HOMOLOGICAL IDENTITIES FOR DIFFERENTIAL GRADED ALGEBRAS

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0. Introduction

Our original motivation for this paper was to answer [4, question (3.10)] on gaps in the sequence of Bass numbers of a Differential Graded Algebra (DGA). We do so in paragraph (3.1).

[4, question (3.10)] asks for a sort of No Holes Theorem for Bass numbers of DGAs. More precisely, it asks for a certain bound on the length of gaps in the sequence of Bass numbers; namely, that if one has $\mu^t \neq 0$ and $\mu^{t+1} = \cdots = \mu^{t+g} = 0$ and $\mu^{t+g+1} \neq 0$, then $g$ is at most equal to the degree of the highest non-vanishing homology of the DGA. This is the best possible bound one can hope for, as shown in [4, exam. (3.9)].

We provide this bound in paragraph (3.1) and thereby answer the question. Our method works for several important classes of DGAs, among them DG fibres of ring homomorphisms, Koszul complexes, and singular chain DGAs of the form $C_*(G; k)$ where $k$ is a field and $G$ a path connected topological monoid with $\dim_k H_k(G; k) < \infty$ (see remark (2.2)). Paragraph (3.1) arises as corollary to a more general Gap Theorem, theorem (2.5), which is the natural generalization to the world of DGAs of the classical No Holes Theorem from homological ring theory (see [10], [12], [18], [23], and [25]).

Since the classical No Holes Theorem lives in the world of so-called homological identities, such as the Auslander-Buchsbaum and Bass Formulae (see [2], [7], [18], [19], [23], and [26]), it seemed natural also to generalize these to DGAs. We do so in theorems (2.3) and (2.4).

We prove theorems (2.3), (2.4), and (2.5) by means of dualizing DG modules (DG module being our abbreviation of Differential Graded module). These are the natural generalization of dualizing complexes from homological ring theory, and were made available in [13] and [14]. As any reader of the ring theoretic literature will know, dualizing complexes can be used to give nice proofs of homological identities; it is hence not
surprising that dualizing DG modules enable us to prove homological identities for DGAs.

Indeed, this is a very simple paper. Our proofs are close in spirit to homological ring theory (see [11], [18], [19], [23], [25], and [26]), and use dualizing DG modules much as ring theory uses dualizing complexes. If anything, our proofs are slightly simpler than the ones from ring theory because they have the benefit of so-called semi-free resolutions, the high tech device from DGA theory which replaces free resolutions.

The classical Auslander-Buchsbaum and Bass Formulae and No Holes Theorem for noetherian local commutative rings are special cases of our results (see paragraph (3.3)). Also, evaluating the Auslander-Buchsbaum Formula for $C_*(G; k)$ proves additivity of homological dimension on $G$-Serre-fibrations for a suitable topological monoid $G$ (see paragraph (3.2)). Hence our Auslander-Buchsbaum Formula is a simultaneous generalization of classical Auslander-Buchsbaum from commutative ring theory and additivity of homological dimension from algebraic topology.

Before ending the introduction, let us make two remarks. First, we will develop our results under some technical conditions (see setups (0.6) and (2.1)). At least some of these are necessary: In paragraph (4.2) we show that the Auslander-Buchsbaum and Bass Formulae can fail for more general DGAs such as $C_*(\Omega X; \mathbb{Q})$. Secondly, while the entire paper deals with chain DGAs, that is, DGAs concentrated in non-negative homological degrees, it is also possible to develop a theory of homological identities for certain cochain DGAs, that is, DGAs concentrated in non-negative cohomological degrees. We do so in the forthcoming [16].

The paper is organized thus: This section ends with a few blanket items. Section 1 introduces some homological invariants for DG modules and proves some elementary properties. Section 2 proves our main results. Section 3 considers some examples. Section 4 shows that the Auslander-Buchsbaum and Bass Formulae can fail for $C_*(\Omega X; \mathbb{Q})$.

(0.1) Some notation. Most of our notation for DGAs and DG modules is standard, in particular concerning derived categories and functors and the various resolutions used to compute them. See [15, sec. 1] for a summary of notation, or see [9, chaps. 3 and 6] or [20]. There are a few items we want to mention explicitly:

We use homological notation in the whole paper, that is, lower indices and differentials of degree $-1$. There is only one exception, in paragraph (4.2). By “degree” we mean homological degree. We visualize DGAs and DG modules with components of high degree at the left end and with differentials pointing to the right. The terms “bounded to the left” and “bounded to the right” are used accordingly.

Let $R$ be a DGA. By $\mathcal{D}(R)$ we denote the derived category of left DG $R$-modules.
By $R^{opp}$ we denote the opposite DGA of $R$, whose product is defined as $s \cdot r = (-1)^{|r||s|}rs$ for graded elements $r$ and $s$. The idea of $R^{opp}$ is that we can identify right DG $R$-modules with left DG $R^{opp}$-modules. So for instance, we will identify $\mathcal{D}(R^{opp})$ with the derived category of right DG $R$-modules. This approach enables us to state many of our definitions and results for left DG $R$-modules only; applying them to left DG $R^{opp}$-modules then takes care of right DG $R$-modules.

Let $M$ be a DG $R$-module. The amplitude of $M$ is defined by

$$\text{amp } M = \sup \{ i \mid H_i(M) \neq 0 \} - \inf \{ i \mid H_i(M) \neq 0 \}.$$  

We operate with the convention $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.

Finally, $R^d$ denotes the graded algebra obtained by forgetting the differential of $R$, and $M^d$ denotes the graded $R^d$-module obtained by forgetting the differential of $M$.

(0.2) **The category fin.** Let $R$ be a DGA for which $H_0(R)$ is a left noetherian ring. Then $\text{fin}(R)$ denotes the full triangulated subcategory of $\mathcal{D}(R)$ which consists of $M$’s so that the homology $H(M)$ is bounded, and so that each $H_i(M)$ is finitely generated as a left $H_0(R)$-module.

(0.3) **Dagger Duality.** In [13] and [14] the theory of dualizing DG modules and the duality they define (“dagger duality”, a term introduced by Foxby) is developed. Here is a brief summary:

Let $R$ be a DGA for which $H_0(R)$ is a noetherian ring, and suppose that $R$ has the dualizing DG module $D$ (see [13] or [14] for the technical definition). For any left DG $R$-module $M$ and any right DG $R$-module $N$ we have the dagger duals

$$M^\dagger = R\text{Hom}_R(M, D) \quad \text{and} \quad N^\dagger = R\text{Hom}_{R^{opp}}(N, D).$$

Strictly speaking, these should be called the dagger duals with respect to $D$, but we always have only a single $D$ around, so there is no risk of confusion.

Dagger duality is now the pair of quasi-inverse contravariant equivalences of categories between $\text{fin}(R)$ and $\text{fin}(R^{opp})$,

$$\xymatrix{ \text{fin}(R) \ar[r]^{(-)\dagger} & \text{fin}(R^{opp}) \ar[l]_{(-)^\dagger} }.$$

Note the slight abuse of notation in that $(-)\dagger$ denotes two different functors.

