Abstract. Higher homological algebra was introduced by Iyama. It is also known as $n$-homological algebra where $n \geq 2$ is a fixed integer, and it deals with $n$-cluster tilting subcategories of abelian categories.

All short exact sequences in such a subcategory are split, but it has well-behaved exact sequences with $n + 2$ objects. This was recently formalised by Jasso in his theory of $n$-abelian categories. There is also a derived version of $n$-homological algebra, formalised by Geiss, Keller, and Oppermann in the theory of $(n+2)$-angulated categories (the reason for the shift from $n$ to $n+2$ is that angulated categories have triangulated categories as the “base case”).

We introduce torsion classes and t-structures into the theory of $n$-abelian and $(n+2)$-angulated categories and prove several results to motivate the definitions. Most of the results concern the $n$-abelian and $(n+2)$-angulated categories $\mathcal{M}(\Lambda)$ and $\mathcal{C}(\Lambda)$ associated to an $n$-representation finite algebra $\Lambda$, as defined by Iyama and Oppermann. We characterise torsion classes in these categories in terms of closure under higher extensions, and give a bijection between torsion classes in $\mathcal{M}(\Lambda)$ and intermediate t-structures in $\mathcal{C}(\Lambda)$ which is a category one can reasonably view as the $n$-derived category of $\mathcal{M}(\Lambda)$. We hint at the link to $n$-homological tilting theory.

0. Introduction

Higher homological algebra was introduced and developed by Iyama in [10], [11], [12]. It is also known as $n$-homological algebra where $n \geq 2$ is an integer.

This paper introduces the notions of torsion class and t-structure into $n$-homological algebra and shows several results to motivate the definitions. The results are mainly foundational, only hinting at the link to $n$-homological tilting theory which was initiated in [13] and [17]. However, as this theory is developed further, we expect that, like classic tilting, it will turn out to be closely linked to torsion classes.

$n$-homological algebra concerns so-called $n$-cluster tilting subcategories of abelian categories, see [11, def. 1.1]. All short exact sequences in such a subcategory are split, but it has well-behaved exact sequences with $n + 2$ objects, also known as $n$-exact sequences. There is also a derived version of the theory focusing on $n$-cluster tilting subcategories of triangulated categories as introduced in [16, sec. 5.1]. There are rich examples of $n$-cluster tilting subcategories, many with interesting links to combinatorics and geometry, see [8], [9], [13], [17].

The abelian version of $n$-homological algebra was recently formalised to the theory of $n$-abelian categories by Jasso in [15, def. 3.1]. In such a category, each morphism $x \to y$ has not a kernel, but an $n$-kernel, that is, there is an exact sequence $0 \to m_n \to \cdots \to m_1 \to x \to y$.
with certain properties. Similarly, \( x \to y \) has an \( n \)-cokernel, and a list of axioms are satisfied. The derived version of \( n \)-homological algebra was formalised to the theory of \((n+2)\)-angulated categories by Geiss, Keller, and Oppermann in [6, def. 2.1].

Let \( n \geq 1 \) be a fixed integer for the rest of the paper (the case \( n = 1 \) is included since \( 1 \)-homological algebra makes sense and reduces to classic homological algebra). The following is our main definition. Note that the suspension functor of an \((n+2)\)-angulated category is denoted by \( \Sigma^n \).

**Definition 0.1** (Torsion classes). Let \( \mathcal{M} \) be an \( n \)-abelian category. A full subcategory \( \mathcal{U} \subseteq \mathcal{M} \) is called a torsion class if, for each \( m \in \mathcal{M} \), there is an \( n \)-exact sequence

\[
0 \to u \xrightarrow{\theta} m \xrightarrow{} v^1 \to \cdots \to v^n \to 0
\]

where \( u \in \mathcal{U} \) and \( v^1 \to \cdots \to v^n \) is in the class \( \mathcal{U} \)-exact defined in Equation (0.1) below.

Let \( \mathcal{C} \) be an \((n+2)\)-angulated category. A full subcategory \( \mathcal{X} \subseteq \mathcal{C} \) is called a torsion class if it is closed under sums and summands and, for each \( c \in \mathcal{C} \), there is a \((n+2)\)-angle

\[
x \xrightarrow{\xi} c \xrightarrow{} y^1 \to \cdots \to y^n \xrightarrow{} \Sigma^n x
\]

where \( x \in \mathcal{X} \) and \( y^1 \to \cdots \to y^n \) is in the class \( \mathcal{X} \)-exact. □

Here \( \mathcal{U} \)-exact is defined by

\[
\mathcal{U} \text{-exact} = \left\{ v^1 \to \cdots \to v^n \text{ is a complex in } \mathcal{M} \mid \begin{array}{l}
0 \to \mathcal{M}(u,v^1) \to \cdots \to \mathcal{M}(u,v^n) \to 0 \\
\text{is exact for each } u \in \mathcal{U}
\end{array} \right\} \tag{0.1}
\]

and \( \mathcal{X} \)-exact is defined similarly. Let us make a few immediate observations: Definition 0.1 implies that \( \mathcal{U} \) is closed under sums and summands, see Lemma 1.2(ii). While \( \mathcal{U} \)-exact and \( \mathcal{X} \)-exact obviously depend on \( n \), we omit \( n \) from the notation since it is fixed throughout the paper. If \( n = 1 \) then \( \mathcal{U} \)-exact = \( \mathcal{U}^\perp \) so the definition specialises to the usual definitions of torsion classes in abelian and triangulated categories, see [1, def. 1.1 and prop. 1.5] and [14, def. 2.2].

The following is a simple example in the \( n \)-abelian category \( \mathcal{M}(\Lambda) \) associated to an \( n \)-representation finite algebra \( \Lambda \), see Section 3, or [13, def. 2.2] and [15, thm. 3.16].

**Theorem 0.2** (=Theorem 8.3). Let \( t \) be an \( n \)-APR tilting module for \( \Lambda \), as introduced in [13, def. 3.1 and obs. 4.1(1)]. Then \( \mathcal{M}(\Lambda) \cap \text{Fac}(t) \) is a splitting torsion class in \( \mathcal{M}(\Lambda) \).

*Splitting* means that the morphism \( \theta \) in Definition 0.1 can be taken to be a split monomorphism, and \( \text{Fac}(t) \) is the class of quotients in \( \text{mod}(\Lambda) \) of modules of the form \( t^\oplus d \). For an example of a torsion class in an \((n+2)\)-angulated category \( \mathcal{C} \), take \( \mathcal{X} = \text{add}(T) \) where \( T \) is a cluster tilting object in \( \mathcal{C} \) as introduced in [17, def. 5.3], and \( \text{add} \) means taking summands of sums.

We will prove a number of results to motivate Definition 0.1. The simplest to state is the following characterisation of torsion classes in the \((n+2)\)-angulated category \( \mathcal{C}(\Lambda) \) associated to \( \Lambda \), see Section 3, or [11, thm. 1.21] and [6, thm. 1].

**Theorem 0.3** (=Theorem 6.1). Let \( \mathcal{X} \subseteq \mathcal{C}(\Lambda) \) be a full subcategory closed under sums and summands. Then

\( \mathcal{X} \) is a torsion class in \( \mathcal{C}(\Lambda) \) ⇔ \( \mathcal{X} \) is left closed under \( n \)-extensions.
For $\mathcal{X}$ to be left closed under $n$-extensions means that, for each morphism $x'' \xrightarrow{\delta} \Sigma^n x'$ with $x', x'' \in \mathcal{X}$, there is an $(n + 2)$-angle $x' \to c^1 \to c^2 \to \cdots \to c^n \to x'' \xrightarrow{\delta} \Sigma^n x'$ with $c^i \in \mathcal{X}$. The theorem is an $n$-homological version of Iyama and Yoshino’s characterisation of torsion classes in triangulated categories, see [14, prop. 2.3(1)]. There is also a characterisation of torsion classes in the $n$-abelian category $\mathcal{M}(\Lambda)$.

**Theorem 0.4** (=Theorem 5.5). Let $\mathcal{U} \subseteq \mathcal{M}(\Lambda)$ be a full subcategory closed under sums and summands. Then

$\mathcal{U}$ is a torsion class in $\mathcal{M}(\Lambda) \Leftrightarrow \mathcal{U}$ has Property $(E)$.

