SPECTRA OF MODULES

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ABSTRACT. This manuscript solves the problem that the so-called “stable category” $\text{Mod} \Lambda$ of an Artin algebra is in general not triangulated. The method is to mimic topology, and hence first form the Spanier-Whitehead category $\text{Stab}(\text{Mod} \Lambda)$, then construct a category $\text{Spectra}$ of “spectra of modules” which completes the compact part of $\text{Stab}(\text{Mod} \Lambda)$ under small coproducts. $\text{Spectra}$ is then a triangulated substitute for $\text{Mod} \Lambda$.

The main results are that $\text{Spectra}$ is a compactly generated triangulated category which contains the compact part of $\text{Stab}(\text{Mod} \Lambda)$ as a full subcategory, and even admits a precise description of its compact objects, which only form a small set of isomorphism classes.

As an application, it is proved that over an Artin algebra, the Gorenstein projective modules form a pre-covering class. This was previously only known for rings satisfying strong homological conditions.

0. INTRODUCTION

0.1. Background. A fundamental object in modern representation theory of finite groups is the homotopy category $\text{Mod}^kG$, see [7] and [23] (which call it “the stable category”). It is obtained from the ordinary module category $\text{Mod}^kG$ by dividing away the morphisms which factor through a projective module.

The homotopy category has many wonderful properties, some of which are due to its resemblance to the homotopy category of spectra from topology. Both categories are compactly generated triangulated categories, which are even closed symmetric monoidal. It has been possible to get inspiration from topology to new, highly useful theorems in representation theory, e.g. on Bousfield localization. This is treated in detail in [23], but many other papers have taken up the idea of lending ideas from topology into representation theory and related fields from algebra (for instance [2], [3], [4], [5], [6], [15], [16], [17], and [21]).

Now consider a more general situation. Suppose that I am interested in the representation theory, not of a finite group, but of an arbitrary Artin algebra, $\Lambda$. It is still possible to construct the homotopy category, $\text{Mod} \Lambda$, by dividing away from $\text{Mod} \Lambda$ the morphisms which factor through a
projective module, and the homotopy category is still an important object in representation theory, see [1, chp. IV]. But now it suffers from a serious defect: It is no longer triangulated, only right- and left-triangulated (see [4]). This means that there are still distinguished triangles and so on, but the right-shifting functor $\Sigma$ is no longer invertible (and neither is the left-shifting functor $\Omega$).

What to do? It would definitely be much nicer to have a triangulated category than merely a right- and left-triangulated category. But how to get one? Fortunately, there is a trick from topology that works: The Spanier-Whitehead category. In algebra, it takes the following form: One constructs a category, $\text{Stab}(\text{Mod} \Lambda)$, whose objects are pairs $(M, m)$ with $M \in \text{Mod} \Lambda$ and $m \in \mathbb{Z}$, and whose morphisms are defined by

$$\text{Hom}_{\text{Stab}(\text{Mod} \Lambda)}((M, m), (N, n)) = \text{colim}_i \text{Hom}_{\text{Mod} \Lambda}(\Sigma^{m+i}M, \Sigma^{n+i}N).$$

This stabilized version of the homotopy category is triangulated.

But now another problem appears: $\text{Stab}(\text{Mod} \Lambda)$ lacks infinite coproducts. A typical way of getting a system without coproduct is to take $(M, 0), (M, -1), (M, -2), \ldots$. So really, nothing great has been gained. The lack of coproducts is a fatal shortcoming. It keeps $\text{Stab}(\text{Mod} \Lambda)$ from having a chance of being compactly generated, and hence obstructs the use of such important tools as the Brown representability theorem and the Neeman-Thomason localization theorem (see [21]).

However, there is a remedy, namely the category of what I call “spectra of modules”, which is what this manuscript is about.

To explain this, let me reveal that there is a much closer analogy between representation theory and topology than said above. This can be made explicit by a dictionary between representation theory and topology:

- $\text{Mod} \Lambda \leftrightarrow$ The category of pointed topological spaces
- $\text{Mod} \Lambda \leftrightarrow$ The homotopy category of pointed topological spaces
- $\text{Stab}(\text{Mod} \Lambda) \leftrightarrow$ The Spanier-Whitehead category of topological spaces
- $????? \leftrightarrow$ The homotopy category of spectra

(see [13, chp. 13] and [3, sec. 3]). The point I want to make is that the passage from $\text{Mod} \Lambda$ through $\text{Mod} \Lambda$ to $\text{Stab}(\text{Mod} \Lambda)$ has an excellent analogy in topology, in the passage from the category of pointed topological spaces, through the homotopy category of pointed topological spaces, to the Spanier-Whitehead category of topological spaces (whose objects are of the form $(X, m)$ where $X$ is a topological space, $m$ an integer). And the Spanier-Whitehead category of topological spaces suffers from exactly the same defect as $\text{Stab}(\text{Mod} \Lambda)$: It is triangulated, but it does not have infinite coproducts. And not only this, but topologists have known the remedy for over 30 years, namely the homotopy category of spectra. This is a completion under coproducts of the compact part of the
Spanier-Whitehead category of topological spaces, and is even (as said above) a compactly generated triangulated category. This is described in great detail in the introduction to [19].

So to remedy the problem in representation theory that $\text{Stab}(\text{Mod}\Lambda)$ lacks infinite coproducts, one just needs to figure out the right category of “spectra of modules” to replace the question marks in the above dictionary.

Fortunately, this figuring has already been done by Grandis, [12, secs. 4.5 to 4.7], who defines the category of spectra of modules which I will use. I shall denote the category by $\text{Spectra}$. However, instead of using Grandis’ own definition of this category, I shall use the one given in [3, thm. 3.11]. It turns out that in the above setup, it simply gives

$\text{Spectra} = \text{the homotopy category of exact complexes of projectives}.$

What I shall do, then, is to show that $\text{Spectra}$ can be used to replace the question marks in the above dictionary. Concretely, I show that $\text{Spectra}$ is a compactly generated triangulated category, which forms a completion of the compact part of $\text{Stab}(\text{Mod}\Lambda)$ under coproducts. Moreover, I shall even describe precisely its compact objects. See the next subsection for details.

Let me remark for the topologists that my results on $\text{Spectra}$ are entirely analogous to the ones from topology described in the introduction to [19].

0.2. This manuscript. After the above general explanation, here is a brief overview of this manuscript.

First note that I will work in slightly higher generality than indicated above: Instead of $\text{Mod}\Lambda$, I will use an abelian category $\mathcal{C}$, and instead of dividing away morphisms factoring through projectives, I will divide away morphisms factoring through objects from a fixed pre-enveloping and pre-covering class $\mathcal{X}$. This defines the homotopy category $\mathcal{C}_\mathcal{X}$. The data $\mathcal{C}$ and $\mathcal{X}$ will be required to satisfy some assumptions, see setup 0.7. The situation from the previous subsection, with an Artin algebra $\Lambda$, can be obtained by letting $\mathcal{C} = \text{Mod}\Lambda$ and $\mathcal{X} = \text{Proj}\Lambda$, see example 0.8 and lemma 4.4.

Now for the contents of the manuscript:

- Section 1 sets up the category $\text{Spectra}$ and an adjoint pair of functors,

$$\begin{array}{ccc}
\text{Spectra} & \xrightarrow{Z_0} & \mathcal{C}_\mathcal{X}, \\
\xleftarrow{s_\mathcal{P}} & & \\
\end{array}$$

which will be fundamental in studying the relationship between $\text{Spectra}$ and $\text{Stab}(\mathcal{C}_\mathcal{X})$. 
• Section 2 proves in theorem 2.7 the first main result: That Spec induces a full embedding
\[ \text{Stab}((C_\mathcal{X})^c) \hookrightarrow \text{Spec} \]
of the compact part of the Spanier-Whitehead category into \text{Spec}. This corresponds perfectly to topology.
• Section 3 proves, under suitable hypotheses, in theorem 3.2 that \text{Spec} is compactly generated, and in theorem 3.7 and corollary 3.8 that \text{Spec}'s compacts are exactly the spectra coming from objects in \text{Stab}((C_\mathcal{X})^c). This again corresponds perfectly to topology.
• Section 4 applies \text{Spec} to a question from representation theory, and proves in theorem 4.14 that over an Artin algebra, the class of so-called Gorenstein projective modules is pre-covering. Previously, this was only known for rings satisfying strong homological conditions, see [10, thm. 2.9] and [11, thm. 3.4].

0.3. Notation and setup. A few things are needed here. See [3], [4], [9], [13, chp. 13], and [18] for background.

Notation 0.1 (Some generalities on categories). The notions of small and large sets ("large set" being synonymous with "class") will play a large roll; see [18, sec. I.6] for some basic facts.

If \( A \) is an additive category, then the category of chain complexes of \( A \)-objects and chain maps is denoted \( \text{Complex}(A) \), while the category of chain complexes of \( A \)-objects and homotopy classes of chain maps is denoted \( K(A) \). If \( a \) is a chain map, then its homotopy class is denoted \([a]\). When \( T \) is a triangulated category, then shifting in \( T \) (also known as suspension) is denoted \((\quad)\)\(^{[1]}\).

If \( A \) is a category with set indexed coproducts (also known as small coproducts), then an object \( A \) in \( A \) is called compact if the functor \( \text{Hom}_A(A, \quad) \) commutes with set indexed coproducts. The full subcategory of compact objects in \( A \) is denoted \( A^c \). \( \square \)

Notation 0.2 (Right- and left-triangulated categories). The main reference for this is [4], which introduces the axioms (LT1) to (LT4) for left-triangulated categories. Right-triangulated categories are characterized by the dual axioms. Such categories are like triangulated categories, only their shifting functors (which some would call suspension functors) are not invertible.

When dealing with an abstract right-triangulated category, I denote its right-shifting functor by \( \Sigma \), and when dealing with an abstract left-triangulated category, I denote its left-shifting functor by \( \Omega \). \( \square \)

Notation 0.3 (Pre-enveloping and pre-covering classes). The main reference for this is [9]. Let \( C \) be an abelian category, \( \mathcal{X} \) a class of objects in \( C \). A morphism \( A \to B \) is called an \( \mathcal{X} \)-monic if, for any \( X \in \mathcal{X} \) and
any morphism \( A \rightarrow X \), a broken arrow exists to give a commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & X \\
\downarrow & & \downarrow \\
B & \rightarrow & X.
\end{array}
\]

An \( X \)-monic \( A \rightarrow X \) with \( X \in X \) is called an \( X \)-pre-envelope of \( A \). If each object in \( C \) has an \( X \)-pre-envelope, then \( X \) is called a \textit{pre-enveloping class}.

If \( X \) is pre-enveloping then one can construct an \( X \)-right-resolution of any \( A \in C \), of the form \( A \xrightarrow{\alpha} X^0 \xrightarrow{d^0_X} X^1 \rightarrow \cdots \), where \( A \xrightarrow{\alpha} X^0 \) is an \( X \)-pre-envelope, and \( X^0 \rightarrow X^1 \) is induced by an \( X \)-pre-envelope \( X^0 / \text{Im}(\alpha) \rightarrow X^1 \), and where each \( X^i \rightarrow X^{i+1} \) is induced by an \( X \)-pre-envelope \( X^i / \text{Im}(d^{i-1}_X) \rightarrow X^{i+1} \) for \( i \geq 1 \).

If \( X \) is pre-enveloping, then an \( X \)-right-assignment for \( C \) is a choice of \( X \)-pre-envelopes for all objects of \( C \). When a choice has been made, the cokernel of \( A \)'s assigned pre-envelope is denoted \( \Sigma A \) and is called \( A \)'s \( X \)-right-syzygy. If one uses an \( X \)-right-assignment to construct \( X \)-right-resolutions, then they are called \( X \)-right-resolutions.

All this can be dualized. The notions dual to \( X \)-monic and \( X \)-pre-envelope are called \( X \)-epic and \( X \)-pre-cover. The notion dual to that of pre-enveloping class is called \textit{pre-covering class}. The notion dual to \( X \)-right-something is called \( X \)-left-something. And the notion dual to \( \Sigma \) is denoted \( \Omega \).

\textbf{Notation 0.4} (The homotopy category). The main reference for this is again [4]. Let \( C \) be an abelian category, \( X \) a pre-enveloping or pre-covering class in \( C \). The category \( C_X \) is obtained from \( C \) by dividing away morphisms which factor through objects from \( X \). So \( C_X \) has the same objects as \( C \), but its morphisms are classes of morphisms in \( C \). I call \( C_X \) the \textit{homotopy category} (of \( C \) with respect to \( X \)). The homomorphism functor in \( C_X \) is denoted by \( \text{Hom}_{C_X} = \pi \).