An alternative way of expressing the duality is to say that

the biduality morphism $M \rightarrow M^{\dagger\dagger}$ is an isomorphism for any $M$ in $\text{fin}(R)$ or $\text{fin}(R^{opp})$.  \hfill (0.3.1)
For $M, N$ in $\text{fin}(R)$ we even have

$$\operatorname{RHom}_{R^{\text{op}}}(N^!, M^!) = \operatorname{RHom}_{R^{\text{op}}}(\operatorname{RHom}_R(N, D), \operatorname{RHom}_R(M, D))$$

\(\approx \operatorname{RHom}_R(M, \operatorname{RHom}_{R^{\text{op}}}(\operatorname{RHom}_R(N, D), D))

\approx \operatorname{RHom}_R(M, N^!)

\approx \operatorname{RHom}_R(M, N), \quad (0.3.2)

where (a) is by the so-called swap isomorphism.

(0.4) **Truncations.** Let $R$ be a chain DGA (that is, $R_i = 0$ for $i < 0$). It is now possible to truncate DG $R$-modules as follows:

First, suppose that $M$ is a DG $R$-module for which $\text{H}(M)$ is bounded to the right, that is, $\text{H}_i(M) = 0$ for $i \ll 0$. Write $v = \inf\{ i \mid \text{H}_i(M) \neq 0 \}$. Then we have the truncation

$$S = \cdots \rightarrow M_{v+2} \rightarrow M_{v+1} \rightarrow \text{Ker} \partial^M_v \rightarrow 0 \rightarrow \cdots,$$

where $\partial^M_v$ denotes the $v$th component of the differential of $M$. This is a DG $R$-submodule of $M$ which is quasi-isomorphic to $M$, and hence isomorphic to $M$ in the derived category of DG $R$-modules. Note that for this to work, it is essential that we have $R_i = 0$ for $i < 0$.

Secondly, suppose that $N$ is a DG $R$-module for which $\text{H}(N)$ is bounded to the left, that is, $\text{H}_i(N) = 0$ for $i \gg 0$. Write $w = \sup\{ i \mid \text{H}_i(N) \neq 0 \}$. Then we have the truncation

$$T = \cdots \rightarrow 0 \rightarrow \text{Im} \partial^{N}_{w+1} \rightarrow N_{w-1} \rightarrow N_{w-2} \rightarrow \cdots.$$

This is a DG $R$-quotient module of $N$ which is quasi-isomorphic to $N$, and hence isomorphic to $N$ in the derived category of DG $R$-modules. Again, for this to work, it is essential that we have $R_i = 0$ for $i < 0$.

Finally, when $\text{H}(R)$ is bounded, the second truncation method described above applies to $R$ and gives a quotient DGA of $R$. Let us denote the quotient morphism by $R \rightarrow T$; since it is a quasi-isomorphism, it induces an equivalence between the derived categories of $R$ and $T$ (see [21, thm. III.4.2]).

(0.5) **Semi-free resolutions.** Let $R$ be a chain DGA (that is, $R_i = 0$ for $i < 0$) for which $\text{H}_0(R)$ is a noetherian ring which is local in the following sense: It has a unique maximal two sided ideal $J$, and $\text{H}_0(R)/J$ is a skew field. We denote the skew field $\text{H}_0(R)/J$ by $k$, and note that $k$ can be viewed as a left-right DG $R$-module concentrated in degree zero.

Let $M$ be a left DG $R$-module with $\text{H}(M)$ bounded to the right and each $\text{H}_i(M)$ finitely generated as an $\text{H}_0(R)$-module. There is now a minimal semi-free resolution $F \rightarrow M$ with

$$F^i \approx \prod_{\beta \leq j} \Sigma^j(R^i)^{(\beta)},$$
where \( v = \inf \{ i \mid H_i(M) \neq 0 \} \) and where each \( \beta_j \) is finite. Here \( \Sigma^j \) denotes the \( j \)'th suspension. In other words, \( F^3 \) is a graded free left \( R^2 \)-module. Minimality of \( F \) means that the differential \( \partial^F \) maps into \( \mathfrak{m}F \), where \( \mathfrak{m} \) is the DG ideal
\[
\cdots \longrightarrow R_2 \longrightarrow R_1 \longrightarrow J \longrightarrow 0 \longrightarrow \cdots.
\]
As consequence, \( \text{Hom}_R(F, k) \) and \( k \otimes_R F \) have vanishing differentials (see [1, prop. 2] and [8, lem. (A.3)(iii)]).

(0.6) **Blanket Setup.** For the rest of this paper, \( R \) denotes a DGA satisfying:

- \( R \) is a chain DGA (that is, \( R_i = 0 \) for \( i < 0 \)).
- \( H_0(R) \) is a noetherian ring which is local in the following sense: It has a unique maximal two sided ideal \( J \), and \( H_0(R)/J \) is a skew field.
- \( R^R \in \text{fin}(R) \) and \( R_R \in \text{fin}(R^{opp}) \).

We denote the skew field \( H_0(R)/J \) by \( k \).

1. INVARIANTS

(1.1) **Definition.** For a left DG \( R \)-module \( M \), we define the \( k \)-projective dimension, the \( k \)-injective dimension, and the depth as
\[
\begin{align*}
k.p.d_R M &= -\inf \{ i \mid H_i(\text{RHom}_R(M, k)) \neq 0 \}, \\
k.i.d_R M &= -\inf \{ i \mid H_i(\text{RHom}_R(k, M)) \neq 0 \}, \\
\text{depth}_R M &= -\sup \{ i \mid H_i(\text{RHom}_R(k, M)) \neq 0 \}.
\end{align*}
\]

(1.2) **Remark.** By the existence of minimal semi-free resolutions (see paragraph (0.5)), it is easy to prove for \( M \) in \( \text{fin}(R) \) that
\[
k.p.d_R M = \sup \{ i \mid H_i(k \otimes_R M) \neq 0 \}.
\]

(1.3) **Definition.** For a left DG \( R \)-module \( M \), we define the \( j \)'th Bass number and the \( j \)'th Betti number as
\[
\begin{align*}
\mu_j^R(M) &= \dim_k H_{-j}(\text{RHom}_R(k, M)), \\
\beta_j^R(M) &= \dim_{R^{opp}} H_{-j}(\text{RHom}_R(M, k)).
\end{align*}
\]
(Note that \( \mu_j^R(M) \) and \( \beta_j^R(M) \) may well equal \( +\infty \).)