Property $(E)$ is stated in Definition 4.1 and we do not reproduce it here, but do wish to point out that it is an $n$-homological version of the classic property of being closed under quotients and extensions, to which it specialises when $n = 1$, see Proposition 4.7.

In classic homological algebra, there is a well-known bijection between torsion classes in an abelian category and so-called intermediate t-structures in the derived category. This is due to [4, thm. 3.1] and [7, prop. 2.1], see [18, prop. 2.1] for a clean statement. We show the following $n$-homological version for the $n$-abelian category $\mathcal{M}(\Lambda)$ and the $(n + 2)$-angulated category $\mathcal{C}(\Lambda)$ which can reasonably be thought of as the $n$-derived category of $\mathcal{M}(\Lambda)$, although we presently lack the definitions to give this sentence a precise meaning.

**Theorem 0.5** (=Theorem 7.5). There is a bijection

$$
\left\{ \mathcal{U} \subseteq \mathcal{M}(\Lambda) \mid \mathcal{U} \text{ is a torsion class} \right\} \leftrightarrow \left\{ \mathcal{X} \subseteq \mathcal{C}(\Lambda) \mid \mathcal{X} \text{ is a left t-structure with } \mathcal{C}^{-n}(\Lambda) \subseteq \mathcal{X} \subseteq \mathcal{C}^0(\Lambda) \right\}.
$$

We refer the reader to Definition 7.1 for the notation $\mathcal{C}^0(\Lambda)$, but state our definition of left t-structures in an $(n + 2)$-angulated category.

**Definition 0.6** (Left t-structures). A left t-structure in an $(n + 2)$-angulated category is a torsion class $\mathcal{X}$ satisfying $\Sigma^n \mathcal{X} \subseteq \mathcal{X}$.

To round off, note that everything we have said can be dualised, resulting in torsion free classes and right t-structures.

The paper is organised as follows. Section 1 is preparatory. Section 2 gives an $(n + 2)$-angulated version of Wakamatsu’s Lemma. Section 3 prepares the rest of the paper by giving some background on the categories $\mathcal{M}(\Lambda)$ and $\mathcal{C}(\Lambda)$. Section 4 introduces Property $(E)$ and shows some initial results about it. Sections 5 through 8 prove Theorems 0.2 through 0.5, though not in that order. Section 9 shows an example of a non-splitting torsion class in $\mathcal{M}(\Lambda)$.

Our terminology is mainly standard, but we do use the words (pre)cover and (pre)envelope due to Enochs [5, sec. 1]. They translate as follows to Auslander and Smalø’s terminology [3]: A precover is a right approximation, a cover is a minimal right approximation, a preenvelope is a left approximation, and an envelope is a minimal left approximation.

1. **Basic lemmas**

Recall that throughout, $n \geq 1$ is a fixed integer. The following lemma is due to Jasso and to Geiss, Keller, and Oppermann.
Lemma 1.1.  

(i) Let \( \mathcal{M} \) be an \( n \)-abelian category with an \( n \)-exact sequence
\[
0 \to m^0 \to m^1 \to \cdots \to m^{n+1} \to 0.
\]

For each \( \tilde{m} \in \mathcal{M} \), the following induced sequence is exact.
\[
0 \to \mathcal{M}(\tilde{m}, m^0) \to \mathcal{M}(\tilde{m}, m^1) \to \cdots \to \mathcal{M}(\tilde{m}, m^{n+1})
\]

In particular, the morphism \( m^0 \to m^1 \) is monic.

(ii) Let \( \mathcal{C} \) be an \((n+2)\)-angulated category with an \((n+2)\)-angle
\[
c^0 \to \cdots \to c^{n+1} \to \Sigma^n c^0.
\]

For each \( \tilde{c} \in \mathcal{C} \), the following induced sequence is exact.
\[
\mathcal{C}(\tilde{c}, c^0) \to \cdots \to \mathcal{C}(\tilde{c}, c^{n+1}) \to \mathcal{C}(\tilde{c}, \Sigma^n c^0)
\]

Proof. Part (i) is built in to Jasso’s definition of \( n \)-abelian categories, see [15, defs. 2.2 and 2.4]. Part (ii) holds by [6, def. 2.1 and prop. 2.5(a)]. \( \square \)

Lemma 1.2.  

(i) In Definition 0.1, the morphism \( u \theta \to m \) is a \( \mathcal{U} \)-cover of \( m \) and \( x \xi \to c \) is an \( \mathcal{X} \)-precover of \( c \).

(ii) A torsion class in an \( n \)-abelian category is closed under sums and summands.

Proof. (i) To show the first claim, note that for \( \tilde{m} \in \mathcal{M} \) the sequence
\[
0 \to \mathcal{M}(\tilde{m}, u) \xrightarrow{\theta} \mathcal{M}(\tilde{m}, m) \to \mathcal{M}(\tilde{m}, v^1) \to \mathcal{M}(\tilde{m}, v^2) \to \cdots \to \mathcal{M}(\tilde{m}, v^n)
\]
is exact by Lemma 1.1(i). If \( \tilde{m} \in \mathcal{U} \) then the map \( \mathcal{M}(\tilde{m}, m) \to \mathcal{M}(\tilde{m}, v^1) \) is zero because \( v^1 \to \cdots \to v^n \) is in \( \mathcal{U} \)-exact, so \( \theta_* \) is surjective. Hence \( u \theta \to m \) is a \( \mathcal{U} \)-precover of \( m \).

Moreover, Lemma 1.1(i) also says that \( \theta \) is monic, so it is in fact a \( \mathcal{U} \)-cover.

The second claim in (i) is proved in the same style, using Lemma 1.1(ii). Note that \( \xi \) is not in general monic, so we can only conclude that it is an \( \mathcal{X} \)-precover, not an \( \mathcal{X} \)-cover.

(ii) Since \( \theta \) is monic, \( \theta_* \) is injective. For \( \tilde{m} \in \text{add}(\mathcal{U}) \) the proof of (i) still shows \( \theta_* \) surjective whence \( \mathcal{M}(\tilde{m}, u) \xrightarrow{\theta} \mathcal{M}(\tilde{m}, m) \) is bijective. That is, the restrictions of the functors \( \mathcal{M}(\cdot, u) \) and \( \mathcal{M}(\cdot, \cdot) \) to \( \text{add}(\mathcal{U}) \) are equivalent. Hence \( m \in \text{add}(\mathcal{U}) \) implies \( m \cong u \in \mathcal{U} \). \( \square \)

2. Wakamatsu’s Lemma for \((n+2)\)-angulated categories

Recall that throughout, \( n \geq 1 \) is a fixed integer. The notion of being left closed under \( n \)-extensions was defined after Theorem 0.3.

Lemma 2.1 \(((n+2)\)-Angulated Wakamatsu’s Lemma). Let \( \mathcal{C} \) be an \((n+2)\)-angulated category, \( \mathcal{X} \subseteq \mathcal{C} \) a full subcategory left closed under \( n \)-extensions. If \( x \xi \to c \) is an \( \mathcal{X} \)-cover, then, in each completion to an \((n+2)\)-angle
\[
x \xi \to c \to y^1 \to \cdots \to y^n \xrightarrow{\xi} \Sigma^n x,
\]
we have \( y^1 \to \cdots \to y^n \) in \( \mathcal{X} \)-exact.

Before the proof an immediate consequence; compare with Definition 0.1.
Corollary 2.2. Let $\mathcal{C}$ be an $(n+2)$-angulated category, $\mathcal{X} \subseteq \mathcal{C}$ a full subcategory closed under sums and summands which is covering and left closed under $n$-extensions. Then $\mathcal{X}$ is a torsion class.

Proof (of Lemma 2.1). Given $\tilde{x} \in \mathcal{X}$, applying the functor $\mathcal{C}(\tilde{x}, -)$ to the $(n+2)$-angle in the lemma gives an exact sequence by Lemma 1.1(ii). Since $\mathcal{C}(\tilde{x}, \xi)$ is surjective, all that remains is to show $\mathcal{C}(\tilde{x}, \varphi \tilde{\xi}) = 0$.

Let $\tilde{x} \overset{\varphi}{\to} y^n$ be given. We must show $\varphi \tilde{\xi} = 0$. Using axioms (F1)(c), (F2), and (F3) of [6, def. 2.1] gives the following commutative diagram, where the first row is a completion of $\varphi \tilde{\xi}$ to an $(n+2)$-angle.

