SPARSENESS OF T-STRUCTURES AND NEGATIVE
CALABI–YAU DIMENSION IN TRIANGULATED
CATEGORIES GENERATED BY A SPHERICAL OBJECT

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Abstract. Let \( k \) be an algebraically closed field and let \( T \) be the \( k \)-linear
algebraic triangulated category generated by a \( w \)-spherical object for an integer
\( w \). For certain values of \( w \) this category is classical. For instance, if \( w = 0 \)
then it is the compact derived category of the dual numbers over \( k \).

As main results of the paper we show that for \( w \leq 0 \), the category \( T \) has no
non-trivial t-structures, but does have one family of non-trivial co-t-structures,
whereas for \( w \geq 1 \) the opposite statement holds.

Moreover, without any claim to originality, we observe that for \( w \leq -1 \),
the category \( T \) is a candidate to have negative Calabi–Yau dimension since
\( \Sigma^w \) is the unique power of the suspension functor which is a Serre functor.

0. Introduction

Let \( k \) be an algebraically closed field, \( w \) an integer, and let \( T \) be a \( k \)-linear
algebraic triangulated category which is idempotent complete and classically
generated by a \( w \)-spherical object.

The categories \( T \), examined initially in [10] for \( w \geq 2 \), have recently been of
considerable interest, see [5], [7], [13], [15], and [18]. The purpose of this paper
is twofold.

First, we show the following main result.

**Theorem A.** If \( w \leq 0 \), then \( T \) has no non-trivial t-structures. It has one family
of non-trivial co-t-structures, all of which are (de)suspensions of a canonical one.

If \( w \geq 1 \), then \( T \) has no non-trivial co-t-structures. It has one family of non-
trivial t-structures, all of which are (de)suspensions of a canonical one.

For \( w \leq 0 \) this is a particularly clean instance of Bondarko’s remark [4, rmk.
4.3.4.4] that there are sometimes “more” co-t-structures than t-structures in a
triangulated category. Note that the case \( w = 2 \) is originally due to Ng [15,
thms. 4.1 and 4.2].

Secondly, without any claim to originality, we observe that if \( w \leq -1 \) then \( T \)
is a candidate for having negative Calabi–Yau dimension, although there does
not yet appear to be a universally accepted definition of this concept. Namely,
the \( w \)'th power of the suspension functor, \( \Sigma^w \), is a Serre functor for \( T \), and \( \Sigma^w \) is the only power of the suspension which is a Serre functor. For \( w \geq 2 \) this is contained in [10, prop. 6.5]. For a general \( w \) it is well known to the experts; we show an easy proof in Proposition 1.8.

The proof of Theorem A occupies Section 4 while Sections 1 to 3 are preparatory. Let us end the introduction by giving some background and explaining the terms used above.

0.a. What is \( T \)?

For certain small values of \( w \), the category \( T \) is well known in different guises: For \( w = 0 \) it is \( \text{D}^c(k[X]/(X^2)) \), the compact derived category of the dual numbers. For \( w = 1 \) it is \( \text{D}^f(k[[X]]) \), the derived category of complexes with bounded finite length homology over the formal power series ring. And for \( w = 2 \) it is the cluster category of type \( A_\infty \), see [7]. For \( w \) negative, \( T \) is less classical.

In general, \( T \) is determined up to triangulated equivalence by the properties stated in the first paragraph of the paper by [13, thm. 2.1]. We briefly explain these properties:

A triangulated category is algebraic if it is the stable category of a Frobenius category; see [6, sec. 9].

An additive category \( A \) is idempotent complete if, for each idempotent \( e \) in an endomorphism ring \( A(a,a) \), we have \( e = \iota \pi \) where \( \iota \) and \( \pi \) are the inclusion and projection of a direct summand of \( a \). Note that \( A(-,-) \) is shorthand for \( \text{Hom}_A(-,-) \).

A \( w \)-spherical object \( s \) in a \( k \)-linear triangulated category \( S \) is defined by having graded endomorphism algebra \( S(s, \Sigma^s s) \) isomorphic to \( k[X]/(X^2) \) with \( X \) placed in cohomological degree \( w \).