To deal with \( C_X \), the notions of right- and left-triangulated categories as explained in notation 0.2 are useful: If \( X \) is pre-enveloping then \( C_X \) is right-triangulated with \( \Sigma \) (which takes right-syzygies as defined in notation 0.3) as well-defined right-shifting functor. If \( X \) is pre-covering then \( C_X \) is left-triangulated with \( \Omega \) (which takes left-syzygies as defined in notation 0.3) as well-defined left-shifting functor.

If \( X \) is both pre-enveloping and pre-covering, then \( (\Sigma, \Omega) \) are adjoint functors on \( C_X \).

If \( f \) is a morphism in \( C \), then \( f \) denotes its class in \( C_X \).
**Notation 0.5** (The Spanier-Whitehead category). Given a right-triangulated category $S$, [3, def. 3.1] constructs what I call the **Spanier-Whitehead category** of $S$, and denote by $\text{Stab } S$ (**Stab** for stabilization). Its objects have the form $(S, m)$ with $S \in S$ and $m \in \mathbb{Z}$. Its morphisms are given by

$$\text{Hom}_{\text{Stab}(S)}((S, m), (T, n)) = \text{colim}_i \text{Hom}_S(S^{m+i}S, S^{n+i}T).$$

(Note that [3, def. 3.1] denotes the Spanier-Whitehead category by $\mathcal{S}(S)$.)

$\text{Stab } S$ is triangulated, and there is a functor $S \rightarrow \text{Stab } S$ given by $S \mapsto (S, 0)$. It sends distinguished right-triangles to distinguished triangles, and is universal among such functors in the sense that if $T$ is any triangulated category and $S \rightarrow T$ is any functor sending distinguished right-triangles to distinguished triangles, then there is a unique triangulated functor $S^*$ making the following diagram commutative:

$$\begin{array}{ccc}
S & \xrightarrow{G} & \text{Stab } S \\
S & \downarrow{s} & \downarrow{s^*} \\
T & & \\
\end{array}$$

**Notation 0.6** (Compactly generated categories). Let $S$ be a right-triangulated category with small coproducts, and let $\mathcal{C}$ be a class of objects in $S$. I say that $\mathcal{C}$ is a **generating small set of compacts** if

- $\mathcal{C}$ is a small set;
- $\mathcal{C}$ consists of compacts;
- $\mathcal{C}$ generates; that is, if $\text{Hom}_S(C, M) = 0$ for some $M$ in $S$ and all $C$ in $\mathcal{C}$, then $M \cong 0$.

If $S$ has a generating small set of compacts, then I say that $S$ is **compactly generated**. This coincides with the usual definition, [21, def. 1.7], if $S$ is triangulated.

**Setup 0.7.** The following are blanket assumptions in sections 1, 2, and 3:

- $\mathcal{C}$ is an abelian category with exact small coproducts;
- $\mathcal{X}$ is a class of objects in $\mathcal{C}$ which is pre-enveloping and pre-covering, and I always suppose that $\mathcal{X}$-right- and $\mathcal{X}$-left-assignments have been made;
- $\mathcal{X}$ is closed under small coproducts and under direct summands;
- Each $X$ in $\mathcal{X}$ is a direct summand in an object of the form $\prod_{\alpha} X_{\alpha}$, where each $X_{\alpha}$ is in $\mathcal{X}$ and is compact in $\mathcal{C}$;
- Each $M$ which is compact in $\mathcal{C}$ has an $\mathcal{X}$-pre-envelope $M \rightarrow X$ for which $X$ is compact in $\mathcal{C}$.

□
Example 0.8. As indicated above, the principal examples satisfying the conditions in setup 0.7 come from the representation theory of Artin algebras: Let $\Lambda$ be an Artin algebra, set $C = \text{Mod} \Lambda$, and set $X = \mathcal{P} \text{proj} \Lambda$, where $\mathcal{P} \text{proj} \Lambda$ denotes the class of projective $\Lambda$-modules. It will be proved in lemma 4.4.1 that this places me in the situation of setup 0.7.

The same is the case if $X$ is replaced with $\mathcal{I} \text{nj} \Lambda$, the injective $\Lambda$-modules, as one can see by arguments similar to those in the proof of lemma 4.4.1.

In case of confusion, I would suggest the reader to keep either of these examples in mind. This makes things look more familiar: If $C = \text{Mod} \Lambda$ and $X = \mathcal{I} \text{nj} \Lambda$, then $X$-right-resolutions are just ordinary injective resolutions, $\Sigma$ is just taking the first syzygy in an injective resolution, and so on. □

Remark 0.9. The blanket assumptions from setup 0.7 have several consequences.

1. The quotient functor $Q : C \longrightarrow C_X$ respects small coproducts by [15, lem. 1.7], and it is easy to check that $Q$ sends compact $C$-objects to compact $C_X$-objects.

2. A small coproduct of $X$-pre-covers is an $X$-pre-cover. This follows because each $X$ in $X$ is a direct summand in an object of the form $\coprod_{\alpha} X_{\alpha}$ where each $X_{\alpha}$ is in $X$ and is compact in $C$.

3. The left-shifting functor $\Omega : C_X \longrightarrow C_X$ respects small coproducts. This follows from (2), and from $C$ having exact small coproducts.

4. The right-shifting functor $\Sigma : C_X \longrightarrow C_X$ sends compact objects to compact objects, because $\pi(\Sigma M, -) = \pi(M, \Omega(-))$, and because $\Omega$ respects small coproducts by (3).

5. The full subcategory $(C_X)^c$ of $C_X$, consisting of compact objects, is a right-triangulated subcategory of $C_X$: If $A \longrightarrow B$ is a morphism of compacts, then (4) says that $\Sigma A$ is compact, and if I take a distinguished triangle $A \longrightarrow B \longrightarrow C \longrightarrow \Sigma A$ in $C_X$, then the five-lemma easily implies that $C$ is compact. □

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1. The functors $Z^0$ and $Sp$

This section introduces Spectra, the category of “spectra of modules” (definition 1.1). There is a canonical functor $Z^0$ from Spectra to the homotopy category $C_X$ (proposition 1.5), and through a number of technicalities (definition 1.6 to lemma 1.16) I aim to prove in theorem 1.17
that there exists a “spectrification functor” $\text{Sp}$, going from the homotopy
category $\mathcal{C}_\mathbb{X}$ to $\text{Spectra}$, which is left-adjoint to $\mathbb{Z}^0$.

The functor $\text{Sp}$ is my main tool for the study of $\text{Spectra}$, so I go on
to prove in theorem 1.19 that $\text{Sp}$ respects distinguished triangles, and in
theorem 1.22 that one can compute $\text{Sp}$ on morphisms using homotopy
colimits.

The following introduces my main object of study. The important part
of the definition is taken from [3, thm. 3.11].

**Definition 1.1.** Let $i$ be an integer. An $i$-spectrum is a complex $A$ in
$\mathbb{K}(\mathbb{X})$ satisfying

$$h^j \text{Hom}_\mathcal{C}(X, A) = 0 \text{ for each } j \leq i \text{ and each } X \text{ in } \mathbb{X}.$$ 

If $A$ is an $i$-spectrum for each $i$, then $A$ is called a spectrum.

The category of spectra and homotopy classes of chain maps is denoted
$\text{Spectra}$; it is a full subcategory of $\mathbb{K}(\mathbb{X})$.

**Remark 1.2.** I am really only interested in spectra; an $i$-spectrum is
just a technical gadget.

If $A$ is an $i$-spectrum, then $A$ is also a $j$-spectrum for any $j \leq i$.

The condition that $A$ in $\mathbb{K}(\mathbb{X})$ is an $i$-spectrum can also be phrased

$$\text{Hom}_{\mathbb{K}(\mathbb{X})}(X, A[j]) = 0 \text{ for each } j \leq i \text{ and each } X \text{ in } \mathbb{X},$$

and it can also be phrased

The canonical map $A^{j-1} \xrightarrow{\sim} Z^j A$ is an $\mathbb{X}$-epic for $j \leq i$.

\[ \Box \]

The category $\text{Spectra}$ has some good elementary properties:

**Proposition 1.3.** Spectra equipped with the distinguished $\mathbb{K}(\mathbb{X})$-triangles
which it contains is a triangulated subcategory of $\mathbb{K}(\mathbb{X})$.

**Proof.** If two vertices in a distinguished triangle in $\mathbb{K}(\mathbb{X})$ are spectra,
then it is easy to see that the third vertex is also a spectrum. \[ \Box \]

**Proposition 1.4.** Let \{\(A_{\alpha}\)\} be a small set of $i$-spectra. Then $\coprod A_{\alpha}$ is
again an $i$-spectrum.

Consequently, when \{\(A_{\alpha}\)\} is a small set of spectra, then $\coprod A_{\alpha}$ is again
a spectrum, so the category $\text{Spectra}$ has small coproducts.

**Proof.** Let $j$ be an integer so that $j \leq i$, and let $X$ be in $\mathbb{X}$. Pick $X'$
in $\mathbb{X}$ so that $X \coprod X' = \coprod_{\beta} X'_{\beta}$ for a small set of $X'_{\beta}$'s which are compact
and in $\mathcal{X}$. Then
\[
\text{Hom}_{K(\mathcal{X})}(X \coprod \alpha A_\alpha[j]) = \text{Hom}_{K(\mathcal{X})}(X_\beta \coprod \alpha A_\alpha[j]) = \prod_\alpha \text{Hom}_{K(\mathcal{X})}(X_\beta, A_\alpha[j]) = \prod_\beta \prod_\alpha \text{Hom}_{K(\mathcal{X})}(X_\beta, A_\alpha[j]) = (*) ,
\]
and each $\text{Hom}_{K(\mathcal{X})}(X_\beta, A_\alpha[j])$ is zero, so $(\ast)$ is zero. But then certainly
\[
\text{Hom}_{K(\mathcal{X})}(X, \coprod \alpha A_\alpha[j]) = 0,
\]
proving that $\prod A_\alpha$ is an $i$-spectrum. \hfill \square

The following proposition introduces the functor $Z^0$ which is basic to my studies of Spectra. In particular, I will soon find myself searching for a left-adjoint to $Z^0$.

**Proposition 1.5.** Taking the zeroth group of cycles of a chain complex gives a functor from the category of complexes and chain maps to the category $\mathcal{C}$,

\[
\text{Complex}(\mathcal{C}) \longrightarrow \mathcal{C}.
\]

If $C$ and $A$ both consist of $\mathcal{X}$-objects, then the corresponding homomorphism

\[
\text{Hom}_{\text{Complex}(\mathcal{C})}(C, A) \longrightarrow \text{Hom}_{\mathcal{C}}(Z^0C, Z^0A)
\]

induces a well-defined homomorphism

\[
\text{Hom}_{K(\mathcal{C})}(C, A) \longrightarrow \pi(Z^0C, Z^0A).
\]

In consequence, taking the zeroth group of cycles gives a well-defined functor

\[
K(\mathcal{X}) \xrightarrow{Z^0} \mathcal{C}_\mathcal{X}.
\]

Moreover, this functor respects small coproducts, and the restriction

\[
\text{Spectra}(\mathcal{X}) \xrightarrow{Z^0} \mathcal{C}_\mathcal{X}
\]

is also a functor respecting small coproducts.

**Proof.** It is a small diagram chase to check that as claimed, when $C$ and $A$ consist of $\mathcal{X}$-objects, then there is an induced, well-defined homomorphism

\[
\text{Hom}_{K(\mathcal{C})}(C, A) \longrightarrow \pi(Z^0C, Z^0A).
\]

To see that the functor

\[
K(\mathcal{X}) \xrightarrow{Z^0} \mathcal{C}_\mathcal{X}
\]

respects coproducts, I note first that taking the zeroth group of cycles at the level of $\text{Complex}(\mathcal{C})$ respects coproducts because $\mathcal{C}$ has exact small
coproducts. Next I observe that coproducts in $K(X)$ are induced by coproducts in $\text{Complx}(C)$, while coproducts in $C_X$ are induced by coproducts in $C$ by remark 0.9.1. But now the desired conclusion is clear.

The proposition’s final observation is clear since coproducts in $\text{Spectra}$ are just given by coproducts in $K(X)$.

\[ \square \]

It will turn out that the left-adjoint of $Z^0 : \text{Spectra} \to C_X$ is a “specification functor” which sends an object from $C_X$ to its corresponding “suspension spectrum”. To perform the necessary constructions, I need some more technical gadgets.