Since $\mathcal{X}$ is left closed under $n$-extensions, we can assume $c^0 \in \mathcal{X}$ whence there exists $c^0 \overset{\theta}{\to} x$ such that $\xi \theta = \gamma$. Hence $\xi \circ \theta \psi = \gamma \psi = \xi$ whence $\theta \psi$ is an isomorphism since $\xi$ is a cover.

It follows from Lemma 1.1(ii) that the composition of two consecutive morphisms in an $(n+2)$-angle is zero, so $\psi \circ \Sigma^{-n}(\varphi \tilde{\xi}) = 0$ and hence $\theta \psi \circ \Sigma^{-n}(\varphi \tilde{\xi}) = 0$. Since $\theta \psi$ is an isomorphism this shows $\Sigma^{-n}(\varphi \tilde{\xi}) = 0$ whence $\varphi \tilde{\xi} = 0$ as desired. □

3. $n$-REPRESENTATION FINITE ALGEBRAS AND THE ASSOCIATED $n$-ABELIAN AND $(n+2)$-ANGULATED CATEGORIES

Recall that throughout, $n \geq 1$ is a fixed integer.

Setup 3.1. In the rest of the paper, $\Lambda$ is a fixed finite dimensional algebra over an algebraically closed field $k$. We assume that $\Lambda$ is $n$-representation finite, that is, $\operatorname{gldim}(\Lambda) \leq n$ and there is an $n$-cluster tilting left $\Lambda$-module $M$, see [13, defs. 2.1 and 2.2]. Note that $M$ is unique by [11, thm. 1.6]. We assume that $\Lambda$ (viewed as a left $\Lambda$-module) and $M$ are basic, that is, without repeated direct summands.

Let $\mathcal{M}(\Lambda) = \operatorname{add}(M)$ be the $n$-cluster tilting subcategory of $\operatorname{mod}(\Lambda)$ corresponding to $M$, see [11, def. 1.1]. Here $\operatorname{mod}(\Lambda)$ is the category of finitely generated left $\Lambda$-modules.

Let $\mathcal{M}(\Lambda) = \operatorname{add}(M)$ be the $n$-cluster tilting subcategory of $\operatorname{mod}(\Lambda)$ corresponding to $M$, see [11, def. 1.1]. Here $\operatorname{mod}(\Lambda)$ is the category of finitely generated left $\Lambda$-modules.

Let

$$\mathcal{C}(\Lambda) = \operatorname{add}\{ \Sigma^{in} \mathcal{M} \mid i \in \mathbb{Z} \}$$

be the $n$-cluster tilting subcategory (see [16, sec. 5.1]) of the bounded derived category $\mathcal{D}^b(\operatorname{mod} \Lambda)$ which exists by [11, thm. 1.21].

Since $\Lambda$ is fixed, we henceforth write $\mathcal{M}$ and $\mathcal{C}$ instead of $\mathcal{M}(\Lambda)$ and $\mathcal{C}(\Lambda)$. □

Remark 3.2. The category $\mathcal{M}$ is $n$-abelian by [15, thm. 3.16], because it is an $n$-cluster tilting subcategory of the abelian category $\operatorname{mod}(\Lambda)$.

The category $\mathcal{C}$ is $(n+2)$-angulated by [6, thm. 1], because it is an $n$-cluster tilting subcategory of the triangulated category $\mathcal{D}^b(\operatorname{mod} \Lambda)$ which satisfies $\mathcal{C} = \Sigma^n \mathcal{C}$.
As remarked in the introduction, it seems reasonable to think of \( C \) as the \( n \)-derived category of \( \mathcal{M} \), although we presently lack the definitions to give this sentence a precise meaning. □

**Lemma 3.3.**

(i) Each full subcategory \( \mathcal{U} \subseteq \mathcal{M} \) closed under sums and summands is functorially finite in each of \( \mathcal{M} \), \( \text{mod}(\Lambda) \), and \( \mathcal{D}^{b}(\text{mod} \Lambda) \).

(ii) Each full subcategory \( \mathcal{X} \subseteq C \) closed under sums and summands is functorially finite in each of \( C \) and \( \mathcal{D}^{b}(\text{mod} \Lambda) \).

**Proof.** (i) Immediate since \( \mathcal{M} = \text{add}(M) \) has only finitely many indecomposable objects.

(ii) Since \( \mathcal{M} \) has only finitely many indecomposable objects and since \( \Lambda \) has finite global dimension, each object in \( \mathcal{D}^{b}(\text{mod} \Lambda) \) has non-zero morphisms only to and only from finitely many indecomposable objects in \( C \). This implies the claim. □

### 4. Property \((E)\)

Recall that throughout, \( n \geq 1 \) is a fixed integer and we are working under Setup 3.1. We think of the following as an \( n \)-homological version of the property of being closed under quotients and extensions. Indeed, it is equivalent to this for \( n = 1 \) under mild assumptions as we will show in Proposition 4.7.

**Definition 4.1 (Property \((E)\)).** Let \( \mathcal{U} \subseteq \mathcal{M} \) be a full subcategory. Consider a triangle

\[
\Sigma^{-n}u'' \oplus m'' \to u' \to e \to \Sigma^{-n+1}u'' \oplus \Sigma m''
\]

in \( \mathcal{D}^{b}(\text{mod} \Lambda) \) where \( u', u'' \in \mathcal{U} \) and \( m'' \in \mathcal{M} \).

If, for each such triangle, the object \( e \in \mathcal{D}^{b}(\text{mod} \Lambda) \) has an \( \mathcal{M} \)-preenvelope \( e \to u \) with \( u \in \mathcal{U} \), then we say that \( \mathcal{U} \) has Property \((E)\). □

Apart from introducing Property \((E)\), the main purpose of this section is to show that it is equivalent to the following which is sometimes handier to use.

**Definition 4.2 (Property \((F)\)).** Let \( \mathcal{U} \subseteq \mathcal{M} \) be a full subcategory. Consider a triangle

\[
\Sigma^{-n}u'' \to b' \to f \to \Sigma^{-n+1}u''
\]

in \( \mathcal{D}^{b}(\text{mod} \Lambda) \) where \( u'' \in \mathcal{U} \) and where \( b' \in \text{mod}(\Lambda) \) is a quotient in \( \text{mod}(\Lambda) \) of a module \( u' \in \mathcal{U} \).

If, for each such triangle, the object \( f \in \mathcal{D}^{b}(\text{mod} \Lambda) \) has an \( \mathcal{M} \)-preenvelope \( f \to u \) with \( u \in \mathcal{U} \), then we say that \( \mathcal{U} \) has Property \((F)\). □

**Remark 4.3.** Recall that each object \( d \in \mathcal{D}^{b}(\text{mod} \Lambda) \) has a standard truncation triangle

\[
\tau^{\leq -1}d \to d \to \tau^{\geq 0}d \to \Sigma \tau^{\leq -1}d
\]

which is determined up to isomorphism by the properties that \( \tau^{\leq -1}d \) has cohomology concentrated in degrees \( \leq -1 \) and \( \tau^{\geq 0}d \) has cohomology concentrated in degrees \( \geq 0 \). □

**Lemma 4.4.** Let \( d \in \mathcal{D}^{b}(\text{mod} \Lambda) \) and \( m \in \mathcal{M} \) be fixed objects. There is an \( \mathcal{M} \)-preenvelope of the form \( d \to m \) if and only if there is an \( \mathcal{M} \)-preenvelope of the form \( \tau^{\geq 0}d \to m \).
Proof. For $\tilde{m} \in \mathcal{M}$, the triangle from Remark (4.3) induces an exact sequence

$$\text{Hom}_{\mathcal{D}b}(\Sigma \tau^\leq -1d, \tilde{m}) \rightarrow \text{Hom}_{\mathcal{D}b}(\tau^> 0d, \tilde{m}) \rightarrow \text{Hom}_{\mathcal{D}b}(d, \tilde{m}) \rightarrow \text{Hom}_{\mathcal{D}b}(\tau^\leq -1d, \tilde{m}).$$

The cohomology of $\tau^\leq -1d$ is concentrated in degrees $\leq -1$ while the cohomology of $\tilde{m}$ is in degree 0, so the two outer terms in the sequence are 0. This implies the lemma. \hfill $\square$

**Lemma 4.5.** Let

$$\Sigma^{-n}u'' \oplus m'' \xrightarrow{(\varphi'', \mu'')} u' \rightarrow e \rightarrow \Sigma^{-n+1}u'' \oplus \Sigma m''$$

be a triangle in $\mathcal{D}^b(\text{mod } \Lambda)$ with $m'', u', u'' \in \text{mod}(\Lambda)$. Set $b' = \text{Coker } \mu''$ and consider the right exact sequence

$$m'' \xrightarrow{\mu''} u' \xrightarrow{\mu'} b' \rightarrow 0.$$

Then each triangle of the form

$$\Sigma^{-n}u'' \xrightarrow{\mu''} b' \rightarrow f \rightarrow \Sigma^{-n+1}u''$$

satisfies $f \cong \tau^> 0e$.

