Triangulated categories were introduced in the mid 1960’s by J.L. Verdier in his thesis, reprinted in [15]. Axioms similar to Verdier’s were independently also suggested in [2]. Having their origins in algebraic geometry and algebraic topology, triangulated categories have by now become indispensable in many different areas of mathematics. Although the axioms might seem a bit opaque at first sight it turned out that very many different objects actually do carry a triangulated structure. Nowadays there are important applications of triangulated categories in areas like algebraic geometry (derived categories of coherent sheaves, theory of motives) algebraic topology (stable homotopy theory), commutative algebra, differential geometry (Fukaya categories), microlocal analysis or representation theory (derived and stable module categories).

It seems that the importance of triangulated categories in modern mathematics is growing even further in recent years, with many new applications only recently found; see B. Keller’s article in this volume for one striking example, namely the cluster categories occurring in the context of S. Fomin and A. Zelevinsky’s cluster algebras which have been introduced only around 2000.

In this chapter we aim at setting the scene for the survey articles in this volume by providing the relevant basic definitions, deducing some elementary general properties of triangulated categories and providing a few examples.

Certainly, this cannot be a comprehensive introduction to the subject. For more details we refer to one of the well-written textbooks on triangulated categories, e.g. [4], [5], [7], [11], [16], and for further topics also to the surveys in this volume.

This introductory chapter should be accessible for a reader with a good background in algebra and some basic knowledge of category theory and homological algebra.

1. ADDITIVE CATEGORIES

In this first section we shall discuss the fundamental notion of an additive category and provide some examples. In particular, the category of complexes over an additive category is introduced which will play a fundamental role in the sequel.

Definition 1.1. A category $\mathcal{A}$ is called an additive category if the following conditions hold:

(A1) For every pair of objects $X,Y$ the set of morphisms $\text{Hom}_\mathcal{A}(X,Y)$ is an abelian group and the composition of morphisms $\text{Hom}_\mathcal{A}(Y,Z) \times \text{Hom}_\mathcal{A}(X,Y) \to \text{Hom}_\mathcal{A}(X,Z)$ is bilinear over the integers.

(A2) $\mathcal{A}$ contains a zero object $0$ (i.e. for every object $X$ in $\mathcal{A}$ each morphism set $\text{Hom}_\mathcal{A}(X,0)$ and $\text{Hom}_\mathcal{A}(0,X)$ has precisely one element).
(A3) For every pair of objects \( X, Y \) in \( \mathcal{A} \) there exists a coproduct \( X \oplus Y \) in \( \mathcal{A} \).

**Remark 1.2.**

(i) A category satisfying (A1) and (A2) is called a *preadditive* category.

(ii) We recall the notion of coproduct from category theory. Let \( \mathcal{C} \) be a category and \( X, Y \) objects in \( \mathcal{C} \). A coproduct of \( X \) and \( Y \) in \( \mathcal{C} \) is an object \( X \oplus Y \) together with morphisms \( \iota_X : X \to X \oplus Y \) and \( \iota_Y : Y \to X \oplus Y \) satisfying the following universal property: for every object \( Z \) in \( \mathcal{C} \) and morphisms \( f_X : X \to Z \) and \( f_Y : Y \to Z \) there is a unique morphism \( f : X \oplus Y \to Z \) making the following diagram commutative

\[
\begin{array}{ccc}
X & \xrightarrow{f_X} & Z \\
\downarrow{\iota_X} & & \downarrow{f} \\
X \oplus Y & \xrightarrow{f_Y} & Y
\end{array}
\]

**Example 1.3.**

(i) Let \( R \) be a ring and consider \( R \) as a category \( \mathcal{C}_R \) with only one object. The unique morphism set is the underlying abelian group and composition of morphisms is given by ring multiplication. Then \( \mathcal{C}_R \) satisfies (A1) and (A2), thus preadditive categories can be seen as generalizations of rings. But \( \mathcal{C}_R \) is not additive in general; in fact the coproduct of the unique object with itself would have to be again this object together with fixed ring elements \( \iota_1, \iota_2 \), and the universal property would mean that for arbitrary ring elements \( f_1, f_2 \) there existed a unique element \( f \) factoring them as \( f_1 = f \iota_1 \) and \( f_2 = f \iota_2 \).