**Definition 1.6.** Given $M \in C$. Let

\[
\begin{array}{ccc}
M & \longrightarrow & C^0 \\
& \longrightarrow & C^1 \\
& \longrightarrow & \ldots
\end{array}
\]

be the assigned $X$-right-resolution of $M$, and let

\[
\begin{array}{ccc}
& \longrightarrow & C^{-2} \\
\ldots & \longrightarrow & C^{-1} \\
& \longrightarrow & M
\end{array}
\]

be the assigned $X$-left-resolution of $M$, using a slightly strange numbering of the $C$’s. Then I let $C_M$ denote the concatenation

\[
\begin{array}{ccc}
\ldots & \longrightarrow & C^{-2} \\
& \longrightarrow & C^{-1} \\
& \longrightarrow & C^0 \\
& \longrightarrow & C^1 \\
& \longrightarrow & \ldots
\end{array}
\]

where the map $C^{-1} \to C^0$ is the composite

\[
C^{-1} \longrightarrow M \longrightarrow C^0.
\]

**Remark 1.7.** It is easy to check that each $C_M$ is a $(-2)$-spectrum. \[ \square \]

Reading the following lemma in the right spirit, one can already see a left-adjoint to $Z^0 : \text{Spectra} \to C_X$ in the making.

**Lemma 1.8.** Given $M \in C$. Then there is a canonical $C$-morphism

\[
\mu : M \longrightarrow Z^0 C_M,
\]

and if $A$ is a $0$-spectrum, then the composition

\[
(*) \quad \text{Hom}_{K(X)}(C_M, A) \xrightarrow{Z^0(-)} \pi(Z^0 C_M, Z^0 A) \xrightarrow{\pi(\mu, Z^0 A)} \pi(M, Z^0 A)
\]

is bijective.

**Proof.** The composition

\[
\begin{array}{ccc}
M & \longrightarrow & C^0 \\
& \longrightarrow & C^1
\end{array}
\]

is zero, and this gives the canonical $C$-morphism $M \xrightarrow{\mu} Z^0 C_M$. It is a diagram chase to see that $(*)$ is bijective. \[ \square \]

I also need a bit of information about other maps out of $C_M$. 
Lemma 1.9. 1. Given a morphism \( M \xrightarrow{g} N \) in \( \mathcal{C} \), there exists a chain map \( C_M \xrightarrow{\beta} C_N \) fitting into a commutative diagram

\[
\begin{array}{ccc}
\cdots & \longrightarrow & C_M^{-1} \\
\downarrow{\beta^{-1}} & & \downarrow{\beta^0} \\
\cdots & \longrightarrow & C_N^{-1} \\
\downarrow{g} & & \downarrow{g} \\
\cdots & \longrightarrow & C_N^0 & \longrightarrow & \cdots \\
\end{array}
\]

2. Given a morphism \( \Sigma M \xrightarrow{f} \Sigma N \) in \( \mathcal{C} \), there exists a chain map \( C_M \xrightarrow{\alpha} C_{\Sigma N}[-1] \) fitting into a commutative diagram

\[
\begin{array}{ccc}
\cdots & \longrightarrow & C_M^0 \\
\downarrow{\alpha^0} & & \downarrow{\alpha^1} \\
\cdots & \longrightarrow & C_{\Sigma N}^0 \\
\downarrow{f} & & \downarrow{f} \\
\cdots & \longrightarrow & C_{\Sigma N}^1 & \longrightarrow & \cdots, \\
\end{array}
\]

and when \( f \) and \( \alpha \) fit together in such a diagram, then \([\alpha]\) is determined uniquely by \( f \).

Proof. (1) When \( g \) is given, I use that \( C_N \)'s left hand part is an \( \mathcal{X} \)-left-resolution of \( N \) to construct \( \beta^{-1}, \beta^{-2}, \ldots \), and I use that \( C_M \)'s right hand part is an \( \mathcal{X} \)-right-resolution of \( M \) to construct \( \beta^0, \beta^1, \ldots \).
(2) When the map \( f \) is given, I use that \( C_{\Sigma N} \)'s left hand part is an \( \mathfrak{X} \)-left-resolution of \( \Sigma N \) to construct \( \alpha^0, \alpha^{-1}, \ldots \), and I use that \( C_M \)'s right hand part is an \( \mathfrak{X} \)-right-resolution of \( M \) to construct \( \alpha^1, \alpha^2, \ldots \).

It is a diagram chase to prove that \( f \) uniquely determines \( [\alpha] \). \( \square \)

Some useful morphisms can now be obtained:

**Definition 1.10.** Using the notation of lemma 1.9.2, let \( N \) equal \( M \) and let \( f = \text{id}_{\Sigma N} \). Then lemma 1.9 gives a chain map \( C_M \xrightarrow{\alpha} C_{\Sigma M}[-1] \) which I will denote by \( \alpha_M \).

Of course, there could be many different possibilities for \( \alpha_M \). But note that \( [\alpha_M] \) is well-defined by lemma 1.9.

The following proposition contains the construction which will turn out to give the functor \( \text{Sp} \).

**Lemma 1.11.** Given \( M \in \mathcal{C} \). Consider the direct system in \( \mathbf{K}(\mathfrak{X}) \),

\[
\begin{array}{ccc}
C_M & \xrightarrow{\alpha_M} & C_{\Sigma M}[-1] \xrightarrow{\alpha_{\Sigma M}[-1]} C_{\Sigma^2 M}[-2] \xrightarrow{\alpha_{\Sigma^2 M}[-2]} & \cdots.
\end{array}
\]

The homotopy colimit of the system is a spectrum.

**Proof.** I must check that

\[
\text{Hom}_{\mathbf{K}(\mathfrak{X})}(X, (\text{hocollim}_i \ C_{\Sigma^i M}[-i])[j]) = 0
\]

for \( X \) in \( \mathfrak{X} \) and each integer \( j \). Pick \( X' \) in \( \mathfrak{X} \) so that \( X \amalg X' = \coprod \beta X_\beta \) for certain \( X_\beta \)'s which are compact and in \( \mathfrak{X} \). Now

\[
\begin{align*}
\text{Hom}_{\mathbf{K}(\mathfrak{X})}(X \amalg X', (\text{hocollim}_i \ C_{\Sigma^i M}[-i])[j]) &= \text{Hom}_{\mathbf{K}(\mathfrak{X})}(\coprod \beta X_\beta[-j], \text{hocollim}_i \ C_{\Sigma^i M}[-i]) \\
&= \prod \beta \text{Hom}_{\mathbf{K}(\mathfrak{X})}(X_\beta[-j], \text{hocollim}_i \ C_{\Sigma^i M}[-i]) \\
&= \prod \beta \text{colim}_i \text{Hom}_{\mathbf{K}(\mathfrak{X})}(X_\beta[-j], C_{\Sigma^i M}[-i]) \\
&= \prod \beta \text{colim}_i \text{Hom}_{\mathbf{K}(\mathfrak{X})}(X_\beta, C_{\Sigma^i M}[j-i]) \\
&= (\ast),
\end{align*}
\]

where “(\( a \))” is by [21, lem. 2.8]. Now, each \( C_{\Sigma^i M} \) is a \((-2)\)-spectrum, so \( \text{Hom}_{\mathbf{K}(\mathfrak{X})}(X_\beta, C_{\Sigma^i M}[j-i]) \) is zero for \( j-i \leq -2 \), that is for \( i \geq j+2 \). Thus each colim appearing in \((\ast)\) is zero, whence

\[
(\ast) = 0.
\]

This obviously implies

\[
\text{Hom}_{\mathbf{K}(\mathfrak{X})}(X, (\text{hocollim}_i \ C_{\Sigma^i M}[-i])[j]) = 0
\]

as desired. \( \square \)

I can now define what \( \text{Sp} \) is on objects.
Definition 1.12. For each $M$ in $C$, I consider the direct system in lemma 1.11 and pick a homotopy colimit which is then a spectrum. I denote this spectrum by $\text{Sp} M$, and denote the canonical morphism $C_M \longrightarrow \text{Sp} M$ in $K(X)$ by $[\text{can}_M]$.

To see that $\text{Sp}$ does what I want it to do, that is, gives a left-adjoint to $Z^0 : \text{Spectra} \longrightarrow C_X$, a few lemmas are handy.

Lemma 1.13. Let $S$ be a right-triangulated category with small coproducts where $\Sigma$ respects small coproducts, and let

$$S_0 \overset{s_0}{\longrightarrow} S_1 \overset{s_1}{\longrightarrow} S_2 \overset{s_2}{\longrightarrow} \cdots$$

be a direct system in $S$.

1. Given $A \in S$. There is a short exact sequence

$$0 \longrightarrow \lim^1 \text{Hom}_S(\Sigma S_i, A) \longrightarrow \text{Hom}_S(\hocolim S_i, A) \longrightarrow \lim \text{Hom}_S(S_i, A) \longrightarrow 0,$$

where the surjection is the canonical homomorphism.

Consequently, if $A$ has the property:

each $\text{Hom}_S(\Sigma S_i, A)$ is surjective,

then the canonical homomorphism

$$\text{Hom}_S(\hocolim S_i, A) \longrightarrow \lim \text{Hom}_S(S_i, A)$$

is an isomorphism.

2. Let $h : S \longrightarrow \text{Ab}$ be a homological functor (i.e. a functor sending distinguished right-triangles to exact sequences) which respects small coproducts. Then the canonical homomorphism

$$\text{colim} h(S_i) \longrightarrow h(\hocolim S_i)$$

is bijective.

Proof. (1) The first part can be proved just like [19, prop. 3.4(b)], and the second part follows from the first part and [19, prop. A.1.10(b)].

(2) This can be proved like [19, prop. 4.1(a)].

Lemma 1.14. Let $M \in C$ be given, and let $A$ be a $1$-spectrum. Then the homomorphism

$$\text{Hom}_{K(X)}(C_{\Sigma M}[-1], A) \overset{\text{Hom}_{K(X)}([\text{can}_M], A)}{\longrightarrow} \text{Hom}_{K(X)}(C_M, A)$$

is an isomorphism.

Proof. Lemma 1.8 gives the two isomorphisms in

$$\text{Hom}_{K(X)}(C_{\Sigma M}[-1], A) \overset{\cong}{\longrightarrow} \pi(\Sigma M, Z^1 A) = \pi(M, Z^1 A) = \pi(M, \omega Z^1 A)$$

and

$$\text{Hom}_{K(X)}(C_M, A) \overset{\cong}{\longrightarrow} \pi(M, Z^0 A).$$
so all I need is to check that the diagram is commutative. But that can be done by examining in detail the way lemma 1.8 obtains its isomorphisms.

\[\]

Lemma 1.13.1 and lemma 1.14 combine to give:

**Lemma 1.15.** Let \(M \in C\) be given, and let \(A\) be a spectrum. The canonical map \(C_M \xrightarrow{\text{can}_M} \text{Sp} M\) induces an isomorphism

\[
\text{Hom}_{K(X)}(\text{Sp} M, A) \xrightarrow{\text{Hom}_{\text{can}_M}, A} \text{Hom}_{K(X)}(C_M, A).
\]

**Proof.** First, by lemma 1.14 and its obvious shifted versions, all the homomorphisms in the inverse system

\[
\text{Hom}_{K(X)}(C_M, A) \leftarrow \text{Hom}_{K(X)}(C_{\Sigma M}[-1], A) \leftarrow \text{Hom}_{K(X)}(C_{\Sigma^2 M}[-2], A) \leftarrow \cdots
\]

are isomorphisms, so the canonical homomorphism

\[
\lim \text{Hom}_{K(X)}(C_{\Sigma^i M}[-i], A) \rightarrow \text{Hom}_{K(X)}(C_M, A)
\]

(1.1)

is an isomorphism.

Secondly, since each homomorphism in the above inverse system is an isomorphism for any spectrum \(A\), I can replace \(A\) by \(A[-1]\) and still get isomorphisms which are in particular surjective. Lemma 1.13.1 then says that the canonical homomorphism

\[
\text{Hom}_{K(X)}(\text{Sp} M, A) = \text{Hom}_{K(X)}(\text{hocolim} C_{\Sigma^i M}[-i], A)
\]

(1.2)

\[
\rightarrow \lim \text{Hom}_{K(X)}(C_{\Sigma^i M}[-i], A)
\]

is an isomorphism.

But the present lemma’s homomorphism is the composition of the homomorphisms from equations (1.2) and (1.1), so is also an isomorphism.

\[\]

I am getting very close to adjointness between \(\text{Sp}\) and \(Z^0\):

**Lemma 1.16.** Let \(M\) be in \(C\), and let \(A\) be a spectrum. Then there is an isomorphism

\[
\text{Hom}_{\text{Spectra}}(\text{Sp} M, A) \rightarrow \pi (M, Z^0 A)
\]

which is natural in \(A\).