**Proof.** The octahedral axiom gives the following commutative diagram where the rows and columns are triangles minus the fourth object, and the first vertical triangle is split.

![Diagram](image)

The long exact cohomology sequence of the second vertical triangle shows that the only non-zero cohomology of $q'$ is

$$H^{-1}(q') = \text{Ker } \mu'' =: a',
H^0(q') = \text{Coker } \mu'' = b',$$

so there is a standard truncation triangle $\Sigma a' \rightarrow q' \xrightarrow{\rho'} b'$ where we observe that $\rho' \pi' = \mu'$. The octahedral axiom gives the following commutative diagram where the rows and columns are triangles minus the fourth object.

![Diagram](image)

Here the third horizontal triangle is isomorphic to the standard truncation triangle $\tau^\leq -1e \rightarrow e \rightarrow \tau^> 0e$, proving the lemma. This holds because the cohomology of $\Sigma a'$ is concentrated in degrees $\leq -1$ and the cohomology of $f$ is concentrated in degrees $\geq 0$; these properties
determine the standard truncation triangle up to isomorphism by Remark 4.3. To verify the statement about the cohomology of \( f \), use the long exact cohomology sequence of the third vertical triangle.

\[ \square \]

**Lemma 4.6.** Let \( \mathcal{U} \subseteq \mathcal{M} \) be a full subcategory. Then
\[ \mathcal{U} \text{ has Property (E) } \iff \mathcal{U} \text{ has Property (F)}. \]

**Proof.** “\( \Rightarrow \):” Consider the triangle from Definition 4.2,
\[ \Sigma^{-n}u'' \xrightarrow{\psi''} b' \rightarrow f \rightarrow \Sigma^{-n+1}u'', \] (4.1)
with \( u'' \in \mathcal{U} \) and \( b' \) a quotient of \( u' \in \mathcal{U} \). We must show there is an \( \mathcal{M} \)-preenvelope \( f \rightarrow u \) with \( u \in \mathcal{U} \).

Pick a right exact sequence
\[ m'' \xrightarrow{\mu''} u' \xrightarrow{\mu'} b' \rightarrow 0 \]
in \( \text{mod}(\Lambda) \) with \( m'' \) projective and note that \( m'' \in \mathcal{M} \) since \( \mathcal{M} \) is \( n \)-cluster tilting, see [11, def. 1.1]. Set \( k' = \text{Ker} \mu' \). There is a short exact sequence \( 0 \rightarrow k' \rightarrow u' \xrightarrow{\mu'} b' \rightarrow 0 \) in \( \text{mod}(\Lambda) \) which induces a triangle \( k' \rightarrow u' \xrightarrow{\mu'} b' \rightarrow \Sigma k' \) in \( \mathcal{D}^b(\text{mod} \Lambda) \). Since \( u'' \) and \( k' \) are in \( \text{mod}(\Lambda) \) we have
\[ \text{Hom}_{\mathcal{D}^b(\text{mod} \Lambda)}(\Sigma^{-n}u'', \Sigma k') \cong \text{Ext}_{\Lambda}^{n+1}(u'', k') = 0 \]
because \( \text{gldim} \Lambda \leq n \), so \( \Sigma^{-n}u'' \xrightarrow{\psi''} b' \) factors as \( \Sigma^{-n}u'' \xrightarrow{\varphi''} u' \xrightarrow{\mu'} b' \).

We have constructed morphisms \( \mu'' \) and \( \varphi'' \) which can be used in the first triangle of Lemma 4.5. Since \( \mu' \varphi'' = \psi'' \), the second triangle of Lemma 4.5 can be taken to be (4.1) so \( f \cong \tau_{\geq 0}e \). By Property (E) there is an \( \mathcal{M} \)-preenvelope \( e \rightarrow u \) with \( u \in \mathcal{U} \), and by Lemma 4.4 there is hence also an \( \mathcal{M} \)-preenvelope \( f \cong \tau_{\geq 0}e \rightarrow u \) with \( u \in \mathcal{U} \) as desired.

“\( \Leftarrow \):” Consider the triangle from Definition 4.1. To show that \( \mathcal{U} \) has Property (E), it is enough by Lemma 4.4 to show that there is an \( \mathcal{M} \)-preenvelope \( \tau_{\geq 0}e \rightarrow u \) with \( u \in \mathcal{U} \). This holds by Lemma 4.5 and Property (F).

\[ \square \]

**Proposition 4.7.** Assume \( n = 1 \) and let \( \mathcal{U} \subseteq \mathcal{M} \) be a full subcategory closed under summands. Then \( \mathcal{U} \) has Property (E) if and only if it is closed under quotients and extensions.

**Proof.** Note that when \( n = 1 \) then \( \Lambda \) is a hereditary algebra of finite representation type and \( \mathcal{M} = \text{mod}(\Lambda) \).

By Lemma 4.6, we can replace Property (E) by Property (F) in the proposition. The triangle in Definition 4.2 has the form
\[ \Sigma^{-1}u'' \rightarrow b' \rightarrow f \rightarrow u'', \]
so \( f \) is an extension in \( \mathcal{D}^b(\text{mod} \Lambda) \) of the modules \( b' \) and \( u'' \), whence \( f \in \text{mod}(\Lambda) \). Since \( \mathcal{M} = \text{mod}(\Lambda) \), an \( \mathcal{M} \)-preenvelope \( f \rightarrow u \) must be split injective, so since \( \mathcal{U} \) is closed under summands, the preenvelope can be chosen with \( u \in \mathcal{U} \) if and only if \( f \in \mathcal{U} \).

It follows that \( \mathcal{U} \) has Property (F) if and only if \( f \in \mathcal{U} \) for each extension \( 0 \rightarrow b' \rightarrow f \rightarrow u'' \rightarrow 0 \) in \( \text{mod}(\Lambda) \) with \( u'' \in \mathcal{U} \) and \( b' \) a quotient in \( \text{mod}(\Lambda) \) of a module \( u' \in \mathcal{U} \). This proves the proposition.

\[ \square \]
5. TORSION CLASSES IN $n$-ABELIAN CATEGORIES ASSOCIATED TO $n$-REPRESENTATION
FINITE ALGEBRAS

Recall that throughout, $n \geq 1$ is a fixed integer and we are working under Setup 3.1. This section proves Theorem 0.4 from the introduction, see Theorem 5.5.

**Lemma 5.1.** Let

$$0 \to m^0 \to \cdots \to m^{n+1} \to 0$$

be an $n$-exact sequence in $\mathcal{M}$. Then, viewed in $\text{mod}(\Lambda)$, it is an exact sequence.