(ii) Let \( R \) be a ring (associative, with unit element). Then the category \( \text{R-Mod} \) of all \( R \)-modules is additive. Similarly, the category \( \text{R-mod} \) of finitely generated \( R \)-modules is additive. In particular, the categories \( \text{Ab} \) of abelian groups and \( \text{Vec}_K \) of vector spaces over a field \( K \) are additive.

(iii) The full subcategory of \( \text{Ab} \) of free abelian groups is additive.

(iv) For a ring \( R \) the full subcategory \( \text{R-Proj} \) of projective \( R \)-modules is additive; similarly for \( \text{R-proj} \), the category of finitely generated projective \( R \)-modules.

1.1. **The category of complexes.** Let \( \mathcal{A} \) be an additive category. A *complex over \( \mathcal{A} \)* is a family \( X = (X_n, d_n^X)_{n \in \mathbb{Z}} \) where \( X_n \) are objects in \( \mathcal{A} \) and \( d_n^X : X_n \to X_{n-1} \) are morphisms such that \( d_n \circ d_{n+1} = 0 \) for all \( n \in \mathbb{Z} \). Usually, a complex is written as a sequence of objects and morphisms as follows

\[
\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots
\]

Let \( X = (X_n, d_n^X) \) and \( Y = (Y_n, d_n^Y) \) be complexes over \( \mathcal{A} \). A *morphism of complexes* \( f : X \to Y \) is a family of morphisms \( f = (f_n : X_n \to Y_n)_{n \in \mathbb{Z}} \) satisfying \( d_n^Y \circ f_n = f_{n-1} \circ d_n^X \) for all \( n \in \mathbb{Z} \), i.e. we have the following commutative diagram

\[
\begin{array}{ccc}
X_{n+1} & \xrightarrow{f_{n+1}} & X_n \xrightarrow{f_n} X_{n-1} \quad \cdots \\
\downarrow{f_{n+1}} & & \downarrow{f_n} \\
Y_{n+1} & \xrightarrow{f_{n-1}} & Y_n \xrightarrow{f_{n-1}} Y_{n-1} \quad \cdots
\end{array}
\]

The complexes over an additive category \( \mathcal{A} \) together with the morphisms of complexes form a category \( \mathcal{C}(\mathcal{A}) \), the *category of complexes over \( \mathcal{A} \)*.
Proposition 1.4. Let $\mathcal{A}$ be an additive category. Then the category of complexes $\text{C}(\mathcal{A})$ is again additive.

Proof. (A1) Addition of morphisms is defined degreewise, i.e. for two morphisms $f = (f_n)_{n \in \mathbb{Z}}$ and $g = (g_n)_{n \in \mathbb{Z}}$ from $X$ to $Y$ their sum is $f + g := (f_n + g_n)_{n \in \mathbb{Z}}$. Using the additive structure of $\mathcal{A}$ it is then easy to check that (A1) holds.

(A2) The zero object in $\text{C}(\mathcal{A})$ is the complex $(0, d)$ where $0_{\mathcal{A}}$ is the zero object of the additive category $\mathcal{A}$ and all differentials are the unique (zero) morphism on the zero object.