**Proof.** By definition, \(\text{Hom}_{\text{Spectra}}(\text{Sp} M, A)\) is \(\text{Hom}_{K(X)}(\text{Sp} M, A)\), so lemma 1.15 gives an isomorphism

\[
\text{Hom}_{\text{Spectra}}(\text{Sp} M, A) \rightarrow \text{Hom}_{K(X)}(C_M, A)
\]

which is natural in \(A\). And from lemma 1.8 I have an isomorphism

\[
\text{Hom}_{K(X)}(C_M, A) \rightarrow \pi (M, Z^0 A)
\]

which is also natural in \(A\). Composing the two gives what I want. \(\square\)

At last, I can prove this section’s main result:
**Theorem 1.17.** The map on objects $M \mapsto \text{Sp } M$ can be supplemented in a unique way with a map on morphisms $\varphi \mapsto \text{Sp } \varphi$ so that

$$\text{Sp} : \mathcal{C}_X \to \text{Spectra}$$

becomes a functor which is left-adjoint to the functor

$$Z^0 : \text{Spectra} \to \mathcal{C}_X.$$

**Proof.** It is well known how this follows formally from lemma 1.16, using Yoneda’s lemma. Having the functor Sp, I will prove two important things about it: First, it respects distinguished triangles (theorem 1.19). Secondly, there is a concrete way of computing what Sp does to a morphism (theorem 1.22).

As one might expect, some technicalities are needed.

**Lemma 1.18.** Consider categories and functors,

$$T \xleftarrow{Z} \xrightarrow{F} S,$$

satisfying:

- $(F, Z)$ is an adjoint pair;
- $T$ is a triangulated category;
- $S$ is a right- and left-triangulated category where the right- and left-shifting functors $\Sigma$ and $\Omega$ form a adjoint pair $(\Sigma, \Omega)$, and where the following holds: If a commutative diagram is given,

$$\begin{array}{ccc}
A & \rightarrow & B & \rightarrow & C & \rightarrow & \Sigma A \\
\downarrow{\alpha} & & \downarrow & & \downarrow{\sigma} & & \\
\Omega P & \rightarrow & M & \rightarrow & N & \rightarrow & P,
\end{array}$$

where the upper row is a distinguished right-triangle and the lower row is a distinguished left-triangle, and where $\alpha$ and $\sigma$ correspond under the adjunction, there exists a morphism $C \to N$ making the whole diagram commutative;

- $Z$ satisfies $Z(-) \circ (-)[-1] \simeq \Omega(-) \circ Z(-)$, and sends distinguished triangles to distinguished left-triangles.

Then $F(-) \circ \Sigma(-) \simeq (-)[1] \circ F(-)$, and $F$ sends distinguished right-triangles to distinguished triangles.

**Proof.** The lemma looks more complicated than it is — its statement is really just that “the left-adjoint of a triangulated functor is triangulated”. To prove the lemma, one can easily adapt the proof of [20, lem. 3.9].
Theorem 1.19. The functor \( S_{\mathcal{S}} : C_{\mathcal{C}} \rightarrow \text{Spectra} \) satisfies \( S_{\mathcal{C}}(-) \circ \Sigma(-) \simeq (-)[1] \circ S_{\mathcal{C}}(-) \), and sends distinguished right-triangles to distinguished triangles.

Proof. I will prove this by using lemma 1.18 on

\[
\begin{array}{c}
\text{Spectra} \\
\xrightarrow{Z^0} \\
\xleftarrow{S_{\mathcal{S}}} C_{\mathcal{C}}.
\end{array}
\]

First, \((S_{\mathcal{S}}, Z^0)\) is an adjoint pair by theorem 1.17, and \text{Spectra} is a triangulated category.

Next, I want to see that lemma 1.18’s conditions on the category \( S \) are satisfied for the category \( C_{\mathcal{C}} \). This is immediate by [2, def. 4.9] and subsequent remarks which tell me that the lemma’s conditions are satisfied for any category of the form \( C_{\mathcal{C}} \), when \( C \) is abelian and \( \mathcal{C} \) is both pre-covering and pre-enveloping.

Finally, I need to check that \( Z^0 \) sends the shift \((-)[-1]\) to the left-shift \( \Omega \), and sends distinguished triangles to distinguished left-triangles. But by [3, thm. 3.11] and its proof, the pair \((\text{Spectra}, Z^0)\) is in fact the costabilization of \( C_{\mathcal{C}} \) with respect to \( \Omega \) (see [3, def. 3.10] for the definition), and the functor \( Z^0 \) must then send shift and distinguished triangles in its source, \text{Spectra}, to shift and distinguished left-triangles in its target, \( C_{\mathcal{C}} \).

\( \square \)

Lemma 1.20. Let \( M \rightarrow N \) be a morphism in \( C \). By lemma 1.9.1, \( \varphi \) gives a (possibly non-unique) chain map \( C_M \rightarrow C_N \). Using the shift \( \Sigma \), I also get a (possibly non-unique) morphism \( \Sigma M \rightarrow \Sigma N \) for which \( \psi = \Sigma(\varphi) \), and using lemma 1.9 again, \( \psi \) gives a (possibly non-unique) chain map \( C_{\Sigma M} \rightarrow C_{\Sigma N} \).

Now the square

\[
\begin{array}{c}
C_M \\
\xrightarrow{[\alpha_M]} \\
\downarrow{[\Phi]} \\
C_N \\
\xrightarrow{[\alpha_N]} \\
\downarrow{[\psi[-1]]} \\
C_{\Sigma M}[-1] \\
C_{\Sigma N}[-1]
\end{array}
\]

is commutative.

Proof. The lemma claims that \( \alpha_N \circ \Phi \) equals \( \Psi[-1] \circ \alpha_M \) modulo homotopy. But both these chain maps fit into commutative diagrams like the one in lemma 1.9.2, with \( \psi \) (or something which has the same class as \( \psi \) in \( C_{\mathcal{C}} \)) in place of \( f \). And by lemma 1.9.2, such a commutative diagram determines the homotopy class of its chain map uniquely. \( \square \)
Construction 1.21. Let \( M \xrightarrow{\varphi} N \) be a morphism in \( C \). Using lemma 1.20 repeatedly, I get a commutative diagram in \( \underline{K} \),

\[
\begin{array}{ccc}
C_M & \xrightarrow{[\alpha_M]} & C_{\Sigma M}[-1] & \xrightarrow{[\alpha_{\Sigma M}[-1]]} & C_{\Sigma^2 M}[-2] & \xrightarrow{[\alpha_{\Sigma^2 M}[-2]]} & \cdots \\
\downarrow{[\Phi]} & & \downarrow{[\Phi]} & & \downarrow{[\Phi]} & & \\
C_N & \xrightarrow{[\alpha_N]} & C_{\Sigma N}[-1] & \xrightarrow{[\alpha_{\Sigma N}[-1]]} & C_{\Sigma^2 N}[-2] & \xrightarrow{[\alpha_{\Sigma^2 N}[-2]]} & \cdots 
\end{array}
\]

This induces a (possibly non-unique) morphism between the homotopy colimits,

\[
\operatorname{Sp} M \xrightarrow{[s]} \operatorname{Sp} N,
\]

which fits into a commutative square in \( \underline{K} \),

\[
\begin{array}{ccc}
C_M & \xrightarrow{\operatorname{can}_M} & \operatorname{Sp} M \\
\downarrow{[\Phi]} & & \downarrow{[s]} \\
C_N & \xrightarrow{\operatorname{can}_N} & \operatorname{Sp} N
\end{array}
\]

where the horizontal morphisms are the canonical ones. \( \square \)

**Theorem 1.22.** Let \( M \xrightarrow{\varphi} N \) be a morphism in \( C \), and let \( \operatorname{Sp} M \xrightarrow{[s]} \operatorname{Sp} N \) be as in construction 1.21. Then \([s]\) is uniquely determined and is \([s] = \operatorname{Sp} \varphi\).

**Proof.** Let \( A \) be a spectrum. I can get a diagram

\[
\begin{array}{ccc}
\operatorname{Hom}_{\operatorname{Spectra}}(\operatorname{Sp} N, A) & \xrightarrow{\operatorname{Hom}_{\operatorname{Spectra}}([s], A)} & \operatorname{Hom}_{\operatorname{Spectra}}(\operatorname{Sp} M, A) \\
\downarrow{\operatorname{Hom}_{\underline{K}}(C_N, A)} & & \downarrow{\operatorname{Hom}_{\underline{K}}(C_M, A)} \\
\pi(N, Z^0 A) & \xrightarrow{\pi([s], Z^0 A)} & \pi(M, Z^0 A),
\end{array}
\]

where the upper square comes from applying \( \operatorname{Hom}_{\underline{K}}(\cdot, A) \) to the commutative square at the end of construction 1.21, and where the two lower slanted arrows come from lemma 1.8.
The upper square is obviously commutative. The lower square is commutative, as one can check by examining in detail how lemma 1.8 performs its construction. So the whole diagram is commutative. And as one can see examining the proof of lemma 1.16, the vertical homomorphisms are adjunction isomorphisms. Finally, the square consisting of the four outer corners is natural with respect to $A$, because the four morphisms in the square are all natural.

But the square consisting of the four outer corners is in fact the device by which $\text{Sp}$ is made into a functor, in that the top vertical homomorphism is $\text{Hom}_{\text{Spectra}}(\text{Sp} \varphi, A)$. So

$$\text{Hom}_{\text{Spectra}}(\text{Sp} \varphi, A) = \text{Hom}_{\text{Spectra}}([s], A)$$

whence

$$[s] = \text{Sp} \varphi.$$

\[ \square \]

2. Embedding the Spanier-Whitehead category in $\text{Spectra}$

Theorem 2.1 below remarks that since $\text{Sp}$ sends distinguished right-triangles in $C_X$ to distinguished triangles in $\text{Spectra}$, it has a unique factorization $\text{Sp}^*$ through $C_X$'s Spanier-Whitehead category $\text{Stab } C_X$. I can hope that $\text{Spectra}$ will be a sort of completion of the compact part of the Spanier-Whitehead category, $\text{Stab}((C_X)^c)$, and indeed, it turns out that after more technicalities (lemmas 2.3 to 2.6), I can prove in theorem 2.7 that $\text{Sp}^*$ embeds $\text{Stab}((C_X)^c)$ as a full triangulated subcategory of $\text{Spectra}$.

**Theorem 2.1.** Consider the Spanier-Whitehead stabilization

$$C_X \xrightarrow{G} \text{Stab } C_X$$

(see notation 0.5). There is a unique triangulated functor $\text{Sp}^* : \text{Stab } C_X \rightarrow \text{Spectra}$ which makes the following diagram commutative:

$$\begin{array}{ccc}
C_X & \xrightarrow{G} & \text{Stab } C_X \\
\downarrow \text{Sp} & & \downarrow \text{Sp}^* \\
\text{Spectra} & & \text{Spectra}.
\end{array}$$

**Proof.** This is immediate by notation 0.5, because I have already proved that $\text{Sp}$ sends distinguished right-triangles to distinguished triangles. \[ \square \]

**Remark 2.2.** By remark 0.9.5, the full subcategory $(C_X)^c$ of $C_X$, consisting of compact objects, is a right-triangulated subcategory of $C_X$. For a moment, let $S$ denote the restriction of $\text{Sp}$ to $(C_X)^c$; then $S$ sends distinguished right-triangles to distinguished triangles, so if $\text{Stab}((C_X)^c)$ is...
the Spanier-Whitehead category of \((C_X)^c\), then there is a unique factorization,

\[
\begin{array}{c}
\begin{array}{c}
(C_X)^c \\
\rightarrow \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\text{Stab}((C_X)^c) \\
\downarrow \\
\text{Spectra.} \\
\end{array}
\end{array}
\end{array}
\]

However, the construction of the stabilization makes it easy to see that \(\text{Stab}((C_X)^c)\) is simply the subcategory of \(\text{Stab} C_X\) consisting of objects \((M, m)\) where \(M\) is in \((C_X)^c\), and that \(S^*\) is simply the restriction to this subcategory of \(\text{Sp}^*\).

I shall frequently commit the sin of denoting \(\text{Sp}^*\)'s restriction to \(\text{Stab}((C_X)^c)\) by \(\text{Sp}^*\).

I need some computations before going for the section’s main result.

**Lemma 2.3.** Let \(M \in (C_X)^c\), \(N \in C_X\), and \(k \in \mathbb{Z}\) be given. Then the canonical homomorphism

\[
\text{colim}_i \text{Hom}_{K(X)}(C_M[-k], C_{\Sigma^i N}[-i]) \xrightarrow{\rho} \text{Hom}_{K(X)}(C_M[-k], \text{hocolim}_i C_{\Sigma^i N}[-i])
\]

is an isomorphism.