A triangulated category \( S \) is classically generated by an object \( s \) if each object in \( S \) can be built from \( s \) using finitely many (de)suspensions, distinguished triangles, and direct summands.

0.b. t-structures and co-t-structures.

To explain these, we first introduce the more fundamental notion of a torsion pair in a triangulated category due to Iyama and Yoshino [9, def. 2.2].

If \( S \) is a triangulated category, then a torsion pair in \( S \) is a pair \((M, N)\) of full subcategories closed under direct sums and summands, satisfying that \( S(M, N) = 0 \) and that \( S = M \star N \) where \( M \star N \) stands for the class of objects \( s \) appearing in distinguished triangles \( m \to s \to n \) with \( m \in M \), \( n \in N \).

A torsion pair \((M, N)\) is called a t-structure if \( \Sigma M \subseteq M \), and a co-t-structure if \( \Sigma^{-1} M \subseteq M \). In each case, the structure is called trivial if it is \((S, 0)\) or \((0, S)\) and non-trivial otherwise.

This is not how t-structures and co-t-structures were first defined by Beilinson, Bernstein, and Deligne in [3, def. 1.3.1], respectively by Bondarko and Pauksztello.
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in [4, def. 1.1.1] and [16, def. 2.4], but it is an economical way to present them
and to highlight their dual natures.

t-structures have become classical objects of homological algebra while co-t-
structures were introduced more recently. They both enable one to “slice” objects
of a triangulated category into simpler bits and they are the subject of vigorous
research.

0.e. Silting subcategories.

We are grateful to Changjian Fu for the following observation: \( \Sigma^w \) is a Serre
functor of \( T \). In the terminology of [1] this means that \( T \) is \( w \)-Calabi–Yau.
Moreover, \( T \) is generated by a \( w \)-spherical object \( s \); in particular, for \( w \leq -1 \)
we have \( T(s, \Sigma^{>0}s) = 0 \). In the terminology of [1], this means that \( s \) is a silting
object.

So for \( w \leq -1 \), the category \( T \) is \( w \)-Calabi–Yau with the silting subcategory
\( \text{add}(s) \). The existence of a category with these properties was left as a question
at the end of [1, sec. 2.1].

It is not hard to check directly that for \( w \leq -1 \), the basic silting objects in \( T \)
are precisely the (de)suspensions of \( s \). This also follows from [1, thm. 2.26].

1. Basic properties of \( T \)

None of the material of this section is original, but not all of it is given explicitly
in the original references [5], [10], and [13]. We give a brief, explicit presentation
to facilitate the rest of the paper.

**Remark 1.1.** The category \( T \) is Krull-Schmidt by [17, p. 52]. Namely, it is
idempotent complete by assumption, and it has finite dimensional Hom spaces
because each object is finitely built from a \( w \)-spherical object \( s \) which in particular
satisfies \( \dim_k T(s, \Sigma^i s) < \infty \) for each \( i \).

We need to compute inside \( T \). Hence a concrete model is more useful than an
abstract characterisation. Let us redefine \( T \) as such a model.

**Definition 1.2.** Set \( d = w - 1 \) and consider the polynomial ring \( k[T] \) as a Differ-
ential Graded (DG) algebra with \( T \) in homological degree \( d \) and zero differential.
We denote this DG algebra by \( A \).

Consider \( D(A) \), the derived category of DG left-\( A \)-modules, and let \( T \) be \( \langle k \rangle \), the
thick subcategory generated by the trivial DG module \( k = A/(T) \) where \( (T) \) is
the DG ideal generated by \( T \).

This is how \( T \) will be defined for the rest of the paper, except in the proof
of Proposition 1.8. It is compatible with the previous definition of \( T \) by the
following result.