(A3) The coproduct of two complexes $X = (X_n, d^n_X)$ and $Y = (Y_n, d^n_Y)$ is defined degreewise by using the coproduct in the additive category $\mathcal{A}$. More precisely $X \oplus Y = (X_n \oplus Y_n, d^n_{X \oplus Y})$ where the differential is obtained by the universal property as in the following diagram.

\[
\begin{array}{ccc}
X_n & \xrightarrow{d_n} & X_{n-1} \\
\downarrow{\iota_X} & & \downarrow{\iota_X} \\
X_n \oplus Y_n & \xrightarrow{d_n} & X_{n-1} \oplus Y_{n-1} \\
\end{array}
\]

From uniqueness in the universal property applied to

\[
\begin{array}{ccc}
X_n & \xrightarrow{d_n \oplus 0} & X_{n-1} \oplus Y_{n-1} \\
\downarrow{\iota_X} & & \downarrow{\iota_X} \\
X_n \oplus Y_n & \xrightarrow{0 \oplus d_Y} & X_{n-1} \oplus Y_{n-1} \\
\end{array}
\]

it follows that $d_{n-1} \circ d_n = 0$. This complex indeed satisfies the properties of a coproduct in the category of complexes $\text{C}(\mathcal{A})$, with morphisms of complexes $\iota_X = (\iota_{X_n})_{n \in \mathbb{Z}} : X \to X \oplus Y$ and $\iota_Y = (\iota_{Y_n})_{n \in \mathbb{Z}} : Y \to X \oplus Y$. For checking the universal property let $Z$ be an arbitrary complex and let $f_X : X \to Z$ and $f_Y : Y \to Z$ be arbitrary morphisms. The unique morphism of complexes satisfying $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$ is $f = (f_n)_{n \in \mathbb{Z}} : X \oplus Y \to Z$, where $f_n$ is obtained from the universal property in degree $n$ as in the following diagram.

\[
\begin{array}{ccc}
Z_n & \xrightarrow{f_n} & (f_Y)_n \\
\downarrow{\iota_Z} & & \downarrow{\iota_Z} \\
X_n \oplus Y_n & \xrightarrow{f_n} & (f_Y)_n \\
\end{array}
\]

Remark 1.5. For complexes over $\mathcal{A} = \text{R-Mod}$ where $R$ is a ring with unit (and other similar examples) the coproduct of two complexes is more easily described on elements as $X \oplus Y = (X_n \oplus Y_n, d^n_{X \oplus Y})$ where the differential is given by $d^n_{X \oplus Y}(x_n, y_n) = (d^n_X(x_n), d^n_Y(y_n))$ for $x_n \in X_n$ and $y_n \in Y_n$, and with morphisms $\iota_X : X \to X \oplus Y$ and $\iota_Y : Y \to X \oplus Y$ being the inclusion maps. The unique morphism of complexes satisfying $f_X = f \circ \iota_X$ and $f_Y = f \circ \iota_Y$ is then given by $f_n(x_n, y_n) = f_X(x_n) + f_Y(y_n)$.\]
1.2. The homotopy category of complexes. Let $\mathcal{A}$ be an additive category. Morphisms $f, g : X \to Y$ in the category $\mathcal{C}(\mathcal{A})$ of complexes are called homotopic, denoted $f \sim g$, if there exists a family $(s_n)_{n \in \mathbb{Z}}$ of morphisms $s_n : X_n \to Y_{n+1}$ in $\mathcal{A}$, satisfying $f_n - g_n = d_{n+1}^Y s_n + s_{n-1} d_n^X$ for all $n \in \mathbb{Z}$.

In particular, setting $g$ to be the zero morphism, we can speak of morphisms being homotopic to zero.

It is easy to check that $\sim$ is an equivalence relation. Moreover, if $f \sim g : X \to Y$ are homotopic and $\alpha : W \to X$ is an arbitrary morphism of complexes, then also the compositions $f\alpha \sim g\alpha$ are homotopic. In fact, $(s_n \alpha_n)_{n \in \mathbb{Z}}$ are homotopy maps since

$$(f_n - g_n)\alpha_n = (d_{n+1}^Y s_n + s_{n-1} d_n^X)\alpha_n = d_{n+1}^Y (s_n \alpha_n) + (s_{n-1} \alpha_{n-1}) d_n^W.$$  

Similarly, if $f, g : X \to Y$ are homotopic and $\beta : Y \to Z$ is a morphism of complexes then $\beta f \sim \beta g$ are homotopic.