**Proof.** The distinguished triangle which defines the homotopy colimit,

\[
\prod C_{\Sigma^i N}[-i] \xrightarrow{\text{id-shift}} \prod C_{\Sigma^i N}[-i] \xrightarrow{\text{hocolim}} C_{\Sigma^i N}[-i],
\]

gives a long exact sequence containing

\[
\text{Hom}(C_M[-k], \prod C_{\Sigma^i N}[-i]) \xrightarrow{\alpha} \text{Hom}(C_M[-k], \prod C_{\Sigma^i N}[-i]) \xrightarrow{\beta} \text{Hom}(C_M[-k], \text{hocolim} C_{\Sigma^i N}[-i]) \xrightarrow{\gamma} \text{Hom}(C_M[-k], \prod C_{\Sigma^i N}[-i+1])
\]

Before starting the proof proper, I need to rewrite \(\alpha\): The Hom at either end of \(\alpha\) can be rewritten,

\[
\text{Hom}(C_M[-k], \prod_{0 \leq i} C_{\Sigma^i N}[-i])
\]

\[
= \text{Hom}(C_M[-k], \prod_{0 \leq i \leq k+1} C_{\Sigma^i N}[-i] \amalg \prod_{k+2 \leq i} C_{\Sigma^i N}[-i])
\]

\[
= \prod_{0 \leq i \leq k+1} \text{Hom}(C_M[-k], C_{\Sigma^i N}[-i]) \amalg \text{Hom}(C_M[-k], \prod_{k+2 \leq i} C_{\Sigma^i N}[-i])
\]

\[
= \prod_{0 \leq i \leq k+1} \text{Hom}(C_M[-k], C_{\Sigma^i N}[-i]) \amalg \text{Hom}(C_M, \prod_{k+2 \leq i} C_{\Sigma^i N}[k-i])
\]

\[
= (\ast).
\]
Now, $C_{\Sigma^N}$ is a $(-2)$-spectrum, so $C_{\Sigma^N}[k-i]$ is a $(-2-k+i)$-spectrum. When $k+2 \leq i$ then $-2-k+i \geq 0$, so $C_{\Sigma^N}[k-i]$ is a 0-spectrum. By proposition 1.4 the same is then true for the coproduct

$$\prod_{k+2 \leq i} C_{\Sigma^N}[k-i].$$

But then lemma 1.8 gives me the first "=" in

$$(\star) = \prod_{0 \leq i \leq k+1} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \amalg \prod_{k+2 \leq i} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[k-i])$$

$$(a) = \prod_{0 \leq i \leq k+1} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \amalg \prod_{k+2 \leq i} \operatorname{Hom}(M, Z^0(C_{\Sigma^N}[k-i]))$$

$$(b) = \prod_{0 \leq i \leq k+1} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \amalg \prod_{k+2 \leq i} \operatorname{Hom}(M, Z^0(C_{\Sigma^N}[k-i]))$$

$$(c) = \prod_{0 \leq i \leq k+1} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \amalg \prod_{k+2 \leq i} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i])$$

$$(\gamma) = \prod_{0 \leq i \leq k+1} \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \amalg \prod_{k+2 \leq i} \operatorname{Hom}(C_M, C_{\Sigma^N}[-i])$$

where "(a)" is because $C$ has exact small coproducts, "(b)" is because $M$ is compact in $C_{\Sigma}$, and "(c)" is by another application of lemma 1.8.

So the coproduct appearing inside the Hom which is source and target of $a$ can be moved outside. Consequently, $a$ can be rewritten

$$\prod_i \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]) \xrightarrow{\text{id-shift}} \prod_i \operatorname{Hom}(C_M[-k], C_{\Sigma^N}[-i]).$$

Of course, the same trick can be applied to $d$.

Now for the proof of the lemma: Using the above rewriting of $a$ shows that the colimit appearing in the lemma is $\operatorname{Coker} a$, and that $b$ gives the lemma's homomorphism $\rho$. Since $\ker b = \operatorname{Im} a$, it follows that $\rho$ is injective.

To see that $\rho$ is surjective, I need to see that $b$ is surjective. This is equivalent to $c$ being zero, which is again equivalent to $d$ being injective. But the alternative way of writing $d$ given above moves the "id-shift" outside the Hom's, so $d$ is clearly injective. \hfill $\Box$

**Lemma 2.4.** Let $M, N \in C$ be given, and look at

$$\begin{array}{cccccc}
M & \longrightarrow & X^0 & \longrightarrow & \cdots & \longrightarrow & X^{i-1} & \longrightarrow & \Sigma^i M \\
f \downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \\
\Omega^i N & \longleftarrow & X'_{i-1} & \longrightarrow & \cdots & \longrightarrow & X'_0 & \longrightarrow & N,
\end{array}$$
where the upper row is part of $M$’s assigned $X$-right-resolution, the lower row is part of $N$’s assigned $X$-left-resolution, and where $f$ is a given homomorphism which gives rise to the successive extensions marked by broken arrows.

Then the rule $f \mapsto g$ gives a well-defined homomorphism

$$\pi(M, \Omega^i N) \to \pi(\Sigma^i M, N)$$

which is equal to the adjunction isomorphism.

Proof. This follows by an easy induction on $i$. 

Lemma 2.5. Let $i \geq 2$ and let $C_M \xrightarrow{[n]} C_{\Sigma^i N}[-i]$ be given. Since $C_{\Sigma^i N}[-i]$ is a 0-spectrum, Lemma 1.8 says that $[\gamma]$ corresponds to $M \xrightarrow{\phi} Z^0(C_{\Sigma^i N}[-i]) = \Omega^i \Sigma^i N$, which again by adjointness corresponds to $\Sigma^i M \xrightarrow{h} \Sigma^i N$. By Lemma 1.9.1, the morphism $h$ induces a (possibly non-unique) chain map $C_{\Sigma^i M} \xrightarrow{[n]} C_{\Sigma^i N}$.

Now the diagram

\[
\begin{array}{ccc}
C_M & \xrightarrow{c} & C_{\Sigma^i M}[-i] \\
\downarrow{[\gamma]} & & \downarrow{[n][-i]} \\
C_{\Sigma^i N}[-i], & & \\
\end{array}
\]

where $[c] = [\alpha_{\Sigma^{i-1} M}[-(i-1)] \circ \cdots \circ [\alpha_M]$, is commutative.
Proof. I look at the commutative diagram

\[
\begin{array}{cccc}
C_{M}^{1} & \xrightarrow{e^{-1}} & C_{\Sigma^{i} M} [-i]^{-1} & \xrightarrow{\eta[-i]^{-1}} & C_{\Sigma^{i} N} [-i]^{-1} \\
C_{M}^{i} & \xrightarrow{e^{i}} & C_{\Sigma^{i} M} [-i]^{i} & \xrightarrow{\eta[-i]^{i}} & C_{\Sigma^{i} N} [-i]^{i} \\
C_{M}^{0} & \xrightarrow{e^{0}} & C_{\Sigma^{i} M} [-i]^{0} & \xrightarrow{\eta[-i]^{0}} & C_{\Sigma^{i} N} [-i]^{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

Using that the left hand complex contains an $X$-right-resolution of $M$, while the right hand complex contains an $X$-left-resolution of $\Sigma^{i} N$, lemma 2.4 says that the $C_{X}$-class of the diagram’s morphism $M \xrightarrow{f} \Omega^{i} \Sigma^{i} N$ corresponds to $\bar{h}$ under the adjunction between $\Sigma^{i}$ and $\Omega^{i}$, so $f = g$.

But then lemma 1.8 applied to the diagram says that $[\eta[-i] \circ [c]]$ is the unique homotopy class of chain maps induced by $g$, in other words,

\[ [\eta[-i] \circ [c]] = [\gamma]. \]

The conclusion to all the preceding calculations is:

**Lemma 2.6.** The triangulated functor

\[ \text{Sp}^{*} : \text{Stab}((C_{X})^{i}) \longrightarrow \text{Spectra} \]

is full and faithful.

Proof. I shall prove this by means of [3, prop. 3.4] which gives simple criteria, in terms of Sp, for $\text{Sp}^{*}$ to be full, respectively faithful.
Namely, to prove that \( \text{Sp}^* \) is faithful, I must check that if \( \varphi \) is a morphism in \((\mathcal{C}_\mathcal{X})^c\) with \( \text{Sp} \varphi = 0 \), then \( \Sigma^i \varphi = 0 \) for \( i \gg 0 \).

And to prove that \( \text{Sp}^* \) is full, I must check that if \( M \) and \( N \) are in \((\mathcal{C}_\mathcal{X})^c\) and \( \text{Sp} M \xrightarrow{[i]} \text{Sp} N \) is a morphism, then there is a shift \([s[i]]\) which can be written \([s[i]] = \text{Sp} \underline{h} \) for some \( \Sigma^i M \xrightarrow{h} \Sigma^i N \).

**Faithfulness:** Using the above remarks, I let \( M \xrightarrow{\varphi} N \) be a morphism in \((\mathcal{C}_\mathcal{X})^c\) satisfying \( \text{Sp} \varphi = 0 \), and must check \( \Sigma^i \varphi = 0 \) when \( i \) is big.

The conclusion to construction 1.21 and theorem 1.22 is that I have a commutative square

\[
\begin{array}{ccc}
C_M & \xrightarrow{[\text{can}_M]} & \text{Sp} M \\
\downarrow{\Phi} & & \downarrow{\text{Sp} \varphi} \\
C_N & \xrightarrow{[\text{can}_N]} & \text{Sp} N,
\end{array}
\]

so under the homomorphism

\[
\text{Hom}_{\mathcal{K}(\mathcal{X})}(\text{Sp} M, \text{Sp} N) \xrightarrow{\text{Hom}_{\mathcal{K}(\mathcal{X})}([\text{can}_M], \text{Sp} N)} \text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, \text{Sp} N),
\]

\( \text{Sp} \varphi \) gets sent to \( \text{Sp}(\varphi) \circ [\text{can}_M] = [\text{can}_N] \circ [\Phi] \), so since \( \text{Sp} \varphi = 0 \), I have \([\text{can}_N] \circ [\Phi] = 0 \).

But lemma 2.3 gives an isomorphism

\[
\text{colim} \text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, C_{\Sigma^i N} [-i]) \xrightarrow{\cong} \text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, \text{Sp} N)
\]

under which, certainly, the element in the colimit represented by \([\Phi]\) gets sent to \([\text{can}_N] \circ [\Phi] \), that is, gets sent to zero.

So \([\Phi]\) represents zero in the colimit, whence its image in

\[
\text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, C_{\Sigma^i N} [-i])
\]

is zero when \( i \) is big. But lemma 1.8 gives the first “\( \cong \)" in

\[
\text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, C_{\Sigma^i N} [-i]) \cong \pi(M, Z^0(C_{\Sigma^i N} [-i])) = \pi(M, \Omega^i \Sigma^i N) \cong \pi(\Sigma^i M, \Sigma^i N)
\]

for \( i \geq 2 \), and it is easy to check that under this isomorphism, \([\Phi]\)'s image in \( \text{Hom}_{\mathcal{K}(\mathcal{X})}(C_M, C_{\Sigma^i N} [-i]) \) gets sent to \( \Sigma^i \varphi \). Consequently, \( \Sigma^i \varphi = 0 \).

**Fullness:** Using the remarks at the beginning of the proof, I let \( M \) and \( N \) be in \((\mathcal{C}_\mathcal{X})^c\) and let \( \text{Sp} M \xrightarrow{[i]} \text{Sp} N \) be a morphism, and need to find \( \Sigma^i M \xrightarrow{h} \Sigma^i N \) so that \( \text{Sp} \underline{h} = [s[i]] \).
To begin with, I observe that the homomorphism

$$\text{Hom}_{K(X)}(\langle \alpha_{\Sigma^i-1,M}[1] \rangle \circ \cdots \circ \langle \alpha_{M}[i] \rangle, (\text{Sp} N)[i])$$

is an isomorphism. This is immediate by \( i \) applications of lemma 1.14.