**Proof.** The projective generator $\Lambda$ of $\text{mod}(\Lambda)$ is in the $n$-cluster tilting subcategory $\mathcal{M}$, so Lemma 1.1(i) implies that $0 \to m^0 \to \cdots \to m^{n+1}$ is exact. A dual argument shows that $m^0 \to \cdots \to m^{n+1} \to 0$ is exact. □

**Lemma 5.2.** Assume $n \geq 2$ and let

$$v^1 \xrightarrow{\psi^1} v^2 \xrightarrow{\psi^2} \cdots \xrightarrow{\psi^{n-2}} v^{n-1} \xrightarrow{\psi^{n-1}} v^n$$

be an exact sequence in $\text{mod}(\Lambda)$ with $v^1, \ldots, v^{n-2} \in \mathcal{M}$. For each $m \in \mathcal{M}$ there is an isomorphism

$$\text{Ext}^{n-1}_\Lambda(m, \text{Ker } \psi^1) \cong \text{Ext}^1_\Lambda(m, \text{Ker } \psi^{n-1}).$$

**Proof.** Break the exact sequence into short exact sequences, write down the induced long exact sequences for $\text{Ext}^i_\Lambda(m, -)$, and use that $\text{Ext}^i_\Lambda(M, M) = 0$ for $i \in \{1, \ldots, n-1\}$ to get isomorphisms between all the $\text{Ext}^{n-1}_\Lambda(m, \text{Ker } \psi^i)$. □

**Lemma 5.3.** Let $\mathcal{U} \subseteq \mathcal{M}$ be a full subcategory and let

$$0 \to u \xrightarrow{\theta} m \to v^1 \to \cdots \to v^n \to 0$$

be an $n$-exact sequence in $\mathcal{M}$ with $u \in \mathcal{U}$ and $v^1, \ldots, v^n$ in $\mathcal{U}$-exact. Write $c = \text{Coker } \theta$, where the cokernel is computed in $\text{mod}(\Lambda)$. Then

(i) $\text{Hom}_\Lambda(\mathcal{U}, c) = 0$,

(ii) $\text{Ext}^{n-1}_\Lambda(\mathcal{U}, c) = 0$.

**Proof.** (i) This is proved similarly to Lemma 1.2(i).

(ii) For $n = 1$ this coincides with (i) so assume $n \geq 2$. The $n$-exact sequence in the lemma is exact in $\text{mod}(\Lambda)$ by Lemma 5.1. We name its morphisms as follows.

$$0 \to u \xrightarrow{\theta} m \to v^1 \xrightarrow{\psi^1} v^2 \to \cdots \to v^{n-1} \xrightarrow{\psi^{n-1}} v^n \to 0 \quad (5.1)$$

Let $\tilde{u} \in \mathcal{U}$ be given. The short exact sequence

$$0 \to \text{Ker } \psi^{n-1} \to v^{n-1} \xrightarrow{\psi^{n-1}} v^n \to 0$$

gives an exact sequence

$$\text{Hom}_\Lambda(\tilde{u}, v^{n-1}) \xrightarrow{\psi^{n-1}} \text{Hom}_\Lambda(\tilde{u}, v^n) \to \text{Ext}^1_\Lambda(\tilde{u}, \text{Ker } \psi^{n-1}) \to \text{Ext}^1_\Lambda(\tilde{u}, v^n).$$

Here $\psi^{n-1}$ is surjective because $v^1 \to \cdots \to v^n$ is in $\mathcal{U}$-exact, and $\text{Ext}^1_\Lambda(\tilde{u}, v^{n-1}) = 0$ because $\tilde{u}$ and $v^{n-1}$ are in $\mathcal{M}$, so $\text{Ext}^1_\Lambda(\tilde{u}, \text{Ker } \psi^{n-1}) = 0$. Combining with the isomorphism in Lemma 5.2 shows $\text{Ext}^{n-1}_\Lambda(\tilde{u}, \text{Ker } \psi^1) = 0$. Finally, $\text{Ker } \psi^1 \cong c$. □
Lemma 5.4. Let $\mathcal{U} \subseteq \mathcal{M}$ be a full subcategory with Property (E). If $u \xrightarrow{\theta} m$ is a $\mathcal{U}$-cover of $m \in \mathcal{M}$, then $\theta$ is a monomorphism in $\mathcal{M}$.

Proof. Lemma 4.6 says that $\mathcal{U}$ has Property (F). Let $u \xrightarrow{\theta} m$ be a $\mathcal{U}$-cover of $m \in \mathcal{M}$, and factorise it in $\text{mod}(\Lambda)$ into a surjection followed by an injection, $u \xrightarrow{s} b' \xrightarrow{i} m$. Property (F) with $u'' = 0$ says that $b'$ has an $\mathcal{M}$-preenvelope $b' \xrightarrow{\beta'} u'$ with $u' \in \mathcal{U}$. Factoring $i$ through $\beta'$ gives the following commutative diagram.

$$
\begin{array}{ccc}
u & \xrightarrow{s} & b' & \xrightarrow{i} & u' \\
\downarrow{\theta} & & \downarrow{i} & & \\
m & & m & & \\
\end{array}
$$

We can factorise $\psi$ through $\theta$, that is, $\psi = \theta \alpha$. But then $\theta \circ \alpha \beta's = \psi \beta's = is = \theta$ whence $\alpha \beta's$ is an isomorphism since $\theta$ is a cover. In particular, $\alpha \beta's$ is injective in $\text{mod}(\Lambda)$. It follows that so are $s$ and $\theta = is$, and then $\theta$ is a monomorphism in $\mathcal{M}$. \hfill \Box

Theorem 5.5. Let $\mathcal{U} \subseteq \mathcal{M}$ be a full subcategory closed under sums and summands. Then $\mathcal{U}$ is a torsion class in $\mathcal{M}$ $\iff$ $\mathcal{U}$ has Property (E).

Proof. $\Rightarrow$" Consider the triangle from Definition 4.1,

$$
\Sigma^{-n} u'' \oplus m'' \to u' \to e \to \Sigma^{-n+1} u'',
$$

with $u', u'' \in \mathcal{U}$ and $m'' \in \mathcal{M}$. By Lemma 3.3(i) there is an $\mathcal{M}$-preenvelope $e \xrightarrow{\varphi} m$.

Since $\mathcal{U}$ is a torsion class, there is an $n$-exact sequence in $\mathcal{M}$,

$$
0 \to u \xrightarrow{\theta} m \to v^1 \to \cdots \to v^n \to 0,
$$

with $u \in \mathcal{U}$ and $v^1 \to \cdots \to v^n$ in $\mathcal{U}$-exact. The sequence is exact in $\text{mod}(\Lambda)$ by Lemma 5.1. Write $c = \text{Coker} \theta$, where the cokernel is computed in $\text{mod}(\Lambda)$. There is a triangle

$$
\Sigma^{-1} c \to u \xrightarrow{\theta} m \to c
$$

in $\mathcal{D}(\text{mod} \Lambda)$. We can construct the following commutative diagram.

$$
\begin{array}{ccc}
\Sigma^{-n} u'' \oplus m'' & \to & u' \to e \to \Sigma^{-n+1} u'' \oplus \Sigma m'' \\
\downarrow{\Sigma^{-1} \sigma} & & \downarrow{\psi} & & \downarrow{\varphi} & & \downarrow{\sigma} \\
\Sigma^{-1} c & \to & u \to m \to c
\end{array}
$$

The morphism $\psi'$ exists because the composition $u' \to e \xrightarrow{\varphi} m \to c$ is zero since $\text{Hom}_\Lambda(u', c) = 0$ by Lemma 5.3(i). The morphism $\sigma$ is obtained by completing the diagram to a morphism of triangles.

However, $\sigma = 0$ because $\text{Hom}(\mathcal{D}(\text{mod} \Lambda), (\Sigma^{-n+1} u'', c) \cong \text{Ext}^n_{\Lambda}(u'', c) = 0$ by Lemma 5.3(ii) and $\text{Hom}(\mathcal{D}(\text{mod} \Lambda), (\Sigma m'', c) = 0$ since $m''$ and $c$ are in $\text{mod}(\Lambda)$. Hence $e \xrightarrow{\varphi'} u \xrightarrow{\theta} m$, and then $e \xrightarrow{\varphi'} u$ is an $\mathcal{M}$-preenvelope with $u \in \mathcal{U}$, showing that $\mathcal{U}$ has Property (E).
Let $m \in \mathcal{M}$ be given and let $u \xrightarrow{\theta} m$ be a $\mathcal{U}$-cover which exists by Lemma 3.3(i). By Lemma 5.4 the morphism $\theta$ is a monomorphism in $\mathcal{M}$, so by [15, def. 3.1] we can complete it to the following $n$-exact sequence in $\mathcal{M}$.

$$0 \to u \xrightarrow{\theta} m \xrightarrow{v^1} v^2 \to \cdots \to v^{n-1} \xrightarrow{\psi^{n-1}} v^n \to 0$$

We complete the proof by showing that $v^1 \to \cdots \to v^n$ is in $\mathcal{U}$-exact. Set $c = \text{Coker } \theta$ and let $u'' \in \mathcal{U}$ be given.