**Lemma 1.3.** The category \( T = \langle k \rangle \) is a \( k \)-linear algebraic triangulated category
which is idempotent complete and classically generated by the \( w \)-spherical object
\( k \).
Proof. The only part which is not clear is that $k$ is $w$-spherical. But there is a distinguished triangle

$$\Sigma^d A \xrightarrow{T} A \longrightarrow k$$

(1)
in $\text{D}(A)$, induced by the corresponding short exact sequence of DG modules. Applying $\text{RHom}_A(-, k)$ gives another distinguished triangle whose long exact homology sequence shows that $k$ is a $w$-spherical object of $\text{D}(A)$. □

Remark 1.4. The distinguished triangle (1) also shows that $k$ is a compact object of $\text{D}(A)$, so $T$ is even a subcategory of the compact derived category $\text{D}^c(A)$.

Definition 1.5. For each $r \geq 0$, the element $T^{r+1}$ of $A$ generates a DG ideal $(T^{r+1})$. Consider the quotient $X_r = A/(T^{r+1})$ as a DG left-$A$-module.

Remark 1.6. There is a distinguished triangle

$$\Sigma^{(r+1)d} A \xrightarrow{T^{r+1}} A \longrightarrow X_r$$
in $\text{D}^c(A)$, induced by the corresponding short exact sequence of DG modules.

Proposition 1.7. The indecomposable objects of $T$ are precisely the (de)suspensions of the objects $X_r$.

Proof. Note that $\text{D}^c(A) = \langle A \rangle$ and that $\text{Hom}_{\text{D}^c(A)}(A, \Sigma^* A)$ is isomorphic to $k[T]$ as a graded algebra, where $T$ is still in homological degree $d$. Since $\text{gr}(k[T]^{\text{op}})$, the abelian category of finitely generated graded right-$k[T]$-modules and graded homomorphisms, is hereditary, [13, thm. 3.6] says that the functor

$$\text{Hom}_{\text{D}^c(A)}(A, \Sigma^*(-)) = H^*(-)$$

induces a bijection between the isomorphism classes of indecomposable objects of $\text{D}^c(A)$ and $\text{gr}(k[T]^{\text{op}})$. This has the following consequences.

If $w \neq 1$ then $d \neq 0$. Then up to isomorphism, the indecomposable objects of $\text{gr}(k[T]^{\text{op}})$ are precisely the graded shifts of the graded modules $k[T]$ and $k[T]/(T^{r+1})$ for $r \geq 0$. So up to isomorphism, the indecomposable objects of $\text{D}^c(A)$ are the (de)suspensions of $A$ and the objects $X_r$ for $r \geq 0$.

Of these objects, precisely the $X_r$ are in $T$, so up to isomorphism the indecomposable objects of $T$ are the (de)suspensions of the objects $X_r$ for $r \geq 0$.

If $w = 1$ then $d = 0$ so $A$ and $k[T]$ are concentrated in degree 0. A graded right-$k[T]$-module is the direct sum of its graded components, and it follows that the indecomposable objects of $\text{gr}(k[T]^{\text{op}})$ are the indecomposable ungraded right-$k[T]$-modules placed in a single graded degree. But up to isomorphism, these are $k[T]$ and $k[T]/(f(T))$ where $f(T)$ is a power of an irreducible, hence first degree, polynomial. So up to isomorphism, the indecomposable objects of $\text{D}^c(A)$ are the (de)suspensions of the objects $A$ and $A/(f(T))$ viewed in $\text{D}^c(A)$.

Again, of these objects, precisely the $X_r$ are in $T$, so up to isomorphism the indecomposable objects of $T$ are the (de)suspensions of the objects $X_r$ for $r \geq 0$. □
It is not hard to see that $A$ is the $w$-Calabi–Yau completion of $k$ in the sense of [11, 4.1]. As a consequence, $\mathcal{T} = \langle k \rangle$ has Serre functor $S = \Sigma^w$. Here we give a direct proof of this fact.

**Proposition 1.8.** The category $\mathcal{T}$ has Serre functor $S = \Sigma^w$, and this is the only power of the suspension which is a Serre functor.