Now, \([s] \circ \text{can}_{M}\) is an element in \(\text{Hom}_{K(X)}(C_M, \text{Sp} N)\). Looking again at the isomorphism in equation (2.1), the colimit contains an element which gets sent to \([s] \circ \text{can}_{M}\). Hence there is an \( i \) and a homotopy class of chain maps \(C_M \xrightarrow{[\gamma]} C_{\Sigma^i,N}[-i]\) such that the element in the colimit represented by \([\gamma]\) gets sent to \([s] \circ \text{can}_{M}\). This says that there is a commutative square,

\[
\begin{array}{ccc}
C_M & \xrightarrow{\text{can}_{M}} & \text{Sp} M \\
\downarrow{[\gamma]} & & \downarrow{[s]} \\
C_{\Sigma^i,N}[-i] & \xrightarrow{\text{can}_{\Sigma^i,N}[-i]} & \text{Sp} N.
\end{array}
\]

And by lemma 2.5 there is a morphism \(\Sigma^i M \xrightarrow{h} \Sigma^i N\) inducing a homotopy class of chain maps \(C_{\Sigma^i,M} \xrightarrow{[\eta]} C_{\Sigma^i,N}\) so that

\[
[\eta][-i] \circ [\alpha_{\Sigma^i-1,M}[-(i-1)]] \circ \cdots \circ [\alpha_{M}] = [\gamma]. \quad (2.2)
\]

Now, on one hand,

\[
\text{Hom}(\langle \alpha_{\Sigma^i-1,M}[1] \rangle \circ \cdots \circ \langle \alpha_{M}[i] \rangle, (\text{Sp} N)[i])\langle [s][i] \circ \text{can}_{\Sigma^i,M}\rangle
\]

\[
= \text{Hom}(\langle \alpha_{\Sigma^i-1,M}[1] \rangle \circ \cdots \circ \langle \alpha_{M}[i] \rangle, (\text{Sp} N)[i])
\]

\[
\circ \text{Hom}(\langle \text{can}_{\Sigma^i,M}\rangle, (\text{Sp} N)[i])\langle [s][i]\rangle
\]

\[
= \text{Hom}(\text{can}_{\Sigma^i,M} \circ \langle \alpha_{\Sigma^i-1,M}[1] \rangle \circ \cdots \circ \langle \alpha_{M}[i] \rangle, (\text{Sp} N)[i])\langle [s][i]\rangle
\]

\[
= \text{Hom}(\text{can}_{\Sigma^i,M}[i], (\text{Sp} N)[i])\langle [s][i]\rangle
\]

\[
= [s][i] \circ \text{can}_{M}[i]
\]

\[
= \text{can}_{\Sigma^i,N} \circ [\gamma][i],
\]

where the last “=” is by the above commutative square. On the other hand,

\[
\text{Hom}(\langle \alpha_{\Sigma^i-1,M}[1] \rangle \circ \cdots \circ \langle \alpha_{M}[i] \rangle, (\text{Sp} N)[i])\langle \text{can}_{\Sigma^i,N} \circ [\eta]\rangle
\]

\[
= \text{can}_{\Sigma^i,N} \circ [\eta] \circ [\alpha_{\Sigma^i-1,M}[1] \circ \cdots \circ [\alpha_{M}[i]]
\]

\[
= \text{can}_{\Sigma^i,N} \circ [\gamma][i],
\]

where the last “=” is by equation (2.2). Comparing the two computations, and using the observation that \(\text{Hom}(\alpha_{\Sigma^i-1,M}[1] \circ \cdots \circ \alpha_{M}[i], (\text{Sp} N)[i])\) is an isomorphism, I get

\[
[s][i] \circ \text{can}_{\Sigma^i,M} = \text{can}_{\Sigma^i,N} \circ [\eta].
\]
This says there is a commutative square
\[ C_{\Sigma^t M} \xrightarrow{[\text{can}_{\Sigma^t M}]} (\text{Sp} M)[i] \]
\[ \downarrow \quad \downarrow \]
\[ C_{\Sigma^t N} \xrightarrow{[\text{can}_{\Sigma^t N}]} (\text{Sp} N)[i]. \]

But arguing like the proof of theorem 1.22, I then get
\[ [s[i]] = \text{Sp} L. \]

□

The following is this section’s main result.

**Theorem 2.7.** \( \text{Sp}^* \) induces an equivalence of triangulated categories
\[ \text{Stab}((C_{X})^c) \cong F, \]
where \( F \) is the full subcategory of \( \text{Spectra} \) given by
\[ F = \{(\text{Sp} M)[i] \mid M \in (C_{X})^c, i \in \mathbb{Z}\}. \]

**Proof.** By lemma 2.6, the functor
\[ \text{Sp}^* : \text{Stab}((C_{X})^c) \rightarrow \text{Spectra} \]
is full and faithful, and it is easy to see that its image is exactly \( F \). So [18, thm. IV.4.1] makes it clear that \( \text{Sp}^* \) induces an equivalence of categories as claimed in the theorem.

And the equivalence respects triangles: Since \( \text{Sp}^* \) is triangulated, the equivalence sends distinguished triangles to distinguished triangles. Conversely, suppose given a distinguished triangle in \( F \),
\[ A \rightarrow B \rightarrow C \rightarrow A[1]. \quad (2.3) \]
I can find a morphism \( (M, m) \rightarrow (N, n) \) in \( \text{Stab}((C_{X})^c) \) which is sent to
\[ A \rightarrow B \text{ by } \text{Sp}^*. \]
Completing to a distinguished triangle,
\[ (M, m) \rightarrow (N, n) \rightarrow (P, p) \rightarrow (M, m + 1), \quad (2.4) \]
\( \text{Sp}^* \) sends (2.4) to a distinguished triangle
\[ A \rightarrow B \rightarrow \text{Sp}^*(P, p) \rightarrow A[1], \]
and it is immediate from the axioms of triangulated categories that this is isomorphic to (2.3). So any distinguished triangle in \( F \) comes from one in \( \text{Stab}((C_{X})^c) \). □
3. Compact objects in Spectra

It is easy to do computations in the category $\text{Stab}((C_X)^c)$. Theorem 2.7 therefore ought to have some consequences for Spectra. Indeed, this section shows that if $C_X$ is compactly generated, then so is Spectra (theorem 3.2). And under more restrictive assumptions, I can even give a precise description of Spectra’s compact objects (theorem 3.7 and corollary 3.8) which turn out to be exactly the objects coming from $\text{Stab}((C_X)^c)$ under $\text{Sp}^*$. This corresponds exactly to topology, where the compacts in the homotopy category of spectra form a subcategory which is equivalent to the part of the Spanier-Whitehead category consisting of compact CW complexes.

The following is a machine for generating compacts in Spectra.

**Proposition 3.1.** If $M$ is compact in $C_X$, then $\text{Sp}M$ is compact in Spectra.

*Proof.* By theorem 1.17, I have

$$\text{Hom}_{\text{Spectra}}(\text{Sp}M, -) \cong \pi(M, Z^0(-)),$$

and by proposition 1.5, the functor $Z^0$ respects small coproducts, so the result is clear. \qed

The machine of proposition 3.1 turns out to be efficient enough for the following main result.

**Theorem 3.2.** Suppose that $\mathcal{C}$ is a generating small set of compacts in $C_X$ (see notation 0.6 for the meaning of this). Then

$$\mathcal{D} = \{(\text{Sp}C)[i] \mid C \in \mathcal{C}, i \in \mathbb{Z}\}$$

is a generating small set of compacts in Spectra.

In particular, if $C_X$ is compactly generated, then Spectra is compactly generated.

*Proof.* It is clear that $\mathcal{D}$ is a small set, and it consists of compacts by proposition 3.1.

To see that $\mathcal{D}$ generates, let $A$ be a spectrum for which

$$\text{Hom}_{\text{Spectra}}((\text{Sp}C)[i], A) = 0$$

for each $C$ in $\mathcal{C}$ and each integer $i$. By theorem 1.17, this says $\pi(C, Z^{-i}A) = 0$ for each $C$ in $\mathcal{C}$ and each $i$. By assumption on $\mathcal{C}$ this says $Z^{-i}A = 0$ in $C_X$ for each $i$. In other words, $Z^{-i}A \in X$ for each $i$.

But as each $A^{i-1} \xrightarrow{s_A^{i-1}} Z^i A$ is $X$-epic, I can then find $Z^j A \xrightarrow{\tilde{\varphi}_j} A^{j-1}$ with $s_A^{i-1} \tilde{\varphi}_j = \text{id}_{Z^j A}$, and this makes it easy to see that $A$ is in fact split exact, hence isomorphic to zero in $K(X)$, hence isomorphic to zero in Spectra. \qed

It is sometimes useful to know not only that a triangulated category is compactly generated, but actually to know the compacts. Before I can
prove more detailed results about the compacts in \( \text{Spectra} \) in some cases, I need a few tools.

**Lemma 3.3.** Let \( T \) be a triangulated category with small coproducts. Then each \( T \) in \( T \) only has a small set of isomorphism classes of direct summands.

*Proof.* Suppose that \( T = T_1 \coprod T_2 \), and let \( T \to T \) be the projection onto \( T_1 \). Then by [6, rem. 3.3], the direct system \( T \to T \to T \to \cdots \) has \( T_1 \) as its homotopy colimit. That is, up to isomorphism I can find \( T_1 \) by knowing \( e \). But \( e \) is an element in the small set \( \text{Hom}_\pi(T, T) \). \( \square \)

**Lemma 3.4.** Let \( T \) be a triangulated category with a full triangulated subcategory \( U \), and let \( \hat{U} \) be the full subcategory

\[
\{ \hat{U} \mid \text{there exists } \hat{V} \text{ s.t. } \hat{U} \coprod \hat{V} \text{ is isomorphic to an object in } U \}
\]

(The coproduct is taken in \( T \), and some would call \( \hat{U} \) the épaisse closure of \( U \).)

Then \( \hat{U} \) is also a full triangulated subcategory of \( T \), and if \( T \) has small coproducts and \( U \) is essentially small, then \( \hat{U} \) is also essentially small.

*Proof.* Let \( X \to Y \) be a morphism in \( \hat{U} \), and look at a distinguished triangle

\[
X \to Y \to Z \to X[1]
\]

in \( T \). I want to see that \( Z \) is in \( \hat{U} \), up to isomorphism. Pick \( X' \) and \( Y' \) so that \( X \coprod X' \) and \( Y \coprod Y' \) are in \( U \). Look at the distinguished triangle

\[
X' \to Y' \to Z' \to X'[1]
\]

in \( T \). The coproduct of the two distinguished triangles is again a distinguished triangle,

\[
X \coprod X' \to Y \coprod Y' \to Z \coprod Z' \to (X \coprod X')[1],
\]

but since \( U \) is triangulated, I now see that \( Z \coprod Z' \) is in \( U \) up to isomorphism, whence \( Z \) is in \( \hat{U} \) up to isomorphism.

Now suppose that \( T \) has small coproducts. Then by lemma 3.3 each \( X \) in \( U \) only has a small set of isomorphism classes of direct summands. So if \( U \) is essentially small, the same obviously holds for \( \hat{U} \). \( \square \)

**Lemma 3.5.** Suppose that \( C_X \) has a generating small set of compacts, \( C \), and that moreover, \( C \) forms a full right-triangulated subcategory of \( C_X \).

Then the compact objects in \( \text{Spectra} \) are exactly the direct summands in objects of the form \( \text{Sp} C[i] \) for \( C \in C \) and \( i \in \mathbb{Z} \). Moreover, there is only a small set of isomorphism classes of compacts in \( \text{Spectra} \).

*Proof.* It is clear that

\[
\text{Stab} C = \{(C, i) \mid C \in C, i \in \mathbb{Z} \}
\]
is a small triangulated subcategory of $\text{Stab}((C_X)^c)$. It is also clear that up to isomorphism, the image of $\text{Stab}\mathcal{C}$ under $\text{Sp}^*$ is the set

$$\mathcal{D} = \{(\text{Sp} C_i)[i] \mid C_i \in \mathcal{C}, i \in \mathbb{Z}\}$$

from theorem 3.2. From these facts follows that

- $\mathcal{D}$ is a triangulated subcategory of $\text{Spectra}$ and
- $\text{Sp}^*|_{\text{Stab}\mathcal{C}} : \text{Stab}\mathcal{C} \rightarrow \mathcal{D}$

is an equivalence of triangulated categories (since $\text{Sp}^*$ on compacts is full and faithful, and $\text{Sp}^*|_{\text{Stab}\mathcal{C}}$ is surjective on objects up to isomorphism when viewed as going into $\mathcal{D}$);

- $\mathcal{D}$ is a generating small set of compact objects in $\text{Spectra}$ (by theorem 3.2).

By proposition 1.4, $\text{Spectra}$ has small coproducts, so from lemma 3.4 follows that $\mathcal{D}$'s épaisse closure, $\hat{\mathcal{D}}$, is an essentially small triangulated subcategory of $\text{Spectra}$. And $\hat{\mathcal{D}}$ clearly consists of compacts, and since it contains $\mathcal{D}$, any set of representatives of $\hat{\mathcal{D}}$'s isomorphism classes forms a generating small set of compact objects of $\text{Spectra}$.

But now the Neeman-Thomason localization theorem, [21, thm. 2.1], applies with $\mathcal{S} = \text{Spectra}$ and $R = \hat{\mathcal{D}}$. Part 2.1.2 of the theorem says $\mathcal{R} = \mathcal{S}$ and part 2.1.3 says $R = R^c$. In other words,

$$\hat{\mathcal{D}} = R = R^c = \mathcal{S}^c = \text{Spectra}^c,$$

proving the lemma’s claim about the form of compact objects in $\text{Spectra}$, and also showing that there is only a small set of isomorphism classes of compacts, since $\hat{\mathcal{D}}$ is essentially small. 