For $n \geq 2$, observe that by Lemma 1.1(i), if we apply the functor $\mathcal{M}(u'', -)$ to the $n$-exact sequence then we get an exact sequence except that the final zero may be missing. Since $\mathcal{M}(u'', \theta)$ is surjective, all that remains is to show $\mathcal{M}(u'', \psi^{n-1})$ surjective. The short exact sequence

$$0 \to \text{Ker } \psi^{n-1} \to v^{n-1} \xrightarrow{\psi^{n-1}} v^n \to 0$$

means that it is sufficient to show $\text{Ext}^1_{\Lambda}(u'', \text{Ker } \psi^{n-1}) = 0$, and combining Lemmas 5.1 and 5.2 shows that this is equivalent to $\text{Ext}^{n-1}_{\Lambda}(u'', \text{Ker } \psi^1) = 0$. Since $\text{Ker } \psi^1 \cong c$ this amounts to $\text{Hom}_{\mathcal{D}^b}(\Sigma^{-n+1}u'', c) = 0$.

For $n = 1$ what we need to show is $\text{Hom}_{\Lambda}(u'', v^1) = 0$. Since we have $v^1 \cong c$ in this special case, this again amounts to $\text{Hom}_{\mathcal{D}^b}(\Sigma^{-n+1}u'', c) = 0$.

So let $\Sigma^{-n+1}u'' \xrightarrow{\delta} c$ be given. There is a triangle $\Sigma^{-1}c \to u \xrightarrow{\theta} m \xrightarrow{\mu} c$ in $\mathcal{D}^b(\text{mod } \Lambda)$, and we can spin it into the following commutative diagram where the rows are triangles, the first one continued by an additional step.

Property (E) says that there is an $\mathcal{M}$-preenvelope $c \xrightarrow{\tilde{\varphi}} \tilde{u}$ with $\tilde{u} \in \mathcal{U}$, and we can augment the diagram with factorisations as follows where $\gamma$ exists because $\theta$ is a $\mathcal{U}$-cover.

Hence $\mu \varphi = \mu \beta \tilde{\varphi} = \mu \theta \gamma \tilde{\varphi} = 0 \circ \gamma \tilde{\varphi} = 0$ whence $\sigma \alpha = 0$. This implies that $\sigma$ factors as $\Sigma^{-n+1}u'' \xrightarrow{\delta} \Sigma u \to c$, but $\text{Hom}_{\mathcal{D}^b}(\Sigma u, c) = 0$ since $u$ and $c$ are in $\text{mod}(\Lambda)$. We conclude $\sigma = 0$ as desired. □
6. TORSION CLASSES IN \((n + 2)\)-ANGULATED CATEGORIES ASSOCIATED TO \(n\)-REPRESENTATION FINITE ALGEBRAS

Recall that throughout, \(n \geq 1\) is a fixed integer and we are working under Setup 3.1. The following is Theorem 0.3 from the introduction. The notion of being left closed under \(n\)-extensions is defined after Theorem 0.3.

**Theorem 6.1.** Let \(\mathcal{X} \subseteq \mathcal{C}\) be a full subcategory closed under sums and summands. Then \(\mathcal{X}\) is a torsion class in \(\mathcal{C}\) \(\iff\) \(\mathcal{X}\) is left closed under \(n\)-extensions.

**Proof.** “\(\Rightarrow\)”: If \(n = 1\) then “torsion class” has the usual meaning, see [14, def. 2.2], and “closed under 1-extensions” means “closed under extensions”, so the result holds by [14, lines after def. 2.2].

Assume \(n \geq 2\). Given a morphism \(x'' \xrightarrow{\delta} \Sigma^nx'\) with \(x',x'' \in \mathcal{X}\), we can complete to an \((n + 2)\)-angle in \(\mathcal{C}\),

\[
\begin{array}{ccccccccccc}
x' & \xrightarrow{\xi'} & c^1 & \xrightarrow{\gamma} & c^2 & \rightarrow & \cdots & \rightarrow & c^n & \xrightarrow{\delta} & x'' & \rightarrow & \Sigma^nx',
\end{array}
\]

where \(\gamma\) can be assumed to be in the radical of \(\mathcal{C}\) by [17, lem. 5.18 and its proof]. We will show \(c^1 \in \mathcal{X}\).

Since \(\mathcal{X}\) is a torsion class, there is an \((n + 2)\)-angle

\[
\begin{array}{ccc}
x & \xrightarrow{\xi} & c^1 \rightarrow d^2 \rightarrow \cdots \rightarrow d^n \xrightarrow{\psi} d^{n+1} \rightarrow \Sigma^nx
\end{array}
\]

in \(\mathcal{C}\) with \(x \in \mathcal{X}\) and \(d^2 \rightarrow \cdots \rightarrow d^{n+1}\) in \(\mathcal{X}\)-exact. By Lemma 1.2(i) the morphism \(\xi'\) factors through \(\xi\) so we can use axiom (F3) of [6, def. 2.1] to get the following commutative diagram.

\[
\begin{array}{cccccccccccc}
x' & \xrightarrow{\xi'} & c^1 & \xrightarrow{\gamma} & c^2 & \rightarrow & \cdots & \rightarrow & c^n & \xrightarrow{\delta} & x'' & \rightarrow & \Sigma^nx' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
x & \xrightarrow{\xi} & c^1 & \rightarrow & d^2 & \rightarrow & \cdots & \rightarrow & d^n & \xrightarrow{\psi} d^{n+1} & \rightarrow & \Sigma^nx
\end{array}
\]

Since \(d^2 \rightarrow \cdots \rightarrow d^{n+1}\) is in \(\mathcal{X}\)-exact, the morphism \(\xi''\) factors through \(\psi\). Combining this with Lemma 1.1(ii), we can construct the following chain homotopy.

\[
\begin{array}{cccccccccccc}
x' & \xrightarrow{\xi'} & c^1 & \xrightarrow{\sigma^1} & c^2 & \rightarrow & \cdots & \rightarrow & c^n & \xrightarrow{\sigma^{n+1}} & x'' & \rightarrow & \Sigma^nx' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
x & \xrightarrow{\xi} & c^1 & \xrightarrow{\sigma^2} & d^2 & \rightarrow & \cdots & \rightarrow & d^n & \xrightarrow{\psi} d^{n+1} & \rightarrow & \Sigma^nx
\end{array}
\]

In particular, \(\text{id}_{c^1} = \sigma^2\gamma + \xi\sigma^1\) whence \(\xi\sigma^1 = \text{id}_{c^1} - \sigma^2\gamma\). The right hand side is invertible because \(\gamma\) is in the radical of \(\mathcal{C}\), so \(\sigma^1\) is a split injection and \(c^1 \in \mathcal{X}\) follows.

“\(\Leftarrow\)”: In view of Lemma 3.3(ii), this holds by Corollary 2.2. \(\square\)

7. BIJECTION BETWEEN INTERMEDIATE LEFT T-STRUCTURES AND \(n\)-ABELIAN TORSION CLASSES

Recall that throughout, \(n \geq 1\) is a fixed integer and we are working under Setup 3.1. This section proves Theorem 0.5 from the introduction, see Theorem 7.5.
Definition 7.1. For \( \ell \in \mathbb{Z} \) set
\[
\mathcal{C}^{\leq \ell n} = \text{add}\{ \Sigma^{\infty} \mathcal{M} \mid i \geq -\ell \}.
\]
In other words, \( \mathcal{C}^{\leq \ell n} \) is the part of \( \mathcal{C} \) in cohomological degrees \( \leq \ell n \).

Remark 7.2. It is not hard to check directly that each \( \mathcal{C}^{\leq \ell n} \) is a left t-structure in \( \mathcal{C} \). This is also a consequence of Theorem 7.5 below.