This implies that we have a well-defined composition of equivalence classes of morphisms modulo homotopy by defining the composition on representatives.

**Definition 1.6.** Let $\mathcal{A}$ be an additive category. The homotopy category $\mathcal{K}(\mathcal{A})$ has the same objects as the category $\mathcal{C}(\mathcal{A})$ of complexes over $\mathcal{A}$. The morphisms in the homotopy category are the equivalence classes of morphisms in $\mathcal{C}(\mathcal{A})$ modulo homotopy, i.e.

$$\text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y) := \text{Hom}_{\mathcal{C}(\mathcal{A})}(X, Y) / \sim.$$  

**Proposition 1.7.** Let $\mathcal{A}$ be an additive category. Then the homotopy category $\mathcal{K}(\mathcal{A})$ is again an additive category.

**Proof.** Addition of morphisms in $\mathcal{K}(\mathcal{A})$ is defined via addition on representatives (it is an easy observation that this is well-defined) and then the sets of morphisms $\text{Hom}_{\mathcal{K}(\mathcal{A})}(X, Y)$ inherit the structure of an abelian group from the category $\mathcal{C}(\mathcal{A})$ of complexes, and also bilinearity of composition. Moreover, the zero object is the same as in $\mathcal{C}(\mathcal{A})$.

It remains to be checked that the universal property of the coproduct $X \oplus Y$ in $\mathcal{C}(\mathcal{A})$ (cf. Proposition 1.4) also carries over to the homotopy category. In fact, the equivalence classes of the morphisms $\iota_X$, $\iota_Y$ and $f$ still make the relevant diagram (cf. Remark 1.2) commutative; for uniqueness we observe that if there is another morphism $g$ making the diagram for the universal property commutative in $\mathcal{K}(\mathcal{A})$, i.e. up to homotopy, then this gives a homotopy between $f$ and $g$.  

### 2. Abelian categories

In this section we shall review the fundamental definition of an abelian category, including the necessary background on the categorical notions of kernels and cokernels. The prototype example of an abelian category will be the category $\textbf{R-Mod}$ of modules over a ring $R$; but we will also see other examples in due course.

We first recall some notions from category theory. Let $\mathcal{A}$ be an additive category; in particular for every pair of objects $X, Y$ there is a zero morphism, namely the composition of the unique morphisms $X \to 0 \to Y$ involving the zero object of $\mathcal{A}$.

The kernel of a morphism $f : X \to Y$ is an object $K$ together with a morphism $k : K \to X$ such that

(i) $f \circ k = 0$
(ii) (universal property) for every morphism \( k' : K' \to X \) such that \( f \circ k' = 0 \), there is a unique morphism \( g : K' \to K \) making the following diagram commutative

![Diagram](image)

By the usual universal property argument, the kernel, if it exists, is unique up to isomorphism; notation: \( \ker f \).

Dually, the \textit{cokernel} of a morphism \( f : X \to Y \) is an object \( C \) together with a morphism \( c : Y \to C \) such that

(i) \( c \circ f = 0 \)

(ii) (universal property) for every morphism \( c' : Y \to C' \) such that \( c' \circ f = 0 \), there is a unique morphism \( g : C \to C' \) making the following diagram commutative

![Diagram](image)

Again, the cokernel, if it exists, is unique up to isomorphism; notation: \( \text{coker} f \).

If the above morphism \( k : \ker f \to X \) has a cokernel in \( \mathcal{A} \), this is called the \textit{coimage} of \( f \), and it is denoted by \( \text{coim} f \).

If the above morphism \( c : Y \to \text{coker} f \) has a kernel in \( \mathcal{A} \), this is called the \textit{image} of \( f \) and it is denoted by \( \text{im} f \).