My aim now is to remove the need for taking direct summands in lemma 3.5.

**Lemma 3.6.** Let $\mathcal{S}$ be a right-triangulated category with small coproducts where $\Sigma$ respects small coproducts. Then each idempotent in $\mathcal{S}$ splits, meaning: If $e \in \text{End}_\mathcal{S}(M)$ is an idempotent, then there exists a biproduct diagram in $\mathcal{S}$,

$$M_1 \xleftarrow{i} M \xrightarrow{q} M_2,$$

where $e = i_1p_1$. (See [18, p. 190] for information about biproduct diagrams.)

**Proof.** It is easy to adapt [6, rem. 3.3] to prove this. 

I can now prove this section’s second main result:

**Theorem 3.7.** Suppose that $C_X$ has a generating small set of compacts, $\mathcal{C}$, and that moreover, $\mathcal{C}$ forms a full right-triangulated subcategory of $C_X$ which is closed under taking direct summands.
Then there is an equivalence of categories
\[ \text{Stab}\, \mathcal{C} \rightarrow \text{Spectra}^c \]
induced by \( \text{Sp}^* \).

Proof. Lemma 3.5’s proof already shows that \( \text{Sp}^* \) induces an equivalence of categories
\[ \text{Stab}\, \mathcal{C} \rightarrow \mathcal{D} = \{ (\text{Sp}\, C)[i] \mid C \in \mathcal{C}, i \in \mathbb{Z} \}, \]
and that \( \text{Spectra}^c \) consists of all direct summands in objects from \( \mathcal{D} \). So I will be done if I can prove that under the present theorem’s assumptions, \( \mathcal{D} \) is closed (up to isomorphism) under taking direct summands in \( \text{Spectra}^c \).

So suppose that \( C \) is in \( \mathcal{C} \) and that \( (\text{Sp}\, C)[i] \) splits as a coproduct, and let
\[ A_1 \xleftarrow{p_1} (\text{Sp}\, C)[i] \xrightarrow{p_2} A_2 \]
be the corresponding biproduct diagram. I want to check that \( A_1 \) is in the subcategory \( \mathcal{D} \).

\( e = i_1 p_1 \) is an idempotent in \( \text{End}_{\text{Spectra}}((\text{Sp}\, C)[i]) \). Since \( (C, i) \) is in \( \text{Stab}(\mathcal{C}_x)^c \) and \( \text{Sp}^* \)'s restriction to \( \text{Stab}(\mathcal{C}_x)^c \) is an equivalence of categories by theorem 2.7, and since \( \text{Sp}^*((C, i)) = (\text{Sp}\, C)[i] \), the idempotent \( e \) corresponds to an idempotent
\[ f \in \text{End}_{\text{Stab}(\mathcal{C}_x)^c}((C, i)) = \text{colim}_j \pi(\Sigma^{i+j} C, \Sigma^{i+j} C) \]
for which \( \text{Sp}^* f = e \). Let \( f \) be represented by \( g \) in \( \pi(\Sigma^{i+n} C, \Sigma^{i+n} C) \); since \( f^2 - f = 0 \) there is a \( p \) so that \( \Sigma^p (g^2 - g) = 0 \). Setting \( h = \Sigma^p g \) gives
\[ h^2 - h = \Sigma^p g \circ \Sigma^p g - \Sigma^p g = \Sigma^p (g^2 - g) = 0, \]
so \( h \) in \( \pi(\Sigma^{i+n+p} C, \Sigma^{i+n+p} C) \) is an idempotent. Note that \( h \) also represents \( f \) whence \( \text{Sp}\, h = e[n + p] \).

By lemma 3.6 there is a biproduct diagram in \( \mathcal{C}_x \),
\[ C_1 \xleftarrow{p_1} \Sigma^{i+n+p} C \xrightarrow{p_2} C_2, \]
for which \( i_1 p_1 = h \). Since \( \mathcal{C} \) is a right-triangulated subcategory of \( \mathcal{C}_x \), the shift \( \Sigma^{i+n+p} C \) is in \( \mathcal{C} \), and since \( \mathcal{C} \) is closed under direct summands, \( C_1 \) and \( C_2 \) are in \( \mathcal{C} \).

Applying \( \text{Sp} \) to the diagram gives a biproduct diagram in \( \text{Spectra} \),
\[ \text{Sp}\, C_1 \xleftarrow{\text{Sp} p_1} (\text{Sp}\, C)[i + n + p] \xrightarrow{\text{Sp} p_2} \text{Sp}\, C_2, \]
where \( \text{Sp} i_1 \circ \text{Sp} p_1 = \text{Sp}\, h = e[n + p] \).
Shifting by $-n - p$ gives a new biproduct diagram,

$$(\text{Sp} C_1)[-n - p] \xrightarrow{e} (\text{Sp} C)[i] \xrightarrow{\text{Sp} p_2} \text{Sp} C_2[-n - p],$$

where

$$e = (\text{Sp} i_2)[-n - p] \circ (\text{Sp} p_1)[-n - p],$$

so this new diagram must be isomorphic to the original biproduct diagram from the start of the proof. So $A_1 \cong (\text{Sp} C_1)[-n - p]$, and this is in the category $\mathcal{D}$. \hfill $\square$

The following extension of theorem 2.7 is now immediate:

**Corollary 3.8.** Suppose that $(C_X)^c$ only has a small set of isomorphism classes, and that a set of representatives of these forms a generating small set of compacts of $C_X$.

Then there is an equivalence of categories

$$\text{Stab}((C_X)^c) \longrightarrow \text{Spectra}^c$$

induced by $\text{Sp}^*$. \hfill $\square$

---

4. The Class of Gorenstein Projectives is Pre-Covering

This section uses the machinery of spectra to prove in theorem 4.14 that over an Artin algebra $\Lambda$, the class $\mathcal{G}$ of Gorenstein projective modules is a pre-covering class in $\text{Mod} \Lambda$. Previously, this was only known for rings satisfying strong homological conditions, see [10, thm. 2.9] and [11, thm. 3.4].

The idea of the proof is first to use Bousfield localization to see in corollary 4.11 that the class $\text{Enochs}$ of Enochs spectra (to be defined in definition 4.6) is a pre-covering class in $\text{Spectra}$, next to lift the result to modules.

Before starting the section proper, let me state the following abstract result on Bousfield localization which I will need:

**Theorem 4.1** (Bousfield localization). Let $\mathcal{T}$ be a compactly generated triangulated category with a small set of isomorphism classes of compact objects, let $A$ be an abelian AB5 category, and let $k : \mathcal{T} \longrightarrow A$ be a covariant homological functor preserving small coproducts. Let

$$E_k = \{ A \in \mathcal{T} \mid k(A[m]) = 0 \text{ for all } m \}.$$  

Then the Verdier localization $\mathcal{T}/E_k$, which is also called the Bousfield localization of $\mathcal{T}$ with respect to $k$, has small Hom sets.
**Remark 4.2.** The Verdier localization $\mathbb{T}/E_k$ is defined as the category $\Sigma_k^{-1}\mathbb{T}$ where all morphisms in the class

$$\Sigma_k = \{ f \text{ is a morphism in } \mathbb{T} \mid f \text{'s mapping cone is in } E_k \}$$

$$= \{ f \text{ is a morphism in } \mathbb{T} \mid k(f|m) \text{ is an isomorphism for each } m \}$$

have been inverted. Such a localization is again a triangulated category, but could have large Hom sets; theorem 4.1 gives a situation where the Hom sets are small. I will not prove theorem 4.1, only remark that one can get a proof by directly adapting Margolis’ proof from [19, chp. 7]. Margolis only states the theorem for the category of topological spectra, but his proof works just as well in the present higher generality. □

Now for the section proper. The following setup explains the notation to be used.

**Setup 4.3.** This section takes place in a specific case of setup 0.7:

First some new notation. $\Lambda$ is a fixed Artin algebra and $R$ is its centre, $S_1, \ldots, S_n$ is a system of representatives of the isomorphism classes of simple $R$-modules, and $J = \bigoplus_{i=1}^n E(S_i)$ is the direct sum of their injective envelopes. The functor

$$D(\cdot) = \text{Hom}_R(\cdot, J) : \text{mod}\Lambda \longrightarrow \text{mod}\Lambda^{\text{opp}}$$

is the usual duality, see [1, sec. II.3].

Next I fix the way setup 0.7’s notation is to be used:

- Setup 0.7’s category $C$ is $\text{Mod}\Lambda$;
- Setup 0.7’s class $X$ is $\text{Proj}\Lambda$, the class of projective $\Lambda$-modules.

Note that this means that $\text{Spectra}$ is the homotopy category of exact complexes of projectives.

Finally some more new notation: The functor

$$k : \text{Spectra} \longrightarrow \text{Ab}$$

is given by $k(\cdot) = h^0(D\Lambda \otimes_\Lambda \cdot)$ (Ab is the category of abelian groups). □

Of course, I need to substantiate that the specific case described in setup 4.3 satisfies the assumptions made in setup 0.7. That and a few other tasks are cleared away by the following lemma.

**Lemma 4.4.** Consider the situation described in setup 4.3.

1. All assumptions made in setup 0.7 hold.
2. $\text{Spectra}$ is compactly generated, the compact objects are exactly the direct summands in objects of the form $(\text{Sp} C)[i]$, where $C$ is a finitely generated $\Lambda$-module and $i \in \mathbb{Z}$, and there is only a small set of isomorphism classes of compact objects.
3. The functor $k : \text{Spectra} \longrightarrow \text{Ab}$ is a covariant homological functor which lands in an abelian AB5 category and preserves small coproducts.
Proof. (1) The assumptions of setup 0.7 are trivial apart from the ones that $\mathcal{P}_{\text{proj}} \Lambda$ is pre-enveloping and that each compact module has a compact $\mathcal{P}_{\text{proj}} \Lambda$-pre-envelope. These can be checked by first noting that the compact modules are exactly the finitely generated ones, and then using the methods of [16, ex. 5.1]. To see that the compact modules are exactly the finitely generated ones, use [22, théorème].

(2) This will follow from theorem 3.2 and lemma 3.5 if I can show that these two results apply to the situation where the set $\mathcal{C}$ is defined as follows: Take for $\mathcal{C}$ a set containing one representative of each isomorphism class of finitely generated $\Lambda$-modules. So taking such a $\mathcal{C}$, what I must check is that it is a generating small set of compacts, and forms a full, right-triangulated subcategory of $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$.

I first check that $\mathcal{C}$ forms a full right-triangulated subcategory of $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$.

Note that by the dual of [4, def. 2.8], if $M \xrightarrow{f} N$ is a morphism between two modules from $\mathcal{C}$, then I can get a distinguished right-triangle containing it by taking a $\mathcal{P}_{\text{proj}} \Lambda$-pre-envelope $M \xrightarrow{g} P$, taking the push out in $\text{Mod} \Lambda$,

\[
\begin{array}{c}
M \xrightarrow{f} N \\
\downarrow g \quad \quad \quad \downarrow h \\
\quad P \xrightarrow{\iota} \Sigma M,
\end{array}
\]

and taking

\[
M \xrightarrow{f} N \xrightarrow{h} C \xrightarrow{\iota} \Sigma M, \tag{4.1}
\]

where $i$ is constructed in an obvious way. Since $M$ is in $\mathcal{C}$, it is finitely generated, so by the methods of [16, ex. 5.1] I can pick $P$ finitely generated. But $N$ is also in $\mathcal{C}$, hence finitely generated, so $C$ is finitely generated. Now I can replace $C$ by the module in $\mathcal{C}$ to which it is isomorphic, and then the distinguished triangle (4.1) consists of objects from $\mathcal{C}$. Hence $\mathcal{C}$ forms a full right-triangulated subcategory of $\mathcal{C}_\Lambda$.

To see that $\mathcal{C}$ forms a small generating set of compacts of $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$, first observe that by remark 0.9.1, $\mathcal{C}$ consists of compacts in $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$.

Now let $M$ be a module such that $\pi(C, M) = 0$ for each $C \in \mathcal{C}$. I need to show $M \cong 0$ in $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$; in other words, I need to show that $M$ is in $\mathcal{P}_{\text{proj}} \Lambda$, i.e. that $M$ is projective.