(1.4) **Remark.** Let \( M \) be a left DG \( R \)-module. It is clear from the definitions that
\[
\begin{align*}
k.p.d_R M &= \sup \{ j \mid \beta_j^R(M) \neq 0 \}, \\
k.i.d_R M &= \sup \{ j \mid \mu_j^R(M) \neq 0 \}, \\
\text{depth}_R M &= \inf \{ j \mid \mu_j^R(M) \neq 0 \}.
\end{align*}
\]
(1.5) **Proposition.** Let $M$ and $N$ be left DG $R$-modules with $H(M)$ bounded to the right and $H(N)$ bounded to the left, and each $H_i(M)$ finitely generated as an $H_0(R)$-module. Then

$$\sup\{ i \mid H_i(R\text{Hom}_R(M, N)) \neq 0 \}$$

$$\leq -\inf\{ i \mid H_i(M) \neq 0 \} + \sup\{ i \mid H_i(N) \neq 0 \}.$$ 

**Proof.** Paragraph (0.5) gives that $M$ admits a semi-free resolution $F \rightarrow M$ with

$$F^i \cong \prod_{v \leq j} \Sigma^j(R^i)^{(\beta_j)},$$

where $v = \inf\{ i \mid H_i(M) \neq 0 \}$ and where each $\beta_j$ is finite. Let $T$ be a truncation of $N$ which is quasi-isomorphic to $N$ and concentrated in degrees smaller than or equal to $\sup\{ i \mid H_i(N) \neq 0 \}$ (see paragraph (0.4)). We then have

$$R\text{Hom}_R(M, N) \cong \text{Hom}_R(F, T).$$

But

$$\text{Hom}_R(F, T)^i = \text{Hom}_R(F^i, T^i)$$

$$\cong \text{Hom}_R\left( \prod_{v \leq j} \Sigma^j(R^i)^{(\beta_j)}, T^i \right)$$

$$\cong \prod_{v \leq j} \Sigma^{-j}(T^i)^{(\beta_j)}$$

is concentrated in degrees smaller than or equal to

$$-v + \sup\{ i \mid H_i(N) \neq 0 \} = -\inf\{ i \mid H_i(M) \neq 0 \} + \sup\{ i \mid H_i(N) \neq 0 \},$$

proving the result. \qed

(1.6) **Lemma.** Let $F$ be a $K$-projective left DG $R$-module with

$$F^i \cong \prod_{j \leq p} \Sigma^j(R^i)^{(\beta_j)},$$

and let $N$ be a left DG $R$-module with $H(N)$ bounded to the right. Then

$$\inf\{ i \mid H_i(\text{Hom}_R(F, N)) \neq 0 \} \geq -p + \inf\{ i \mid H_i(N) \neq 0 \}.$$ 

**Proof.** This is just like the proof of proposition (1.5): Let $S$ be a truncation of $N$ which is quasi-isomorphic to $N$ and concentrated in degrees larger than or equal to $\inf\{ i \mid H_i(N) \neq 0 \}$ (see paragraph (0.4)). We then have a quasi-isomorphism

$$\text{Hom}_R(F, N) \cong \text{Hom}_R(F, S).$$
But
\[
\operatorname{Hom}_R(F, S)^i \cong \operatorname{Hom}_R(F^i, S^i) \\
\cong \operatorname{Hom}_R(\prod_{j \leq p} \Sigma^j(R^j)^{(\beta_j)}, S^i) \\
\cong \prod_{j \leq p} \Sigma^{-j}(S^i)^{(\beta_j)}
\]
is concentrated in degrees larger than or equal to
\[-p + \inf\{ i \mid H_i(N) \neq 0 \},
\]
proving the result. \(\square\)

(1.7) **Lemma.** Let \(M\) be a left DG \(R\)-module and suppose that \(F \xrightarrow{\approx} M\) is a minimal \(K\)-projective resolution with
\[
F^i \cong \prod_{v \leq j} \Sigma^j(R^j)^{(\beta_j)},
\]
where each \(\beta_j\) is finite. Then
\[
\beta_j^R(M) = \begin{cases} 
0 & \text{for } j < v, \\
\beta_j & \text{for } j \geq v,
\end{cases}
\]
and we have
\[
k.pd_R \ M = \sup\{ j \mid \beta_j^R(M) \neq 0 \} = \sup\{ j \mid \beta_j \neq 0 \} \quad (1)
\]
and
\[
\inf\{ i \mid H_i(M) \neq 0 \} = \inf\{ j \mid \beta_j^R(M) \neq 0 \} = \inf\{ j \mid \beta_j \neq 0 \}. \quad (2)
\]

**Proof.** To see (1), note that we have
\[
H(\operatorname{RHom}_R(M, k)) \cong H(\operatorname{Hom}_R(F, k)) \\
\cong \operatorname{Hom}_R(F, k)^i \\
\cong \operatorname{Hom}_R(F^i, k^i) \\
= (\ast),
\]
where (a) is because \(F\) is minimal whence \(\operatorname{Hom}_R(F, k)\) has zero differential. But
\[
(\ast) = \operatorname{Hom}_R(\prod_{v \leq \ell} \Sigma^\ell(R^i)^{(\beta_i)}, k^i) \cong \prod_{v \leq \ell} \Sigma^{-\ell}(k^i)^{(\beta_i)},
\]
so
\[
H_{-j}(\operatorname{RHom}_R(M, k)) \cong \begin{cases} 
0 & \text{for } j < v, \\
k^{(\beta_j)} & \text{for } j \geq v,
\end{cases}
\]
and (1) follows.

In (2), the first = is known from remark (1.4), and the second = is clear from (1).
As for (3), the second equality is again clear from (1). We will therefore be done if we can prove \( \inf \{ i \mid H_i(M) \neq 0 \} = \inf \{ j \mid \beta_j \neq 0 \} \), and this is equivalent to

\[
\inf \{ i \mid H_i(F) \neq 0 \} = \inf \{ j \mid \beta_j \neq 0 \}.
\]

(b)

So let \( u = \inf \{ j \mid \beta_j \neq 0 \} \). Then we have

\[
F^i \cong \prod_{u \leq j} \Sigma^j(R^l)^{(\beta_j)}, \quad \beta_u \neq 0
\]

which easily implies (b) because \( F \) is minimal.

\( \square \)

(1.8) Proposition. Let \( M \) and \( N \) be left DG \( R \)-modules with \( H(M) \) and \( H(N) \) bounded to the right, and each \( H_i(M) \) and each \( H_i(N) \) finitely generated as an \( H_0(R) \)-module. Suppose that \( k \text{pd}_R M \) is finite. Then

\[
\inf \{ i \mid H_i(R \text{Hom}_R(M, N)) \neq 0 \} = -k \text{pd}_R M + \inf \{ i \mid H_i(N) \neq 0 \}.
\]

Proof. If \( N \cong 0 \) then both sides of the equation are \(+\infty\), so we can assume \( N \neq 0 \).

First, we let \( F \xrightarrow{\sim} M \) be a semi-free resolution. By paragraph (0.5) we can pick \( F \) minimal with \( F^i \cong \prod_{v \leq j} \Sigma^j(R^l)^{(\beta_j)} \) and all \( \beta_j \) finite, and by lemma (1.7)(2) we then even have

\[
F^i \cong \prod_{v \leq j \leq p} \Sigma^j(R^l)^{(\beta_j)}
\]

with \( p = k \text{pd}_R M \) and \( \beta_p \neq 0 \).

Secondly, we write \( u = \inf \{ i \mid H_i(N) \neq 0 \} \) and let \( S \) be a truncation of \( N \) which is quasi-isomorphic to \( N \) and concentrated in degrees larger than or equal to \( u \) (see paragraph (0.4)).