**Example 2.1.** Let \( R \) be a ring. In the category \textbf{R-Mod} of all \( R \)-modules the categorical kernels and cokernels are the usual ones, i.e., for a morphism \( f : X \to Y \) we have \( \ker f = \{ x \in X \mid f(x) = 0 \} \) and \( \text{coker} f = Y / \text{im} f \) where \( \text{im} f = \{ f(x) \mid x \in X \} \) is the usual image of \( f \).

**Remark 2.2.** Suppose that for a morphism \( f \) both the coimage and the image exist. Then we claim that it follows from the universal properties that there is a natural morphism \( \text{coim} f \to \text{im} f \).

In fact, the image of \( f \) is the kernel of \( c : Y \to \text{coker} f \), hence there is a morphism \( \tilde{k} : \text{im} f \to Y \) such that \( \circ \tilde{k} = 0 \) and by the universal property there exists a unique morphism \( \tilde{g} : X \to \text{im} f \) making the following diagram commutative

![Diagram](image)
Note that $\tilde{k} \circ \tilde{g} \circ k = f \circ k = 0$, which implies that $\tilde{g} \circ k : \ker f \to \im f$ must be zero, by using the uniqueness in the diagram

Then we can consider the following diagram for the universal property of the cokernel

and deduce that there is a unique morphism $\text{coim } f \to \im f$, as desired.

**Definition 2.3.** An additive category $A$ is called an abelian category if the following axioms are satisfied:

(A4) Every morphism in $A$ has a kernel and a cokernel.

(A5) For every morphism $f : X \to Y$ in $A$, the natural morphism $\text{coim } f \to \im f$ is an isomorphism.

**Example 2.4.**

(i) Let $R$ be a ring. The category $R\text{-Mod}$ of all $R$-modules is an abelian category. In fact, (A5) follows directly from the isomorphism theorem for $R$-modules.

However, the subcategory $R\text{-mod}$ of finitely generated modules is not abelian in general since kernels of homomorphisms between finitely generated modules need not be finitely generated. Indeed we have that $R\text{-mod}$ is an abelian category if and only if $R$ is Noetherian.

In particular, the category of finite-dimensional vector spaces over a field is abelian, and the category of finitely generated abelian groups is abelian.

(ii) The subcategory of $\textbf{Ab}$ consisting of free abelian groups is not abelian.
On the other hand, for a prime number \( p \), the abelian \( p \)-groups form an abelian subcategory of \( \text{Ab} \) (an abelian group is called a \( p \)-group if for every element \( a \) we have \( p^k a = 0 \) for some \( k \)).

(iii) For finding examples of additive categories satisfying (A4) but failing to be abelian, the following observation can be useful. Suppose \( f : X \to Y \) is a morphism with \( \ker f = 0 \) and \( \coker f = 0 \), i.e. a monomorphism and an epimorphism. Then the coimage of \( f \) is the identity on \( X \), the image of \( f \) is the identity on \( Y \) and hence the natural morphism \( \text{coim } f \to \text{im } f \) is just \( f \) itself. So in this special case the axiom (A5) states that a morphism which is a monomorphism and an epimorphism must be invertible.

(iv) Explicit examples of additive categories where axiom (A5) fails for the above reason are the category of topological abelian groups (with continuous group homomorphisms) or the category of Banach complex vector spaces (with continuous linear maps). In such categories the cokernel of a morphism \( f : X \to Y \) is of the form \( Y/\overline{\text{im} f} \) where \( \overline{\text{im} f} \) is the closure of the usual set-theoretic image of \( f \). In particular, the natural morphism \( \text{coim } f \to \text{im } f \) is the inclusion of the usual image of \( f \) into its closure, and this is in general not an isomorphism.

**Proposition 2.5.** Let \( \mathcal{A} \) be an abelian category. Then the category of complexes \( C(\mathcal{A}) \) is also abelian.

**Proof.** We have seen in Proposition 1.4 that \( C(\mathcal{A}) \) is an additive category, so it remains to verify the axioms (A4) and (A5).