**Proof.** For this proof only, it is convenient to use another model for $\mathcal{T}$. Consider the dual numbers $k[U]/(U^2)$ and view them as a DG algebra with $U$ placed in cohomological degree $w$ and zero differential. Denoting this DG algebra by $B$, it is immediate that $B$ is a $w$-spherical object of $D(B)$, the derived category of DG left-$B$-modules, and so the thick subcategory $\langle B \rangle$ generated by $B$ is equivalent to $\mathcal{T}$. This is the model we will use. In fact, $\langle B \rangle$ is equal to the compact derived category $D^c(B)$.

For $X, Y \in D^c(B)$ we have the following natural isomorphisms where $D(-) = \text{Hom}_k(-, k)$.

\[
D \text{RHom}_B(Y, DB \otimes_B X) \cong D\left( \text{RHom}_B(Y, DB) \otimes_B X \right) \\
\cong \text{RHom}_{B^{op}} \left( \text{RHom}_B(Y, DB), DX \right) \\
\cong \text{RHom}_{B^{op}}(DY, DX) \\
\cong \text{RHom}_B(X, Y).
\]

Here (a) holds for $X = B$ and hence for the given $X$ because it is finitely built from $B$. The isomorphisms (b) and (c) are by adjointness of $\otimes$ and $\text{RHom}$. And (d) is duality.

Taking zeroth homology of the above formula shows that $DB \otimes_B -$ is a right Serre functor of $D^c(B)$. But direct computation shows $DB \cong \Sigma^w B$ as DG $B$-bimodules, so $\Sigma^w$ is a right Serre functor. Since it is an equivalence of categories, it is even a Serre functor.

Finally, no other power of $\Sigma$ is a Serre functor of $D^c(B)$: If $\Sigma^i$ is a Serre functor then $\Sigma^i \cong \Sigma^w$ whence $\Sigma^{i-w} \cong \text{id}$. This implies $i = w$ since already $\Sigma^{i-w} B \cong B$ implies $i = w$ as one sees by taking homology. \qed

**Remark 1.9.** The AR translation of $\mathcal{T}$ is $\tau = \Sigma^{-1} S = \Sigma^{w-1} = \Sigma^d$.

**Proposition 1.10.**

(i) If $w \neq 1$ then the AR quiver of $\mathcal{T}$ consists of $|d|$ copies of $ZA_\infty$. One copy is shown in Figure 1 and the others are obtained by applying $\Sigma, \Sigma^2, \ldots, \Sigma^{|d|-1}$.

(ii) If $w = 1$ then the AR quiver of $\mathcal{T}$ consists of countably many homogeneous tubes. One tube is shown in Figure 2 and the others are obtained by applying all non-zero powers of $\Sigma$. \qed
Proof. For \( w \geq 2 \) this is [10, thm. 8.13].

For \( w \) general, the shape of the AR quiver is given in [5, sec. 3.3]. For \( w \leq 0 \), to see that the \( |d| \) copies of \( \mathbb{Z}A_\infty \) look as claimed, one can compute the AR triangles of \( T \) by methods similar to those of [10, sec. 8].

Finally, for \( w = 1 \) we have \( d = 0 \). The AR translation is \( \tau = \Sigma^0 = \text{id} \) by Remark 1.9, so for each \( X_r \) there is an AR triangle \( X_r \to Y \to X_r \). The long exact homology sequence shows that if \( r = 0 \) then \( Y = X_1 \) and if \( r \geq 1 \) then \( Y = X_{r-1} \oplus X_{r+1} \). Hence the homogeneous tube in Figure 2 is a component of the AR quiver as claimed. For each \( i \), applying \( \Sigma^i \) to Figure 2 gives a component of the AR quiver. The components obtained in this fashion contain all indecomposable objects of \( T \) so form the whole AR quiver. \( \square \)

2. Morphisms in \( T \)

This section computes the Hom spaces between indecomposable objects in the category \( T \).

**Definition 2.1.** Suppose that \( w \neq 0 \) so the AR quiver of \( T \) consists of copies of \( \mathbb{Z}A_\infty \) by Proposition 1.10(i). Let \( t \in T \) be an indecomposable object. Figure 3 defines two sets \( F^\pm(t) \) consisting of indecomposable objects in the same component of the AR quiver as \( t \). Each set can be described as a rectangle stretching off to infinity in one direction; it consists of the objects inside the indicated boundaries including the ones on the boundaries. In particular we have \( t \in F^+(t) \cup F^-(St) \).

