text{h}^i(M')$ a graded torsion module for each $i$. So

$$\sup \text{RHom}_A(X, M) = \sup \text{RHom}_A(X, M''')$$

(a)$$= \sup((M' \otimes_A X'))$$

(b)$$= -\inf M' \otimes_A X$$

$$= -\inf M' - \text{depth } X + \text{depth } A$$

$$= \sup M - \text{depth } X + \text{depth } A,$$

where (a) is by adjunction and (b) is by lemma 2.1. $\square$

The following is the special case of theorem 2.2 where $X$ and $M$ are concentrated in degree zero, that is, where $X$ and $M$ are graded modules.

Theorem 2.3 (Ext vanishing). Assume that $A$ has depth $A < \infty$ and satisfies that each $\text{Ext}^i_A(k, A)$ is a graded torsion $A$-right-module.

Let $X$ in $\text{Gr}(A)$ have $\text{fd} X < \infty$, and let $M$ be in $\text{gr}(A)$. Then

$$\text{Ext}^i_A(X, M) = 0 \text{ for } i > \text{depth } A - \text{depth } X.$$ 

If $\text{depth } X < \infty$ and $M \neq 0$ also hold, then

$$\text{Ext}^i_A(X, M) \neq 0 \text{ for } i = \text{depth } A - \text{depth } X.$$ 

This says that for $\text{fd} X < \infty$, the number $\text{depth } A - \text{depth } X$ plays the role of projective dimension of $X$, but only with respect to finitely generated graded modules $M$.

Of course, this fails when $M$ is general, as illustrated by the following example.

Example 2.4. Let $A$ be the polynomial algebra $k[x]$. Then the conditions of theorem 2.3 are satisfied, and it is classical that depth $A$ is 1.

Let $X$ be $k[x, x^{-1}]$. Then depth $X \geq 1$ because $X$ is a graded torsion free module, so depth $A - \text{depth } X \leq 0$ and theorem 2.3 gives

$$\text{Ext}^i_A(X, M) = 0 \text{ for } i > 0$$

for $M$ in $\text{gr}(A)$.

However, this must fail when $M$ is general, for otherwise $X$ would be a projective object of $\text{Gr}(A)$ which it is certainly not.
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REFERENCES


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