Definition 7.3. If \( \mathcal{U} \subseteq \mathcal{M} \) is a full subcategory then set
\[
\mathcal{X}(\mathcal{U}) = \text{add}(\mathcal{U} \cup \mathcal{C}^{\leq -n})\]  

Remark 7.4. By construction, \( \mathcal{X}(\mathcal{U}) \) is “intermediate” in the sense \( \mathcal{C}^{\leq -n} \subseteq \mathcal{X}(\mathcal{U}) \subseteq \mathcal{C}^{\leq 0} \). Moreover, \( \Sigma^n \mathcal{X}(\mathcal{U}) \subseteq \mathcal{X}(\mathcal{U}) \), so \( \mathcal{X}(\mathcal{U}) \) has a shot at being what we can reasonably call an intermediate left t-structure.

Indeed, the following is the main result of this section.

Theorem 7.5. The assignment \( \mathcal{U} \mapsto \mathcal{X}(\mathcal{U}) \) defines a bijection
\[
\left\{ \mathcal{U} \subseteq \mathcal{M} \mid \mathcal{U} \text{ is a torsion class} \right\} \to \left\{ \mathcal{X} \subseteq \mathcal{C} \mid \mathcal{X} \text{ is a left t-structure with } \mathcal{C}^{\leq -n} \subseteq \mathcal{X} \subseteq \mathcal{C}^{\leq 0} \right\}.
\]

Proof. It is clear that \( \mathcal{U} \mapsto \mathcal{X}(\mathcal{U}) \) defines a bijection
\[
\left\{ \mathcal{U} \subseteq \mathcal{M} \mid \mathcal{U} \text{ is a full subcategory closed under sums and summands} \right\} \to \left\{ \mathcal{X} \subseteq \mathcal{C} \mid \mathcal{X} \text{ is a full subcategory closed under sums and summands with } \mathcal{C}^{\leq -n} \subseteq \mathcal{X} \subseteq \mathcal{C}^{\leq 0} \right\}.
\]

To prove the theorem, we must hence let \( \mathcal{U} \subseteq \mathcal{M} \) be a full subcategory closed under sums and summands and show that \( \mathcal{U} \) is a torsion class if and only if \( \mathcal{X}(\mathcal{U}) \) is a left t-structure. Since \( \Sigma^n \mathcal{X}(\mathcal{U}) \subseteq \mathcal{X}(\mathcal{U}) \) holds by construction, it is enough to show that \( \mathcal{U} \) is a torsion class if and only if \( \mathcal{X}(\mathcal{U}) \) is a torsion class. By Theorems 5.5 and 6.1 this amounts to
\( \mathcal{U} \) has Property (E) \( \Leftrightarrow \mathcal{X}(\mathcal{U}) \) is left closed under \( n \)-extensions,  

and we will show the two implications.

\( \Rightarrow \) in Equation (7.1): Let \( x'' \xrightarrow{\delta} \Sigma^n x' \) be a morphism with \( x', x'' \in \mathcal{X}(\mathcal{U}) \). We must show that there is an \((n+2)\)-angle in \( \mathcal{C} \),
\[
x' \to c^1 \to c^2 \to \cdots \to c^n \to x'' \xrightarrow{\delta} \Sigma^n x',
\]
with \( c^1 \in \mathcal{X}(\mathcal{U}) \). According to the construction in [6, proof of thm. 1], we can get \( c^1 \) by constructing a triangle
\[
\Sigma^{-n} x'' \xrightarrow{\Sigma^{-n} \delta} x' \to d \to \Sigma^{-n+1} x''
\]
in \( \mathcal{D}(\text{mod}\Lambda) \) and letting \( d \to c^1 \) be a \( \mathcal{C} \)-preenvelope. Note that because
\[
\mathcal{C} = \text{add}\{ \Sigma^{\infty} \mathcal{M} \mid i \in \mathbb{Z} \},
\]
we can let
\[
c^1 = \bigoplus_{i \in \mathbb{Z}} \Sigma^{\infty} m(i)
\]
where \( m(i) \in \mathcal{M} \) and \( d \to \Sigma^{\infty} m(i) \) is a \((\Sigma^{\infty} \mathcal{M})\)-preenvelope for each \( i \). To show that we can get \( c^1 \in \mathcal{X}(\mathcal{U}) \), it is then enough to show that we can let
(i) $m(i) = 0$ for $i \leq -1$,
(ii) $m(0) \in \mathcal{U}$.

Here (i) can be achieved because if $i \leq -1$ then $\text{Hom}_{\mathcal{D}b}(d, \Sigma^i \mathcal{M}) = 0$ for degree reasons. Specifically, the long exact cohomology sequence of the triangle (7.2) shows that the cohomology of $d$ is concentrated in degrees $\leq n - 1$, while the cohomology of an object of $\Sigma^i \mathcal{M}$ is concentrated in degree $-in \geq n$.

To achieve (ii) we must show that there is an $\mathcal{M}$-preenvelope $d \rightarrow m(0)$ with $m(0) \in \mathcal{U}$. By definition of $\mathcal{X}(\mathcal{U})$ we have $x' = u' \oplus c'$ and $x'' = u'' \oplus c''$ with $u', u'' \in \mathcal{U}$ and $c', c'' \in \mathcal{C}^{\leq -n}$. The octahedral axiom gives the following commutative diagram. The rows and columns are triangles minus the fourth object, the second vertical triangle is split, and the second horizontal triangle is (7.2).

Here $\text{Hom}_{\mathcal{D}b}(c', \mathcal{M}) = 0$ for degree reasons, so any morphism from $d$ to an object of $\mathcal{M}$ factors through $d'$, so it is enough to see that there is an $\mathcal{M}$-preenvelope $d' \rightarrow m(0)$ with $m(0) \in \mathcal{U}$.

Since $c' \in \mathcal{C}^{\leq -n}$ we have $\Sigma^{-n}c'' \in \mathcal{C}^{\leq 0}$ so we can write $\Sigma^{-n}c'' = m'' \oplus \tilde{c}$ with $m'' \in \mathcal{M}$ and $\tilde{c} \in \mathcal{C}^{\leq -n}$, whence $\Sigma^{-n}(u'' \oplus c'') = \Sigma^{-n}u'' \oplus m'' \oplus \tilde{c}$. We have $\text{Hom}_{\mathcal{D}b}(\tilde{c}, u') = 0$ for degree reasons, so the octahedral axiom gives the following commutative diagram. The rows and columns are triangles minus the fourth object, the first vertical triangle is split, and the second horizontal triangle is the third horizontal triangle from the previous diagram.

Here $\text{Hom}_{\mathcal{D}b}(\Sigma \tilde{c}, \mathcal{M}) = 0$ for degree reasons, so as above it is enough to see that there is an $\mathcal{M}$-preenvelope $d'' \rightarrow m(0)$ with $m(0) \in \mathcal{U}$. But this is true because $\mathcal{U}$ has Property $(E)$ and $d''$ appears in the third horizontal triangle.

“$\Leftarrow$” in Equation (7.1): Consider the triangle of Definition 4.1,

$$\Sigma^{-n}u'' \oplus m'' \xrightarrow{\psi''} u' \xrightarrow{\psi} e \xrightarrow{\psi} \Sigma^{-n+1}u'' \oplus \Sigma m''$$

with $u', u'' \in \mathcal{U}$ and $m'' \in \mathcal{M}$. We must show that there is an $\mathcal{M}$-preenvelope $e \rightarrow u$ with $u \in \mathcal{U}$.
The subcategory $\mathcal{X}(U)$ of $C$ is left closed under $n$-extensions, and since $u'$ and $u'' \oplus \Sigma^m u''$ are in $\mathcal{X}(U)$, there is an $(n+2)$-angle in $C$,

$$
u' \xrightarrow{\theta'} x^1 \xrightarrow{y^2} \cdots \xrightarrow{y^n} u'' \oplus \Sigma^m u'' \xrightarrow{\Sigma^n u''} \Sigma^n u',$$

with $x^1 \in \mathcal{X}(U)$. By definition of $\mathcal{X}(U)$ we have $x^1 = u \oplus c$ with $u \in U$ and $c \in C_{\leq -n}$. Let $x^1 \xrightarrow{\xi} u$ be the projection onto the summand. It follows from Lemma 1.1(ii) that $\theta' \circ \psi'' = 0$, so we get the following commutative diagram.

$$
\begin{array}{ccc}
\Sigma^{-n} u'' \oplus m'' & \xrightarrow{\psi''} & u' \\
\downarrow \downarrow & & \downarrow \phi \\
\Sigma^{-n} u'' \oplus m'' & \xrightarrow{\psi''} & x^1 \\
\end{array}
$$

We will show that $e \xrightarrow{\xi \phi} u$ is an $M$-preenvelope.