**Proposition 2.2.** Suppose that \( w \neq 0,1 \). Let \( t,u \) be indecomposable objects in \( T \). Then
\[
\dim_k T(t,u) = \begin{cases} 
1 & \text{for } u \in F^+(t) \cup F^-(St), \\
0 & \text{otherwise},
\end{cases}
\]
where \( S = \Sigma^w \) is the Serre functor of \( T \); see Proposition 1.8.
Figure 3. The regions $F^\pm(t)$ for $w \neq 1$

Proposition 2.3. Suppose that $w = 0$. Let $t, u$ be indecomposable objects in $T$. Then

$$
\dim_k T(t, u) = \begin{cases} 
2 & \text{for } u = t, \\
1 & \text{for } u \in (F^+(t) \cup F^-(t)) \setminus t, \\
0 & \text{otherwise.}
\end{cases}
$$

Proposition 2.4. Suppose that $w = 1$. Let $u$ be an indecomposable object of $T$. Then

$$
\dim_k T(X_r, u) = \begin{cases} 
\min\{r, s\} + 1 & \text{for } u = X_s \text{ or } u = \Sigma X_s, \\
0 & \text{for all other } u.
\end{cases}
$$

Note that in Proposition 2.2, the sets $F^+(t)$ and $F^-(St)$ are disjoint. For $w \neq 2$, they even sit in different components of the AR quiver, while for $w = 2$ we have $d = w-1 = 1$ and the AR quiver has only one component. In Proposition 2.3, the sets $F^+(t)$ and $F^-(t)$ have intersection $t$. In this case $w = 0$ so $d = w - 1 = -1$ and the AR quiver has only one component.

Proof of Propositions 2.2, 2.3, and 2.4.

Propositions 2.2 and 2.3 give the dimensions of Hom spaces in a conceptual way using the regions $F^\pm$. Unfortunately we do not have a conceptual proof.

The proof we have is pedestrian: Applying $\mathbb{R}Hom_A(-, X_s)$ to the distinguished triangle from Remark 1.6 gives a new distinguished triangle whose long exact homology sequence contains

$$
H^{i-1}(X_s) \to H^{i-(r+1)d-1}(X_s) \to \mathbb{R}Hom_A(X_r, X_s) \to H^i(X_s) \to H^{i-(r+1)d}(X_s).
$$

The middle term is isomorphic to $T(X_r, \Sigma X_s)$. The DG module $X_s$ is $A/(T^s+1)$, so the four outer terms are easily computable. The first and last maps are induced by $\cdot T^{r+1}$ and can also be computed. Hence the middle term can be determined.

For $w \neq 1$, combining the dimensions of Hom spaces with the detailed structure of the AR quiver as described by Proposition 1.10 proves Propositions 2.2 and 2.3, and for $w = 1$ one gets Proposition 2.4 directly. □
3. T- AND CO-T-STRUCTURES

This section gives some easy properties of t- and co-t-structures. Lemmas 3.1 and 3.2 are valid in general triangulated categories.

Recall that if \((X, Y)\) is a t-structure then the heart is \(H = X \cap \Sigma Y\), and if \((A, B)\) is a co-t-structure then the co-heart is \(C = A \cap \Sigma^{-1} B\).

Lemma 3.1. Let \((X, Y)\) be a t-structure and \((A, B)\) a co-t-structure with heart and co-heart \(H\) and \(C\).

(i) \(\text{Hom}(H, \Sigma^{<0} H) = 0\).
(ii) \(\text{Hom}(C, \Sigma^{>0} C) = 0\).
(iii) \(X = \Sigma X\) if and only if \(H = 0\).
(iv) \(A = \Sigma A\) if and only if \(C = 0\).