We now have \( R \text{Hom}_R(M, N) \cong \text{Hom}_R(F, S) \), and the proposition’s equation amounts to

\[
\inf \{ i \mid H_i(\text{Hom}_R(F, S)) \neq 0 \} = -p + u.
\]

Now, lemma (1.6) gives

\[
\inf \{ i \mid H_i(\text{Hom}_R(F, S)) \neq 0 \} \geq -p + u,
\]

so we will be done when we have proved

\[
H_{-p+u}(\text{Hom}_R(F, S)) \neq 0.
\]

(a)

As \( S_u \) is the right-most non-zero component of \( S \), there is a surjection of left \( R \)-modules \( S_u \twoheadrightarrow H_u(S) \). Moreover, since \( H_u(S) \cong H_u(N) \) is finitely generated as a left \( H_0(R) \)-module, Nakayama’s lemma gives that there is a surjection of left \( H_0(R) \)-modules \( H_u(S) \twoheadrightarrow k \). Altogether, there is a surjection of left \( R \)-modules,

\[
S_u \twoheadrightarrow k.
\]
It is clear how this gives rise to a surjection of left DG $R$-modules $S \rightarrow \Sigma^u k$, and denoting the kernel by $S'$, there is a short exact sequence of left DG $R$-modules,
\[ 0 \rightarrow S' \rightarrow S \rightarrow \Sigma^u k \rightarrow 0. \] (b)

Note that
\[ \inf \{ i \mid H_i(\text{Hom}_R(F, S')) \neq 0 \} \geq -p + \inf \{ i \mid H_i(S') \neq 0 \} \geq -p + u, \] (c)

where the first $\geq$ is by lemma (1.6), and the second $\geq$ is because $S'$ is a DG submodule of $S$, hence concentrated in degrees larger than or equal to $u$.

As $F$ is semi-free, acting with the functor $\text{Hom}_R(F, -)$ on (b) gives a new short exact sequence
\[ 0 \rightarrow \text{Hom}_R(F, S') \rightarrow \text{Hom}_R(F, S) \rightarrow \text{Hom}_R(F, \Sigma^u k) \rightarrow 0 \]
whose homology long exact sequence contains
\[ H_{-p+u}(\text{Hom}_R(F, S)) \rightarrow H_{-p+u}(\text{Hom}_R(F, \Sigma^u k)) \rightarrow H_{-p+u-1}(\text{Hom}_R(F, S')). \]

The last term is zero because of (c), so if we can prove that the middle term is non-zero then it will follow that the first term is non-zero, proving (a) as required. But by minimality of $F$, we have the first $\cong$ in
\[ H(\text{Hom}_R(F, \Sigma^u k)) \cong \text{Hom}_R(F, \Sigma^u k)^{1} \cong \text{Hom}_R(F^i, (\Sigma^u k)^{1}) \cong \text{Hom}_R(\prod_{v \leq j \leq p} \Sigma^j(R^i)^{\otimes j}, (\Sigma^u k)^{1}) \cong \prod_{v \leq j \leq p} \Sigma^{-j+\nu}(k^i)^{\otimes j}, \]
and as we have $\beta_p \neq 0$, this is non-zero in degree $-p + u$. \hfill \Box

(1.9) **Proposition.** Suppose that $R$ has a dualizing DG module $D$ satisfying the extra conditions
\[ \text{RHom}_{R}(Rk, R D_R) \cong k_R \quad \text{and} \quad \text{RHom}_{R}(k_R, R D_R) \cong Rk. \]

Let $M$ be in $\text{fin}(R)$. Then
\[ \mu^j_R(M) \text{ and } \beta_j^{\text{rev}}(M^i) \text{ are zero simultaneously,} \]
and we have
\[ k \cdot \text{id}_R M = k \cdot \text{pd}_{R^{\text{rev}}} M^i \]
and
\[ \text{depth}_R M = \inf \{ i \mid H_i(M^i) \neq 0 \}. \] (3)
Proof. To see (1), note that the proposition’s extra conditions on \( D \) can also be expressed
\[
(Rk)^\dagger \cong k_R \quad \text{and} \quad (k_R)^\dagger \cong Rk.
\] (a)
Thus,
\[
\text{RHom}_R(Rk, RM) \cong \text{RHom}_R((k_R)^\dagger, (RM)^\dagger) \cong \text{RHom}_R^{\text{opp}}((R^\dagger)^\dagger, k_R),
\]
where the first \( \cong \) follows from equations (a) and (0.3.1), and the second \( \cong \) follows from equation (0.3.2). Hence we get isomorphisms of abelian groups,
\[
\text{H}_j(\text{RHom}_R(k, M)) \cong \text{H}_j(\text{RHom}_R^{\text{opp}}(M^\dagger, k)),
\]
and (1) follows.
As for (2), it follows immediately from (1) and remark (1.4).
To see (3), we can compute,
\[
\text{depth}_R M \overset{(b)}{=} \inf \{ j \mid \mu_j^R(M) \neq 0 \} \\
\overset{(c)}{=} \inf \{ j \mid \beta_j^{R^{\text{opp}}}(M^\dagger) \neq 0 \} \\
\overset{(d)}{=} \inf \{ i \mid \text{H}_i(M^\dagger) \neq 0 \},
\]
where (b) is by remark (1.4) and (c) is by (1), while (d) is by lemma (1.7)(3) because \( M^\dagger \) is in \( \text{fin}(R^{\text{opp}}) \) and hence by paragraph (0.5) admits a resolution as required in lemma (1.7). \qed

(1.10) Corollary. Suppose that \( R \) has a dualizing DG module \( D \) satisfying the extra conditions
\[
\text{RHom}_R(Rk, RD_R) \cong k_R \quad \text{and} \quad \text{RHom}_R^{\text{opp}}(k_R, RD_R) \cong Rk.
\]
Then
\[
\text{depth}_R R = \inf \{ i \mid \text{H}_i(D) \neq 0 \} = \text{depth}_R^{\text{opp}} R.
\]
Proof. The corollary’s first can be proved as follows,
\[
\text{depth}_R R = \inf \{ i \mid \text{H}_i((RD_R)^\dagger) \neq 0 \} \\
= \inf \{ i \mid \text{H}_i(\text{RHom}_R(R, D)) \neq 0 \} \\
= \inf \{ i \mid \text{H}_i(D) \neq 0 \},
\]
where the first \( = \) is by proposition (1.9)(3). The corollary’s second \( = \) follows by an analogous computation. \qed

2. Identities

(2.1) Setup. Recall that \( R \) denotes a DGA satisfying the conditions of setup (0.6). In the rest of the paper, we also require:
• \( R \) has a dualizing DG module \( D \) satisfying
\[
\text{RHom}_R(Rk, RD_R) \cong k_R \quad \text{and} \quad \text{RHom}_R^{\text{opp}}(k_R, RD_R) \cong Rk.
\]
(2.2) Remark. From [13] we know that, in suitable circumstances, one can get a dualizing DG module for \( R \) by coinducing a dualizing complex from a commutative central base ring \( A \). That is, if \( A \) has the dualizing complex \( C \), then

\[
D = \text{RHom}_A(R, C)
\]

is a dualizing DG module for \( R \).

A small computation with the pattern

\[
\text{RHom}_R(k, D) = \text{RHom}_R(k, \text{RHom}_A(R, C)) \\
\approx \text{RHom}_A(R \otimes_R k, C) \\
\approx \text{RHom}_A(k, C) \\
\approx k
\]

proves frequently that such a dualizing DG module \( D \) also satisfies the extra conditions of setup (2.1). (Some care is needed when making this concrete; for instance, we have made no conditions on the behaviour of \( k \) viewed as an \( A \)-module, so the last \( \approx \) does not necessarily apply.)