Let $e \xrightarrow{\xi} m$ be a morphism with $m \in M$. We must show that it factors through $e \xrightarrow{\xi \phi} u$. Since $\chi \psi' \circ \psi'' = \chi \circ \psi' \psi'' = \chi \circ 0 = 0$, the dual of Lemma 1.1(ii) implies that there is $x^1 \xrightarrow{\alpha} m$ with $\alpha \theta' = \chi \psi'$. Hence $(\chi - \alpha \varphi) \psi' = \chi \psi' - \alpha \theta' = 0$ so there is $\Sigma^{-n+1} u'' \oplus \Sigma m'' \xrightarrow{\beta} m$ with $\beta \psi = \chi - \alpha \varphi$. However, this implies

$$
\chi = \alpha \varphi
$$

because $\beta = 0$ since $\text{Hom}_{\text{gr}}(\Sigma^{-n+1} u'' \oplus \Sigma m'', m) = 0$. Namely,

$$
\text{Hom}_{\text{gr}}(\Sigma^{-n+1} u'', m) \cong \text{Ext}_{\text{A}}^{-n-1}(u'', m) = 0
$$

because $u''$ and $m$ are in the $n$-cluster tilting subcategory $M$, and

$$
\text{Hom}_{\text{gr}}(\Sigma m'', m) = 0
$$

because $m''$ and $m$ are in $M$, hence in $\text{mod}(A)$.

Now consider $x^1 \xrightarrow{\alpha} m$. Here $x^1 = u \oplus c$ with $c \in C_{\leq -n}$, and $\text{Hom}_{\text{gr}}(c, m) = 0$ for degree reasons, so $\alpha$ factors through the projection $x^1 \xrightarrow{\xi} u$. That is,

$$
\alpha = \gamma \xi,
$$

and combining with Equation (7.3) shows $\chi = \gamma \circ \xi \phi$. So $\chi$ factors through $\xi \phi$ as desired. \(\square\)

8. Example: Splitting torsion classes arising from $n$-APR tilting modules

Recall that throughout, $n \geq 1$ is a fixed integer and we are working under Setup 3.1.

In classic tilting theory, if $t$ is a tilting module then $\text{Fac}(t)$, the set of all quotients of modules of the form $t \oplus d$, is a torsion class. If $t$ is a so-called APR tilting module, see [2, thm. 1.11] or [1, exa. VI.2.8(c)], then $\text{Fac}(t)$ is splitting in the sense that for each $m \in \text{mod}(A)$, the short exact sequence $0 \to u \to m \to v \to 0$ with $u \in \text{Fac}(t), v \in \text{Fac}(t)^\perp$ is split. We will show an $n$-homological version of this.
**Setup 8.1.** Let \( p \) be a simple projective, non-injective left \( \Lambda \)-module and write \( \Lambda = p \oplus q \) as left \( \Lambda \)-modules. The corresponding \( n \)-APR tilting module, introduced in [13, def. 3.1 and obs. 4.1(1)], is

\[ t = (\tau_n p) \oplus q \]

where \( \tau_n = \text{Tr} \Omega^{n-1} D \) is the composition of the transpose, \( \text{Tr} \), the \((n-1)\)st syzygy in a minimal projective resolution, \( \Omega^{n-1} \), and \( k \)-linear duality, \( D \). See [1, sec. IV.2] for the definition of the transpose.

**Remark 8.2.** The module \( t \) is a tilting module for \( \Lambda \) in the usual sense by [13, thm. 3.2], and \( t \in \mathcal{M} \) by [13, thm. 4.2(1)].

For the following theorem, recall that the notion of a torsion class being splitting was defined after Theorem 0.2.

**Theorem 8.3.** The full subcategory \( \mathcal{M} \cap \text{Fac}(t) \) is a splitting torsion class in \( \mathcal{M} \).

**Proof.** We have \( \mathcal{M} = \text{add}(M) \) for a basic \( n \)-cluster tilting module \( M \in \text{mod}(\Lambda) \). The indecomposable projective module \( p \) is a direct summand of \( M \), see e.g. [13, def. 2.1], so we can write \( M = M' \oplus p \) where \( p \) is not a direct summand of \( M' \). We first show

\[ \mathcal{M} \cap \text{Fac}(t) = \text{add}(M'). \]

The inclusion \( \supseteq \) follows from [13, lem. 4.4] and the second bullet in [13, lem. 3.5], and the inclusion \( \subseteq \) follows from [13, prop. 3.3(2)] which implies

\[ \text{Hom}_\Lambda(t, p) = 0, \]

whence the indecomposable \( p \) is not in \( \mathcal{M} \cap \text{Fac}(t) \).

To complete the proof, we show that \( \text{add}(M') \) is a splitting torsion class. For \( m \in \mathcal{M} \) there is a split short exact sequence \( 0 \to u \to m \to v \to 0 \) with \( u \in \text{add}(M') \) and \( v \in \text{add}(p) \). It gives an \( n \)-exact sequence

\[ 0 \to u \xrightarrow{\theta} m \to v \to 0 \to \cdots \to 0 \]

in \( \mathcal{M} \) with \( \theta \) a split monomorphism. Here \( v \to 0 \to \cdots \to 0 \) is in \( \text{add}(M') \)-exact because \( \text{Hom}_\Lambda(M', p) = 0 \); this follows from the first two displayed equations of the proof.

9. **Example: A non-splitting torsion class**

Recall that throughout, \( n \geq 1 \) is a fixed integer and we are working under Setup 3.1. We keep Setup 8.1 in place.

**Theorem 9.1.** The full subcategory \( \mathcal{U} = \text{add}(p) \) is a non-splitting torsion class in \( \mathcal{M} \).

**Proof.** By Theorem 5.5 and Lemma 4.6, it is enough to show that \( \mathcal{U} \) has Property \((F)\).

Since \( p \) is indecomposable, each module \( u \in \mathcal{U} \) has the form \( u = p^{\oplus i} \). Since \( p \) is simple, each quotient of \( u \) in \( \text{mod}(\Lambda) \) has the form \( p^{\oplus j} \). Hence, in the triangle of Definition 4.2,

\[ \Sigma^{-n} u'' \xrightarrow{\varphi} b' \to f \to \Sigma^{-n+1} u'', \]

we have \( u'' = p^{\oplus i} \) and \( b' = p^{\oplus j} \). Since \( p \) is projective and \( n \geq 1 \) we have \( \varphi = 0 \) so

\[ f = b' \oplus \Sigma^{-n+1} u'' = p^{\oplus j} \oplus \Sigma^{-n+1} p^{\oplus i}. \]
If \( n = 1 \) then \( f = p^{j+i} \). This is an object of \( \mathcal{U} \), so \( f \) has an \( \mathcal{M} \)-preenvelope in \( \mathcal{U} \), namely \( f \) itself. If \( n \geq 2 \) then \( \text{Hom}_{\mathcal{D}^b}(\Sigma^{-n+1}p^{j+i}, \mathcal{M}) = 0 \) for degree reasons, so to get an \( \mathcal{M} \)-preenvelope of \( f \) we can take one of \( p^{j+i} \). But \( p^{j+i} \) is an object of \( \mathcal{U} \), so \( f \) has an \( \mathcal{M} \)-preenvelope in \( \mathcal{U} \), namely \( p^{j+i} \). This verifies Property (\( F \)).

Finally, to show that \( \mathcal{U} \) is non-splitting, let \( E(p) \) be the injective envelope of \( p \) in \( \text{mod}(\Lambda) \). Then \( E(p) \in \mathcal{M} \), see e.g. [11, def. 1.1], so there is an \( n \)-exact sequence

\[
0 \to u \to E(p) \to v^1 \to \cdots \to v^n \to 0
\]

as in Definition 0.1. By Lemma 1.2(i) the morphism \( \theta \) is a \( \mathcal{U} \)-cover so must be non-zero since the object \( p \in \mathcal{U} \) has a non-zero morphism \( p \to E(p) \). If \( \mathcal{U} \) were splitting then we could assume that \( \theta \) was split whence \( u \) would be injective. Since \( u = p^{j+i} \), this would show \( p \) injective contradicting Setup 8.1. \( \square \)

References


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