Proof. (i) We have \(H \subseteq X\) and \(\Sigma^{<0} H \subseteq \Sigma^{<0} \Sigma Y = \Sigma^{\leq 0} Y \subseteq Y\). The last \(\subseteq\) is a well known property of t-structures and follows from \(\Sigma \geq 0 X \subseteq X\) by taking right perpendicular categories; cf. [9, remark after def. 2.2] by which \(X^\perp = Y\). Here \(\perp\) is as defined in [9, start of sec. 2]. But \(\text{Hom}(X, Y) = 0\) so \(\text{Hom}(H, \Sigma^{<0} H) = 0\) follows.

(ii) Dual to part (i).

(iii) \(\Rightarrow\): Suppose \(h \in H\). Then \(\Sigma^{-1} h \in \Sigma^{-1} H \subseteq \Sigma^{-1} X = X\) so \(h = \Sigma(\Sigma^{-1} h) \in \Sigma X\). We also have \(h \in H \subseteq \Sigma Y\). But \(\text{Hom}(\Sigma X, \Sigma Y) = \text{Hom}(X, Y) = 0\) so \(\text{Hom}(h, h) = 0\) proves \(h = 0\).

\(\Leftarrow\): For \(x \in X\) consider the distinguished triangle

\[
x' \to \Sigma^{-1} x \to y'
\]

with \(x' \in X\), \(y' \in Y\) which exists because \((X, Y)\) is a torsion pair. It gives a distinguished triangle \(x \to \Sigma y' \to \Sigma^2 x'\) with \(x, \Sigma^2 x' \in X\). But \(X\) is closed under extensions since it is equal to \(^\perp Y\) by [9, remark after def. 2.2] again, so \(\Sigma y' \in X\).

We also have \(\Sigma y' \in \Sigma Y\) so \(\Sigma y' \in H\) and hence \(\Sigma y' = 0\). But then the distinguished triangle (2) shows \(\Sigma^{-1} x \cong x' \in X\). Hence \(\Sigma^{-1} X \subseteq X\), and since we also know \(\Sigma X \subseteq X\) it follows that \(\Sigma X = X\).

(iv) Dual to part (iii).

A torsion pair \((M, N)\) with \(\Sigma M = M\) (and consequently \(\Sigma N = N\)) is called a stable t-structure; see [14, p. 468]. In this case, \(M\) and \(N\) are thick subcategories of \(T\).

Lemma 3.2. If \((M, N)\) and \((M', N')\) are torsion pairs with \(M \subseteq M'\) and \(N \subseteq N'\), then \((M, N) = (M', N')\).

Proof. The inclusion \(N \subseteq N'\) implies \(^\perp N \supseteq ^\perp N'\), but this reads \(M \supseteq M'\) by [9, remark after def. 2.2] so we learn \(M = M'\). Hence also \(N = M^\perp = M'^\perp = N'\).

Lemma 3.3. A stable t-structure in \(T\) is trivial.
Proof. Let \((X, Y)\) be a stable t-structure in \(T\) with \(X \neq 0\). Then \(X\) contains an indecomposable object \(x\). But \(X\) is a thick subcategory of \(T\), and it is easy to see from the AR quiver of \(T\) that hence \(X = T\). 

\[\square\]

4. PROOF OF THEOREM A

4.a. **Proof of Theorem A for t-structures, \(w \leq -1\).**

Let \((X, Y)\) be a t-structure in \(T\) with heart \(H = X \cap Y\) and let \(h \in H\). Serre duality gives

\[\text{Hom}_k(T(h, h), k) \cong T(h, Sh) \cong T(h, \Sigma^w h) = 0\]

where “= 0” is by Lemma 3.1(i) because \(w \leq -1\). This implies \(h = 0\) so \(H = 0\). But then \((X, Y)\) is a stable t-structure by Lemma 3.1(iii) and hence trivial by Lemma 3.3.

4.b. **Proof of Theorem A for t-structures, \(w = 0\).**

In this case we have \(d = w - 1 = -1\). The AR quiver consists of \(|d| = 1\) copy of \(\mathbb{Z}A_\infty\) by Lemma 1.10(i); see Figure 1.

Assume that \((X, Y)\) is a non-trivial t-structure in \(T\). By Lemmas 3.3 and 3.1(iii) the heart \(H\) is non-zero so contains an indecomposable object.