Summing up, when this method works, the conditions of setup (2.1) hold for \( R \), and hence the results of this section apply to \( R \).

In particular, the DGAs in the following list satisfy the standing conditions of setup (0.6), and the method we have sketched shows that they also satisfy the conditions of setup (2.1). Hence the results of this section apply to them:

- The DG fibre \( F(\alpha') \), where \( A' \xrightarrow{\alpha'} A \) is a local ring homomorphism of finite flat dimension between noetherian local commutative rings \( A' \) and \( A \), and where \( A \) has a dualizing complex (see [6, (3.7)]).
- The Koszul complex \( K(a) \), where \( a = (a_1, \ldots, a_n) \) is a sequence of elements in the maximal ideal of the noetherian local commutative ring \( A \), and where \( A \) has a dualizing complex (see [24, exer. 4.5.1])
- The singular chain DGA \( C_*(G; k) \) where \( k \) is a field and \( G \) a path connected topological monoid with \( \dim_k H_*(G; k) < \infty \) (see [9, chap. 8]).

Finally, let us mention a “degenerate” case: Let \( A \) be a noetherian ring which is local in the following sense: It has a unique maximal two sided ideal \( J \), and \( A/J \) is a skew field. We can then consider \( A \) as a DGA concentrated in degree zero, and \( A \) falls under setup (0.6). So if \( A \) satisfies the conditions of setup (2.1), then the results of this section apply to \( A \).

A special case of this is that \( A \) is even a noetherian local commutative ring. Then “dualizing DG module” just means “dualizing complex” by [14], and if \( D \) is a dualizing complex for \( A \) then the extra conditions of setup (2.1) hold automatically by [17, prop. V.3.4] (we might need to
replace $D$ by some $\Sigma^i D)$. So we can extend the list above: The results of this section also apply to

- The noetherian local commutative ring $A$, where $A$ has a dualizing complex.

(2.3) Theorem (Auslander-Buchsbaum Formula). Recall that we work under the standing conditions of setups (0.6) and (2.1). Let $M$ be in $\text{fin}(R)$ and suppose that $k.pd_R M$ is finite. Then

$$ k.pd_R M + depth_R M = depth_R R. $$

Proof. Proposition (1.8) applies to $R\text{Hom}_R(M, D)$: We have that $M$ is in $\text{fin}(R)$ by assumption, so $M$ satisfies the proposition’s finiteness conditions. And $R_R$ is in $\text{fin}(R^{opp})$ by setup (0.6), so

$$(R_R)^{\dagger} = R\text{Hom}_{R^{opp}}(R_R, R_D) \cong R_D$$

is in $\text{fin}(R)$, so $R_D$ also satisfies the proposition’s finiteness conditions. Finally, we have $k.pd_R M < \infty$ by assumption.

We can now compute,

$$ depth_R M = \inf \{ i \mid H_i(M^{\dagger}) \neq 0 \} $$

$$ = \inf \{ i \mid H_i(R\text{Hom}_R(M, D)) \neq 0 \} $$

$$ = -k.pd_R M + \inf \{ i \mid H_i(D) \neq 0 \} $$

$$ = -k.pd_R M + depth_R R, $$

where (a) is by proposition (1.9)(3) and (b) is by proposition (1.8), while (c) is by corollary (1.10). \hfill \square

(2.4) Theorem (Bass Formula). Recall that we work under the standing conditions of setups (0.6) and (2.1). Let $N$ be in $\text{fin}(R)$ and suppose that $k.id_R N$ is finite. Then

$$ k.id_R N + \inf \{ i \mid H_i(N) \neq 0 \} = depth_R R. $$

Proof. From the duality (0.3) we know that $N^{\dagger}$ is in $\text{fin}(R^{opp})$, and from proposition (1.9)(2) we have $k.pd_{R^{opp}} N^{\dagger} = k.id_R N$, so $k.pd_{R^{opp}} N^{\dagger}$ is finite. Now

$$ k.id_R N = k.pd_{R^{opp}} N^{\dagger} $$

$$ \overset{(a)}{=} depth_{R^{opp}} R - depth_{R^{opp}} N^{\dagger} $$

$$ \overset{(b)}{=} depth_{R^{opp}} R - \inf \{ i \mid H_i(N^{\dagger}) \neq 0 \} $$

$$ \overset{(c)}{=} depth_{R^{opp}} R - \inf \{ i \mid H_i(N) \neq 0 \} $$

$$ \overset{(d)}{=} depth_R R - \inf \{ i \mid H_i(N) \neq 0 \}, $$
where (a) is by the Auslander–Buchsbaum Formula, theorem (2.3), and (b) is by proposition (1.9)(3), while (c) is by equation (0.3.1) and (d) is by corollary (1.10).

(2.5) **Gap Theorem.** Recall that we work under the standing conditions of setups (0.6) and (2.1). Let $M$ be in $\text{fin}(R)$ and let $g$ in $\mathbb{Z}$ satisfy $g > \text{amp } R$. Assume that the sequence of Bass numbers of $M$ has a gap of length $g$, in the sense that there exists $\ell$ in $\mathbb{Z}$ so that

$$
\mu^\ell_R(M) \neq 0, \quad \mu^{\ell+1}_R(M) = \cdots = \mu^{\ell+g}_R(M) = 0, \quad \mu^{\ell+g+1}_R(M) \neq 0.
$$

Then we have

$$
\text{amp } M \geq g + 1.
$$

**Proof.** By paragraph (0.4) there is a quasi-isomorphic truncation $T$ of $R$ which is concentrated between degrees 0 and $\sup \{i \mid H_i(R) \neq 0\} = \text{amp } R$, and the derived categories of $R$ and $T$ are equivalent. Let us therefore replace $R$ with $T$ and transport $M$ through the equivalence. Then we are in a situation where the conditions of the theorem still hold, but where $R$ is concentrated between degrees 0 and $\text{amp } R$.

Observe from the duality (0.3) that $M^!$ is in $\text{fin}(R^{\text{opp}})$. So paragraph (0.5) gives that $M^!$ admits a minimal semi-free resolution $F \xrightarrow{\sim} M^!$ with

$$
F^! \cong \bigoplus_{v \leq j} \Sigma^j(R^!)(\beta_j),
$$

where $v = \inf \{ i \mid H_i(M^!) \neq 0 \}$ and where each $\beta_j$ is finite. Lemma (1.7)(1) yields

$$
\beta^\text{Repr}_{j}(M^!) = \begin{cases} 
0 & \text{for } j < v, \\
\beta_j & \text{for } j \geq v.
\end{cases} \hspace{1cm} (a)
$$

Note that we have $F \cong M^!$ in $\mathcal{D}(R^{\text{opp}})$ and $F^! \cong M$ in $\mathcal{D}(R)$.