I use the criterion of [8, chp. VI, ex. 6] to prove that $M$ is flat, whence $M$ is also projective because $\Lambda$ is an Artin algebra: Let $m_1, \ldots, m_n$ be elements in $M$ and let $\lambda_1, \ldots, \lambda_n$ be elements of $\Lambda$ so that $\sum \lambda_i m_i = 0$. Let $M' = \Lambda m_1 + \cdots + \Lambda m_n$. Since $M'$ is finitely generated, it is isomorphic to a module in $\mathcal{C}$ so $\pi(M', M) = 0$. In particular, the inclusion $M' \hookrightarrow M$
equals zero in \( \pi(M', M) \), so factors as \( M' \xrightarrow{\gamma} P \xrightarrow{\varphi} M \) where \( P \) is projective, hence flat. Now \( \sum_i \lambda_i \gamma(m_i) = 0 \), so by [8, chp. VI, ex. 6], there exist elements \( p_j \) in \( P \) and \( s_{ij} \) in \( \Lambda \) so that \( \gamma(m_i) = \sum_j s_{ij}p_j \) and \( \sum_i \lambda_is_{ij} = 0 \). But by the first of these equations,

\[
m_i = \iota(m_i) = \varphi\gamma(m_i) = \varphi(\sum_j s_{ij}p_j) = \sum_j s_{ij}\varphi(p_j).
\]

And this shows that the \( s_{ij} \) and the \( \varphi(p_j) \) do what is necessary by [8, chp. VI, ex. 6] to show that \( M \) is flat.

(3) Off hand, \( D\Lambda \otimesdığımız \Lambda \) is a triangulated functor from \( K(\Lambda) \) to \( K(\Lambda) \) which clearly respects small coproducts. So off hand, the functor \( h^0(D\Lambda \otimes gpointer) \) is a covariant homological functor from \( K(\Lambda) \) to \( \text{Mod} \Lambda \) which respects small coproducts.

But note that by construction, Spectra is a full triangulated subcategory of \( K(\Lambda) \). So restricting \( h^0(D\Lambda \otimesBạn) \) to Spectra and tacitly composing it with the forgetful functor \( \text{Mod} \Lambda \to \text{Ab} \) certainly gives a homological functor which respects small coproducts. And \( \text{Ab} \) is an abelian AB5 category.

After this excursion, I am in a position to use the theory of sections 1 to 3.

Let me first recall from [10, sec. 1] the concept of Gorenstein projectives:

**Definition 4.5.** A module \( M \) is called Gorenstein projective if there exists a complex of projective modules, \( P \), satisfying:

- \( M = \ker d^0_P \);
- \( P \) is exact;
- \( \text{Hom}_\Lambda(P, Q) \) is exact when \( Q \) is a projective module.

I denote the class of Gorenstein projective modules by \( \mathcal{G} \). Note that it contains \( \mathcal{P}_{\text{proj}} \Lambda \).

This gives one the idea for the following definition:

**Definition 4.6.** A complex \( E \in K(\mathcal{P}_{\text{proj}} \Lambda) \) is called an Encho complex if it satisfies: For \( X \in \mathcal{P}_{\text{proj}} \Lambda \), both \( \text{Hom}_\Lambda(X, E) \) and \( \text{Hom}_\Lambda(E, X) \) are exact.

The category of Encho complexes and homotopy classes of chain maps is denoted \( \text{E} \).

Clearly, Encho consists of exact complexes of projectives, so is a subcategory of Spectra. It is direct from the definitions that I have

**Lemma 4.7.** The Gorenstein projectives are exactly the modules of the form \( Z^0E \) for \( E \in \text{E} \).

The following alternative description of Encho is the motivation for introducing the functor \( k \) of setup 4.3:
Proposition 4.8. Enochs can be obtained as
\[
\text{Enochs} = \{ A \in \text{Spectra} \mid k(A[m]) = 0 \text{ for all } m \}.
\]

Proof. For a moment, let me think of \( k(-) = h^0(DA \otimes_A -) \) as taking values in \( \text{Mod } \Lambda \). Then
\[
\text{Hom}_R(k(A[m]), J) = \text{Hom}_R(h^m(DA \otimes_A A), J)
= h^{-m} \text{Hom}_R(DA \otimes_A A, J)
= h^{-m} \text{Hom}_A(A, \text{Hom}_R(DA, J))
= h^{-m} \text{Hom}_A(A, A),
\]
so the proposition’s category
\[
\{ A \in \text{Spectra} \mid k(A[m]) = 0 \text{ for all } m \}
\]
consists of exactly those spectra \( A \) for which \( \text{Hom}_A(A, \Lambda) \) is exact. But this is equivalent to \( A \) being in Enochs:

On one hand, by definition of Enochs, if \( A \) is in Enochs, then \( \text{Hom}_A(A, \Lambda) \) is certainly exact. On the other hand, suppose that \( A \) is a spectrum for which \( \text{Hom}_A(A, \Lambda) \) is exact. Then \( \text{Hom}_A(A, \prod I \Lambda) \) is exact for any small index set \( I \). But by [14, thm. 8.1(ii’)], \( \Lambda \) is \( \Sigma \)-pure injective as a \( \Lambda \)-module, so by [14, thm. 8.1(ii)], \( \prod I \Lambda \) is a direct summand in \( \prod I \Lambda \) for each small index set \( I \). So when \( \text{Hom}_A(A, \prod I \Lambda) \) is always exact, then so is \( \text{Hom}_A(A, \prod I \Lambda) \), and then clearly, so is \( \text{Hom}_A(A, P) \) for any projective module \( P \). Hence \( A \) is in Enochs. \( \square \)

I can now prove a key result:

Proposition 4.9. Each of the inclusion functors
\[
i_* : \text{Enochs} \hookrightarrow \text{Spectra} \quad \text{and} \quad j_* : \text{Spectra} \hookrightarrow \text{K Proj } \Lambda
\]
has a right-adjoint. Consequently, the inclusion
\[
j_* i_* : \text{Enochs} \hookrightarrow \text{K Proj } \Lambda
\]
also has a right-adjoint.

Proof. By lemma 4.4.2, \( \text{Spectra} \) is compactly generated and only has a small set of isomorphism classes of compacts.

Using theorem 4.1 with \( S = \text{Spectra} \) and \( A = \text{Ab} \) and \( k \) equal to the present section’s concrete \( k \), and using that by proposition 4.8 the present section’s category Enochs equals theorem 4.1’s \( E_k \), I get that the Verdier localization \( \text{Spectra} / \text{Enochs} = S / E_k \) has small Hom sets, whence [17, lem. 3.5] says that the inclusion \( i_* \) has a right adjoint.

And \( j_* \) is a triangulated functor defined on a compactly generated triangulated category, which respects small coproducts, so \( j_* \) has a right adjoint by the Brown adjoint functor theorem, [21, thm. 4.1].

Finally, the right-adjoint of \( j_* i_* \) is the composition of the right-adjoints of \( i_* \) and \( j_* \). \( \square \)

The following proposition takes note of a trivial consequence of the existence of right-adjoints to inclusions.
Proposition 4.10. Let $K$ be a category, let $E$ be a class of objects in $K$, think of $E$ as a full subcategory of $K$, and let $i_* : E \rightarrow K$ be the inclusion functor. Suppose that $i_*$ has a right-adjoint $i^* : K \rightarrow E$.

If $k$ is in $K$, then the counit $i_* i^* k \xrightarrow{\epsilon_k} k$ is an $E$-pre-cover of $k$.

Proof. Given $e \in E$ and $k \in K$, the composition
\[
\text{Hom}_E(e, i^* k) \xrightarrow{i_*} \text{Hom}_K(i_* e, i_* i^* k) \rightarrow \text{Hom}_K(i_* e, k)
\]
is the adjunction isomorphism, so $\text{Hom}_K(i_* e, \epsilon_k)$ must be surjective. In other words, any morphism $i_* e \rightarrow k$ lifts through $\epsilon_k$. \qed

So now I can prove:

Corollary 4.11. Viewed as a class of objects, Enochs is pre-covering in $K(\text{Proj}_\Lambda)$.

Proof. Combine propositions 4.9 and 4.10. \qed

As said at the beginning of this section, the idea now is to “lift” this corollary to the module level to get a proof that the class of Gorenstein projectives is pre-covering. First two technical results:

Lemma 4.12. Let $E$ be an Enochs complex. Then each $E^j \xrightarrow{s^j_E} Z^{j+1} E$ is $\text{Proj}_\Lambda$-epic (i.e. surjective) and each $Z^{j+1} E \xrightarrow{i_E^{j+1}} E^{j+1}$ is $\text{Proj}_\Lambda$-monic.

Proof. This follows easily because $\text{Hom}_\Lambda(X, E)$ and $\text{Hom}_\Lambda(E, X)$ are exact when $X$ is projective and $E$ is an Enochs complex. \qed

Lemma 4.13. Let $E$ be an Enochs complex, let $M$ be in $\text{Mod}_\Lambda$, let $\cdots \rightarrow X^{-2} \rightarrow X^{-1} \xrightarrow{\xi} M$ be an unconventionally indexed $\text{Proj}_\Lambda$-left-resolution of $M$ (i.e. a projective resolution), and write $X$ for the complex
\[
\cdots \rightarrow X^{-2} \rightarrow X^{-1} \rightarrow 0 \rightarrow \cdots.
\]
The homomorphisms $E^{-1} \xrightarrow{s^{-1}_E} Z^0 E$ and $X^{-1} \xrightarrow{\xi} M$ are both $\text{Proj}_\Lambda$-epic, i.e. surjective, so any chain map $E \rightarrow X$ induces a homomorphism $Z^0 E \rightarrow M$.

This induces a well-defined isomorphism, natural with respect to $E \in \text{Enochs}$,
\[
\text{Hom}_K(\text{Proj}_\Lambda)(E, X) \rightarrow \pi(Z^0 E, M).
\]

Proof. This is a small diagram chase. \qed

Now I am ready to prove this section’s main result:

Theorem 4.14. The class $G$ of Gorenstein projective $\Lambda$-modules is pre-covering in $\text{Mod}_\Lambda$.

Proof. I start by showing that $G$ is a pre-covering class in $(\text{Mod}_\Lambda)_{\text{Proj}_\Lambda}$. Let $M$ be in $(\text{Mod}_\Lambda)_{\text{Proj}_\Lambda}$, let $\cdots \rightarrow X^{-2} \rightarrow X^{-1} \xrightarrow{\xi} M$ be an
unconventionally indexed $\mathcal{P}_{\text{proj}} \Lambda$-left-resolution of $M$ (i.e. a projective resolution), and write $X$ for the complex

$$\cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow 0 \longrightarrow \cdots .$$

Using corollary 4.11, pick an Enochs-pre-cover $E \xrightarrow{[e]} X$ of $X$ in $K(\mathcal{P}_{\text{proj}} \Lambda)$. By lemma 4.13 there is an induced morphism $Z^0 E \xrightarrow{[m]} M$, and I will prove this to be a $\mathcal{G}$-pre-cover of $M$ in $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$.

For this, suppose given a morphism from a $\mathcal{G}$-object to $M$. By lemma 4.7, it has the form $Z^0 F \xrightarrow{[m]} M$ for some Enochs complex $F$. By lemma 4.13, $\underline{n}$ is induced by some $F \xrightarrow{[f]} M$. But since $[e]$ is a pre-cover, there exists $F \xrightarrow{[g]} E$ so that $[e] \circ [g] = [f]$. And the naturality of lemma 4.13's isomorphism then says

$$m \circ Z^0 [g] = n,$$

whence $\underline{n}$ has been factored through $m$ as desired.

I now show that $\mathcal{G}$ is also a pre-covering class in $\text{Mod} \Lambda$. Let $M$ be in $\text{Mod} \Lambda$, think of $M$ as an object in $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$, and use the above to pick a $\mathcal{G}$-pre-cover $G \xrightarrow{[m]} M$ of $M$ in $(\text{Mod} \Lambda)_{\mathcal{P}_{\text{proj}} \Lambda}$. Pick also a surjection $P \xrightarrow{\underline{p}} M$ from a projective to $M$. I will prove that $G \oplus P \xrightarrow{(g,p)} M$ is a $\mathcal{G}$-pre-cover of $M$ in $\text{Mod} \Lambda$.

First, $P$ is certainly Gorenstein projective, so $P \oplus G$ is Gorenstein projective. Secondly, let $H$ be Gorenstein projective, and let $H \xrightarrow{\underline{h}} M$ be given. I can factor $\underline{h}$ through $g$; in other words, there exists $H \xrightarrow{\underline{f}} G$ so that $g \underline{f} = \underline{h}$. This again means that $h - gf$ factors through a projective, hence in particular factors through the projective pre-cover $p$. So there exists $H \xrightarrow{\underline{f}} P$ so that $h - gf = pf'$. But now I can write down the homomorphism

$$\begin{pmatrix} f \\ f' \end{pmatrix} : H \longrightarrow G \oplus P,$$

which satisfies

$$(g,p) \circ \begin{pmatrix} f \\ f' \end{pmatrix} = gf + pf' = h,$$

so $h$ has been factored through $(g,p)$, proving my claim. $\square$

References


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