However, if \(t\) is an indecomposable object not on the base line of the AR quiver then \(\tau t \in F^-(t)\); see Figure 3. Hence \(T(t, \tau t) \neq 0\) by Proposition 2.3, and by Remark 1.9 this reads \(T(t, \Sigma^{-1} t) \neq 0\). But \(T(H, \Sigma^{-0} H) = 0\) by Lemma 3.1(i), so each indecomposable object in \(H\) is forced to be on the base line of the AR quiver.

Moreover, if \(h \in H\) is indecomposable then \(H\) cannot contain another indecomposable object \(h'\): Both objects would have to be on the base line of the AR quiver which has only one component, so we would have \(h' = \tau^i h\) for some \(i \neq 0\), that is, \(h' = \Sigma^{-i} h\). But this contradicts \(T(H, \Sigma^{-0} H) = 0\).

It follows that \(H = \text{add}(h)\) for an indecomposable object \(h\) on the base line of the AR quiver, and \(h = \Sigma^i X_0\) for some \(i\). Direct computation shows that \(h\) is 0-spherical, so there is a non-zero, non-invertible morphism \(h \to h\). But this morphism is easily verified not to have a kernel in \(H\), and this is a contradiction since the heart of a t-structure is abelian.

4.c. **Proof of Theorem A for t-structures, \(w = 1\).**

Here the AR quiver of \(T\) consists of countably many stable tubes as detailed in Proposition 1.10(ii).

By [5, sec. 3.1], an alternative model of \(T\) is \(D^f(k[[X]])\), the derived category of complexes with bounded finite length homology over the ring \(k[[X]]\). This shows that \(T\) has a canonical t-structure.

Assume that \((X, Y)\) is a non-trivial t-structure in \(T\). In particular, \(X\) is closed under extensions. The components of the AR quiver of \(T\) are homogeneous tubes and the AR triangles of \(T\) can be read off. The triangles imply that if \(X\) contains

\[\text{\textbf{}}\]
an indecomposable object \( t \) then it contains the whole component of \( t \). So \( X \) is equal to add of a collection of components of the AR quiver. Now let \( Q \) be a component such that \( Q \subseteq X \) but \( \Sigma^{-1}Q, \Sigma^{-2}Q, \ldots \not\subseteq X \). Such a \( Q \) exists because \( X \) is closed under \( \Sigma \) and not equal to 0 or \( T \). It is then clear that

\[
X = \text{add}(Q \cup \Sigma Q \cup \cdots).
\]

The right hand side only depends on the component \( Q \) of the AR quiver, and since all other components have the form \( \Sigma_i Q \) (see Proposition 1.10(ii)), this implies that all non-trivial t-structures are (de)suspensions of each other, and hence (de)suspensions of the canonical t-structure.

4.d. **Proof of Theorem A for t-structures, \( w \geq 2 \).**

Recall that \( A \) is \( k[T] \) viewed as a DG algebra with \( T \) in homological degree \( d = w - 1 \) and zero differential. Each object of \( T \) is a direct sum of finitely many (de)suspensions of the objects \( X_r = A/(T^{r+1}) \) by Remark 1.1 and Proposition 1.7. In particular, each object of \( T \) is isomorphic to a DG module \( t \) which is finite dimensional over \( k \).

Since \( w \geq 2 \) we have \( d \geq 1 \) which means that \( A \) is a chain DG algebra. So for each DG left-\( A \)-module \( t \) there is a distinguished triangle \( t_{(\geq 0)} \to t \to t_{(<0)} \) in \( D(A) \) induced by the following (vertical) short exact sequence of DG modules.

\[
\cdots \to t_2 \to t_1 \to \ker \partial_0 \to 0 \to 0 \to \cdots
\]

\[
\cdots \to t_2 \to t_1 \to t_0 \to a_0 \to t_{-1} \to t_{-2} \to \cdots
\]

\[
\cdots \to 0 \to 0 \to t_0/\ker \partial_0 \to t_{-1} \to t_{-2} \to \cdots
\]