By assumption we have

$$
\mu^\ell_R(M) \neq 0, \quad \mu^{\ell+1}_R(M) = \cdots = \mu^{\ell+g}_R(M) = 0, \quad \mu^{\ell+g+1}_R(M) \neq 0.
$$

By proposition (1.9)(1) this translates to

$$
\beta^\text{Repr}_{\ell}(M^!) \neq 0, \quad \beta^\text{Repr}_{\ell+1}(M^!) = \cdots = \beta^\text{Repr}_{\ell+g}(M^!) = 0, \quad \beta^\text{Repr}_{\ell+g+1}(M^!) \neq 0.
$$

And by equation (a) this says

$$
\beta_\ell \neq 0, \quad \beta_{\ell+1} = \cdots = \beta_{\ell+g} = 0, \quad \beta_{\ell+g+1} \neq 0. \hspace{1cm} (b)
$$

But then the graded right $R^!$-module $F^!$ splits as

$$
F^! \cong F^!_1 \amalg F^!_2,
$$
where the summands have the form
\[ F_1 = \prod_{v \leq 1 \leq \ell} \Sigma^j (R^1)^{(j)}, \quad (c) \]
\[ F_2 = \prod_{\ell + g + 1 \leq j} \Sigma^j (R^1)^{(j)}. \quad (d) \]

Now observe that
- The left-most summand in \( F_1 \) has index \( j = \ell \), so is concentrated between degrees \( \ell \) and \( \ell + \text{amp } R < \ell + g \) because \( R^1 \) itself is concentrated between degrees 0 and \( \text{amp } R \).
- The right-most summand in \( F_2 \) has index \( j = \ell + g + 1 \), so has its right-most component in degree \( \ell + g + 1 \) (and continues to the left).

In other words, the summands \( F_1 \) and \( F_2 \) are separated by at least one zero in degree \( \ell + g \) so the differential of \( F \) cannot map between \( F_1 \) and \( F_2 \). Hence the splitting of \( F \) is induced by a splitting of the right DG \( R \)-module \( F \),

\[ F \cong F_1 \oplus F_2. \]

Clearly, both \( F_1 \) and \( F_2 \) are minimal \( K \)-projective, as \( F \) itself is. Also, we have \( F_1 \not\cong 0 \) and \( F_2 \not\cong 0 \) in \( D(R^{op}) \), as one sees easily from \( \beta_\ell \not= 0 \) and \( \beta_{\ell + g + 1} \not= 0 \) (see equation (b)).

The rest of the proof consists of computations with \( \text{RHom}_{R^{op}}(F_1, F_2) \). Let us first check that we have

\[ \text{RHom}_{R^{op}}(F_1, F_2) \not\cong 0. \quad (e) \]

From \( F \cong M^f \) we get

\[ \text{H}(F_1) \oplus \text{H}(F_2) \cong \text{H}(F) \cong \text{H}(M^f), \]

so it is clear that \( F_1 \) and \( F_2 \) are in \( \text{fin}(R^{op}) \). Moreover, we know \( \beta_\ell \not= 0 \) from equation (b), so equation (c) and lemma (1.7)(2) give

\[ k \cdot \text{pd}_{R^{op}} F_1 = \ell, \]

in particular \( k \cdot \text{pd}_{R^{op}} F_1 < \infty \). Finally, \( F_2 \) is bounded to the right and has \( F_2 \not\cong 0 \), so \( \inf \{ i \mid H_i(F_2) \not= 0 \} \) is finite. Proposition (1.8) can now be applied and shows

\[ \inf \{ i \mid H_i(\text{RHom}_{R^{op}}(F_1, F_2)) \not= 0 \} \]
\[ = -k \cdot \text{pd}_{R^{op}} F_1 + \inf \{ i \mid H_i(F_2) \not= 0 \} \]
\[ = -\ell + \inf \{ i \mid H_i(F_2) \not= 0 \}, \quad (f) \]

and this is a finite number, so (e) follows.

To proceed, let us focus on the number

\[ \inf \{ i \mid H_i(\text{RHom}_{R^{op}}(F_1, F_2)) \not= 0 \} \quad (g) \]
which appeared in (f). It is easy to establish a lower bound: Equation (d) gives
\[
\inf\{ i \mid H_i(F_2) \neq 0 \} \geq \ell + g + 1,
\]
so starting with equation (f) we get
\[
\inf\{ i \mid H_i(\text{RHom}_R(F_1, F_2)) \neq 0 \} = -\ell + \inf\{ i \mid H_i(F_2) \neq 0 \} \\
\geq -\ell + \ell + g + 1 \\
= g + 1. \tag{h}
\]

Next we want to establish an upper bound on the number (g). From \(F^1 \cong M\) we get
\[
H(F^1_1) \cong H(F^1_2) \cong H(F^1) \cong H(M), \tag{i}
\]
so it is clear that \(F^1_1\) and \(F^1_2\) are in \(\text{fin}(R)\). Hence
\[
\inf\{ i \mid H_i(\text{RHom}_R(F_1, F_2)) \neq 0 \} \\
\leq \sup\{ i \mid H_i(\text{RHom}_R(F_1, F_2)) \neq 0 \} \\
= \sup\{ i \mid H_i(F_1^1, F_1^1) \neq 0 \} \\
\leq -\inf\{ i \mid H_i(F_1^1) \neq 0 \} + \sup\{ i \mid H_i(F_1^1) \neq 0 \} \\
\leq -\inf\{ i \mid H_i(F_2^1) \neq 0 \} + \sup\{ i \mid H_i(M) \neq 0 \}, \tag{n}
\]
where (j) holds because of (e), and (k) is by equation (0.3.2), while (l) is by proposition (1.5), and (m) is because (i) implies
\[
\sup\{ i \mid H_i(F_2^1) \neq 0 \} \leq \sup\{ i \mid H_i(M) \neq 0 \}.
\]

The lower and upper bounds allow us to complete the proof: Combining (h) and (n) we may write
\[
g + 1 \leq \inf\{ i \mid H_i(\text{RHom}_R(F_1, F_2)) \neq 0 \} \\
\leq -\inf\{ i \mid H_i(F_2^1) \neq 0 \} + \sup\{ i \mid H_i(M) \neq 0 \},
\]

hence
\[
\sup\{ i \mid H_i(M) \neq 0 \} \geq \inf\{ i \mid H_i(F_2^1) \neq 0 \} + (g + 1). \tag{o}
\]

Finally, from equation (i) we also get
\[
\inf\{ i \mid H_i(M) \neq 0 \} \leq \inf\{ i \mid H_i(F_2^1) \neq 0 \}. \tag{p}
\]

Subtracting (p) from (o) we get
\[
\text{amp } M = \sup\{ i \mid H_i(M) \neq 0 \} - \inf\{ i \mid H_i(M) \neq 0 \} \\
\geq \inf\{ i \mid H_i(F_2^1) \neq 0 \} + (g + 1) - \inf\{ i \mid H_i(F_2^1) \neq 0 \} \\
= g + 1.
\]
\[\square\]
3. Examples

(3.1) **Gaps in Bass series.** Let us start this paragraph with a short recap on [4, question (3.10)]: As above, we say that the sequence of Bass numbers of the left DG $R$-module $M$ has a gap of length $g$ if there exists an $\ell$ with

$$
\mu_R^\ell(M) \neq 0, \quad \mu_R^{\ell+1}(M) = \cdots = \mu_R^{\ell+g}(M) = 0, \quad \mu_R^{\ell+g+1}(M) \neq 0.
$$

Now, [4, question (3.10)] asks whether the length of gaps in the sequence of Bass numbers of $R$ itself is bounded by $\amp R$.

Indeed, using theorem (2.5) we can prove even more: Let $M$ be any left DG $R$-module in $\fin(R)$ with $\amp M \leq \amp R + 1$. If there were a gap of length $g$ in the sequence of Bass numbers of $M$, with $g > \amp R$, then theorem (2.5) would give $\amp M \geq g + 1 > \amp R + 1 > \amp R$, hence $\amp M \geq \amp R + 2$, a contradiction. So we must have:

\[
\text{The length of gaps in the sequence of Bass numbers of } M \text{ is bounded by } \amp R.
\]

Note by remark (2.2) that the DGAs for which we have now answered [4, question (3.10)] include DG fibres, Koszul complexes, and DGAs of the form $C_\ast(G;k)$ where $k$ is a field and $G$ a topological monoid with $\dim_k H_\ast(G;k) < \infty$.

Note also that in the case of the DG fibre of a local ring homomorphism of finite flat dimension, one can prove the stronger result that there are no gaps in the sequence of Bass numbers of the DG fibre by using [5, (7.2) and thm. (7.4)].

(3.2) **G-Serre-fibrations.** We will now evaluate the Auslander-Buchsbaum Formula (theorem (2.3)) for the singular chain DGA $C_\ast(G;k)$. This turns out to result in additivity of homological dimension on G-Serre-fibrations: Let $k$ be a field, $G$ a path connected topological monoid, and

$$
G \longrightarrow P \longrightarrow X
$$

a G-Serre-fibration with $G$ acting on $P$ from the left (see [9, chap. 2]). Assume that $H_\ast(G;k), H_\ast(P;k)$ and $H_\ast(X;k)$ are finite dimensional over $k$. The composition in $G$ turns $C_\ast(G;k)$ into a DGA (which is potentially highly non-commutative), and the action of $G$ on $P$ turns $C_\ast(P;k)$ into a left DG $C_\ast(G;k)$-module (see [9, chap. 8]).

By remark (2.2), the conditions of setups (0.6) and (2.1) hold for $C_\ast(G;k)$, so the results of section 2 also hold, in particular the Auslander-Buchsbaum Formula.

In fact, note that by remark (2.2), the conditions of setup (2.1) are satisfied with the dualizing DG module

$$
c_\ast(G;k)D_{C_\ast(G;k)} = \RHom_k(c_\ast(G;k) C_\ast(G;k)c_\ast(G;k), k).
$$
Dagger dualization with respect to this $D$ is particularly simple: For a left DG $R$-module $M$ we have

$$M^\dagger = \text{RHom}_{C_\ast(G;k)}(M, D)$$

$$= \text{RHom}_{C_\ast(G;k)}(M, \text{RHom}_k(C_\ast(G;k), k))$$

$$\cong \text{RHom}_k(C_\ast(G; k) \otimes_{C_\ast(G;k)} M, k)$$

$$\cong \text{RHom}_k(M, k), \quad (a)$$

that is, dagger dualization is just dualization with respect to $k$.

We now want to use the Auslander-Buchsbaum Formula on the left DG $C_\ast(G;k)$-module $C_\ast(P;k)$. Clearly $C_\ast(P;k)$ is in $\text{fin}(C_\ast(G;k))$. Next note that

$$k \otimes_{C_\ast(G;k)} C_\ast(P;k) \cong C_\ast(X;k)$$

by [9, thm. 8.3], so using remark (1.2) we may compute,

$$k \cdot \text{pd}_{C_\ast(G;k)} C_\ast(P;k) = \sup \{ i \mid H_i(k \otimes_{C_\ast(G;k)} C_\ast(P;k)) \neq 0 \}$$

$$= \sup \{ i \mid H_i(C_\ast(X;k)) \neq 0 \}$$

$$= \sup \{ i \mid H_i(X;k) \neq 0 \}.$$

This is finite by assumption. Thus we may apply the Auslander-Buchsbaum Formula, and get

$$k \cdot \text{pd}_{C_\ast(G;k)} C_\ast(P;k) + \text{depth}_{C_\ast(G;k)} C_\ast(P;k)$$

$$= \text{depth}_{C_\ast(G;k)} C_\ast(G;k).$$

Substituting the above expression for $k \cdot \text{pd}_{C_\ast(G;k)} C_\ast(P;k)$, this becomes

$$\sup \{ i \mid H_i(X;k) \neq 0 \} + \text{depth}_{C_\ast(G;k)} C_\ast(P;k)$$

$$= \text{depth}_{C_\ast(G;k)} C_\ast(G;k). \quad (b)$$

Finally, for any $M$ in $\text{fin}(C_\ast(G;k))$ we have

$$\text{depth}_{C_\ast(G;k)} M \overset{(c)}{=} \inf \{ i \mid H_i(M) \neq 0 \} \overset{(d)}{=} -\sup \{ i \mid H_i(M) \neq 0 \},$$

where (c) is by proposition (1.9)(3) and (d) follows from (a).

Using this in equation (b) and rearranging terms, we finally get

$$\sup \{ i \mid H_i(P;k) \neq 0 \} = \sup \{ i \mid H_i(G;k) \neq 0 \} + \sup \{ i \mid H_i(X;k) \neq 0 \},$$

stating that homological dimension is additive on $G$-Serre-fibrations.

This result is well known; in a slightly different form it also follows from the Leray-Serre spectral sequence (see [22, exam. 5.B]). It is handy because of the restrictions it imposes on the fibre $G$ and the base $X$ in terms of the total space $P$. For instance, if $\sup \{ i \mid H_i(P;k) \neq 0 \}$ is zero, then both $\sup \{ i \mid H_i(G;k) \neq 0 \}$ and $\sup \{ i \mid H_i(X;k) \neq 0 \}$ must be zero (see also [22, thm. 5.7]).
(3.3) **Commutative rings.** We noted already in remark (2.2) that the results of section 2 apply to a noetherian local commutative ring with a dualizing complex, since such a ring can be viewed as a DGA concentrated in degree zero.

Indeed, let us show that for any noetherian local commutative ring $A$, the classical Auslander-Buchsbaum and Bass Formulae and the No Holes Theorem (see [2], [7], [10], [12], and [23]) follow from theorems (2.3), (2.4), and (2.5):

First, to prove the three classical results for $A$, it suffices to prove them for the completion $\hat{A}$, so we can assume that $A$ is complete. Hence $A$ has a dualizing complex $D$ by [17, p. 299], and by remark (2.2) the results of section 2 apply to $A$.

The classical Auslander-Buchsbaum Formula now follows from theorem (2.3) since our notions of $k.pd$ and depth coincide with the classical notions of projective dimension and depth for complexes in $D^b(A)$ by [3, prop. 5.5].

The classical Bass Formula likewise follows from theorem (2.4) since our notion of $k.id$ coincides with the classical notion of injective dimension for complexes in $D^b(A)$, again by [3, prop. 5.5].