Each of \( t_{(\geq 0)} \) and \( t_{(<0)} \) is also finite dimensional over \( k \) which implies that they can be built in finitely many steps from the DG module \( k \); that is, they belong to \( \langle k \rangle = T \). Hence the distinguished triangle is in \( T \), so \( (T_{(\geq 0)}, T_{(<0)}) \) is a t-structure in \( T \) where

\[
T_{(\geq 0)} = \{ t \in T \mid H_*(t) \text{ is in homological degrees } \geq 0 \},
\]

\[
T_{(<0)} = \{ t \in T \mid H_*(t) \text{ is in homological degrees } < 0 \}.
\]

(3)

We refer to this t-structure as canonical. Note that it was first constructed, in higher generality, in [8, thm. 1.3], and in the DG case in [2, lem. 2.2] and [12, lem. 5.2].

It is easy to check that \( T_{(\geq 0)} \) is the smallest subcategory of \( T \) which contains \( X_0 \) and is closed under \( \Sigma \), extensions, and direct summands. Likewise, \( T_{(<0)} \) is the smallest subcategory of \( T \) which contains \( \Sigma^{-1}X_0 \) and is closed under \( \Sigma^{-1} \), extensions, and direct summands.

Now assume that \( (X,Y) \) is a non-trivial t-structure in \( T \). By Lemmas 3.3 and 3.1(iii) the heart \( H \) is non-zero so contains an indecomposable object \( h \). We have \( T(h, \Sigma^{<0}h) = 0 \) by Lemma 3.1(i).
If $t$ is an indecomposable object not on the base line of the AR quiver then $	au^{-1}t \in F^+(t)$; see Figure 3. Hence $T(t, \tau^{-1}t) \neq 0$ by Proposition 2.2, and by Remark 1.9 this reads $T(t, \Sigma^{-d}t) \neq 0$. Hence $h$ is forced to be on the base line of the AR quiver. Suspending or desuspending the t-structure, we can assume $h = X_0$.

We have $h \in X$ and $h \in \Sigma Y$ whence $\Sigma^{-1}h \in Y$. That is, $X_0 \in X$ and $\Sigma^{-1}X_0 \in Y$.

However, $X$ is closed under $\Sigma$, extensions, and direct summands, and since $T_{(\geq 0)}$ is the smallest subcategory of $T$ with these properties which contains $X_0$, we get $T_{(\geq 0)} \subseteq X$. Similarly, $T_{(<0)} \subseteq Y$.

By Lemma 3.2 this forces $(X, Y) = (T_{(\geq 0)}, T_{(<0)})$, and we have shown that as desired, up to (de)suspension, any non-trivial t-structure in $T$ is the canonical one.

4.e. Proof of Theorem A for co-t-structures, $w \leq 0$.

In the proof for t-structures, $w \geq 2$, we showed a canonical t-structure. Tweaking the method slightly in the present case produces a canonical co-t-structure. Each object of $T$ is still isomorphic to a DG module $t$ which is finite dimensional over $k$. Since $A$ is $k[T]$ with $T$ in homological degree $d = w - 1$, and since $w \leq 0$ and $d \leq -1$, we have that $A$ is a cochain DG algebra. So there is a distinguished triangle $t_{\leq 0} \to t \to t_{>0}$ in $D(A)$ where the subscripts indicate hard truncations in the relevant homological degrees. Each of $t_{\leq 0}$ and $t_{>0}$ is also finite dimensional over $k$ and is therefore in $T$. Hence $(T_{(\leq 0)}, T_{(>0)})$ is a co-t-structure in $T$ where

\[ T_{(\leq 0)} = \{ t \in T \mid H_*(t) \text{ is in homological degrees } \leq 0 \}, \]
\[ T_{(>0)} = \{ t \in T \mid H_*(t) \text{ is in homological degrees } > 0 \}. \]

The rest of the proof is dual to the proof for t-structures, $w \geq 2$.

4.f. Proof of Theorem A for co-t-structures, $w \geq 1$.

This is dual to the proof for t-structures, $w \leq -1$.

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