The classical No Holes Theorem can be obtained as follows from theorem (2.5): Consider $M$ in $D^b(A)$ and suppose that there is a “hole” in the sequence of Bass numbers of $M$, that is, we have $\mu^2_A(M) = 0$, but there are non-zero Bass numbers both below and above $\mu^2_A(M)$. In the terminology of theorem (2.5), this says that the sequence of Bass numbers of $M$ has a gap. If we let $g$ be the length of the gap, then we have $g > 0$ whence $g > \text{amp } A$ since $\text{amp } A = 0$, so theorem (2.5) states

$$\text{amp } M \geq g + 1 > 1,$$

so $M$ is certainly not an ordinary $A$-module since its homology is not concentrated in one degree. So if $M$ is an ordinary $A$-module, then there are no holes in the sequence of Bass numbers of $M$.

(3.4) **Non-commutative rings.** The method of paragraph (3.3) could also be used on a suitable non-commutative noetherian ring, and, when successful, would recover the Auslander-Buchsbaum and Bass Formulae and the No Holes Theorem (see [25] and [26]).

However, the question of existence of a suitable dualizing DG module satisfying the conditions of setup (2.1) is much more delicate in this case (see [25] and [14]), so we prefer to leave the matter with this remark.

4. **A Counterexample**

(4.1) **Remark.** By section 2, in particular remark (2.2), the Auslander-Buchsbaum and Bass Formulae hold for DGAs of the form $C_*(G; k)$ where $k$ is a field and $G$ a path connected topological monoid with
\( \dim_k H_*(G; k) < \infty \). The following paragraph shows that the formulae can fail if we drop the condition \( \dim_k H_*(G; k) < \infty \), even if we keep the weaker condition \( \dim_k H_i(G; k) < \infty \) for each \( i \).

\[ (4.2) \textbf{Loop space homology.} \text{ Let } X \text{ be a finite simply connected CW complex, and write} \]
\[ d = \sup\{ i \mid H_i(X; \mathbb{Q}) \neq 0 \}. \]

This is a finite number.

The Moore loop space \( \Omega X \) is a topological monoid (see [9, p. 29, exam. 1]). As \( X \) is simply connected, \( \Omega X \) is path connected. We will consider \( C_*(\Omega X; \mathbb{Q}) \) which is a DGA.

Observe that since \( X \) is a finite CW complex, \( \dim_\mathbb{Q} H_i(X; \mathbb{Q}) < \infty \) holds for each \( i \). So we also have \( \dim_\mathbb{Q} \pi_i(X) \otimes \mathbb{Q} < \infty \) for each \( i \) by [9, p. 208, rmk. 1], and therefore

\[ \dim_\mathbb{Q} H_i(\Omega X; \mathbb{Q}) < \infty \]

for each \( i \) because of [9, formula (33.7)]. So \( C_*(\Omega X; \mathbb{Q}) \) satisfies the same conditions as the \( C_*(G; k) \)'s we have considered before, except that it can have homology in infinitely many degrees.

We will show that if Poincaré duality over \( \mathbb{Q} \) fails for \( X \), then both the Auslander-Buchsbaum and the Bass Formula fails for \( C_*(\Omega X; \mathbb{Q}) \). We do so by contraposition. So we suppose that the Auslander-Buchsbaum Formula or the Bass Formula holds for \( C_*(\Omega X; \mathbb{Q}) \), and show that \( X \) has Poincaré duality over \( \mathbb{Q} \).

The path space fibration
\[ \Omega X \longrightarrow PX \longrightarrow X \]
(see [9, p. 29, exam. 1]) can be inserted into [9, thm. 8.3] and gives
\[ C_*(X; \mathbb{Q}) \cong C_*(PX; \mathbb{Q}) \otimes_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} = (*), \]
and as \( PX \) is contractible we have
\[ (* \cong \mathbb{Q} \otimes_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q}. \]

Combining this with remark (1.2) gives
\[ k.\text{pd}_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} = \sup\{ i \mid H_i(C_*(X; \mathbb{Q})) \neq 0 \} \]
\[ = \sup\{ i \mid H_i(X; \mathbb{Q}) \neq 0 \} \]
\[ = d. \]

As
\[ k.\text{pd}_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} = -\inf\{ i \mid H_i(\text{RHom}_{C_*(\Omega X; \mathbb{Q})}(\mathbb{Q}, \mathbb{Q})) \neq 0 \} \]
\[ = k.\text{id}_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} \]
holds by definition, we even get
\[ k.\text{pd}_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} = k.\text{id}_{C_*(\Omega X; \mathbb{Q})} \mathbb{Q} = d. \]
Since $d$ is finite, this shows that the Auslander-Buchsbaum and Bass Formulae both apply to $Q$ over $C_*(\Omega X; \mathbb{Q})$.

Using a minimal semi-free resolution of $Q$ proves

$$\text{depth}_{C_*} (\Omega X; \mathbb{Q}) Q = 0,$$

and

$$\inf \{ i \mid H_i(Q) \neq 0 \} = 0$$

is clear, so the Auslander-Buchsbaum and Bass Formulae for $Q$ both amount to

$$d = \text{depth}_{C_*} (\Omega X; \mathbb{Q}) C_* (\Omega X; \mathbb{Q}). \quad (a)$$

Equation (a) states that $\text{Ext}_{C_*} (\Omega X; \mathbb{Q}) (\mathbb{Q}, C_* (\Omega X; \mathbb{Q}))$ sits in cohomological degree $d$ and higher. (Warning: This is the only time in the paper we use cohomological rather than homological terminology!) Now, [8, thm. 2.1] states

$$\text{Ext}_{C_*} (\Omega X; \mathbb{Q}) (\mathbb{Q}, C_* (\Omega X; \mathbb{Q})) \cong \text{Ext}_{C_*} (X; \mathbb{Q}) (\mathbb{Q}, C_* (X; \mathbb{Q})) = (**)$$

and dualizing with respect to $Q$ gives

$$(**) \cong \text{Ext}_{(C_* (X; \mathbb{Q}))^{op}} (\text{Hom}_Q (C_* (X; \mathbb{Q}), \mathbb{Q}), \mathbb{Q}) = (***)$$

So equation (a) states that $(***)$ sits in cohomological degree $d$ and higher.

On the other hand, the cohomology of $\text{Hom}_Q (C_* (X; \mathbb{Q}), \mathbb{Q})$ sits between cohomological degrees $-d$ and 0. Consider a minimal semi-free resolution of $\text{Hom}_Q (C_* (X; \mathbb{Q}), \mathbb{Q})$ which starts in cohomological degree $-d$ and continues to the right, to higher cohomological degrees. Using this to compute $(***)$ shows that $(***)$ sits in cohomological degree $d$ and lower.

Altogether, $(***)$ is hence concentrated in cohomological degree $d$, so the same holds for $(**)$. However, it is elementary that $(**)$ is finite dimensional over $Q$ in each degree, so this shows that the whole of $(**)$ is finite dimensional over $Q$:

$$\dim_Q \text{Ext}_{C_*} (X; \mathbb{Q}) (\mathbb{Q}, C_* (X; \mathbb{Q})) < \infty.$$ 

But then $X$ has Poincaré duality over $Q$ by [8, cor. 4.5].

References

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