(A4) Let \( f : X \to Y \) be a morphism in \( C(\mathcal{A}) \), i.e. \( f = (f_n)_{n \in \mathbb{Z}} \) with \( f_n : X_n \to Y_n \) morphisms in \( \mathcal{A} \). We show the existence of a kernel and leave the details of the dual argument for the cokernel as an exercise.

Since \( \mathcal{A} \) is abelian, each morphism \( f_n : X_n \to Y_n \) has a kernel \( K_n := \ker f_n \) in \( \mathcal{A} \), coming with a morphism \( k_n : K_n \to X_n \) satisfying the above universal property. Note that for every \( n \in \mathbb{Z} \) we have \( f_{n-1} \circ d_n^X \circ k_n = d_n^Y \circ f_n \circ k_n = 0 \). Then it follows by the universal property of kernels that there is a unique morphism \( d_n^K : K_n \to K_{n-1} \) such that \( k_{n-1} \circ d_n^K = d_n^X \circ k_n \). Note that

\[
k_{n-1} \circ d_n^K \circ d_{n+1}^K = d_n^X \circ k_n \circ d_{n+1}^K = d_n^X \circ d_{n+1}^X = k_{n+1} = 0
\]

since \( X \) is a complex. By uniqueness of the map in the universal property of \( K_{n-1} \) it follows that \( d_n^K \circ d_{n+1}^K = 0 \), i.e. \( (K_n, d_n^K) \) is a complex.

Combining the universal properties of the kernels \( K_n \) it easily follows that the complex \((K_n, d_n^K)\) indeed satisfies the universal property for the kernel of \( f \) in \( C(\mathcal{A}) \).

(A5) The crucial observation is that a morphism of complexes \( f = (f_n) : X \to Y \) is an isomorphism in \( C(\mathcal{A}) \) if and only if each \( f_n \) is an isomorphism in \( \mathcal{A} \). In fact, if each \( f_n \) is an isomorphism, with inverse \( g_n \), then the family \( g = (g_n) \) is automatically a morphism of complexes (and hence clearly an inverse to \( f \) in \( C(\mathcal{A}) \)): for all \( n \in \mathbb{Z} \) we have

\[
d_{n+1}^X \circ g_{n+1} = g_n \circ f_n \circ d_{n+1}^X \circ g_{n+1} + g_n \circ d_{n+1}^Y \circ f_{n+1} \circ g_{n+1} = g_n \circ d_{n+1}^Y.
\]

The reverse implication is obvious.

For axiom (A5) now consider the natural morphism \( \text{coim } f \to \text{im } f \). In the proof of (A4) above we have seen that kernels and cokernels in \( C(\mathcal{A}) \), and hence also the morphism \( \text{coim } f \to \text{im } f \), are obtained degreewise. But since \( \mathcal{A} \) is abelian by assumption, we know that for every \( n \) the natural morphism \( \text{coim } f_n \to \text{im } f_n \) in
\(A\) is indeed an isomorphism. Then, by the introductory remark, the morphism of complexes \((\text{coim} f_n \to \text{im} f_n)_{n \in \mathbb{Z}}\) is an isomorphism in \(\text{C}(A)\). □

An important observation is that the homotopy category \(\text{K}(A)\) is not abelian in general, even if \(A\) is abelian.

**Example 2.6.** We provide an explicit example for the failure of axiom (A4) in a homotopy category. Consider the abelian category \(A = \text{Ab}\) of abelian groups.

Let \(f : X \to Y\) be the following morphism of complexes of abelian groups, with non-zero entries in degrees 1 and 0,

\[
\begin{array}{cccccccc}
\cdots & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow \text{id} & & \downarrow \text{id} & & \\
\cdots & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots 
\end{array}
\]

In the category \(\text{C}(\text{Ab})\) of complexes \(f\) is non-zero and has the zero complex as kernel (cf. the proof of Proposition 2.5). However, \(f\) is homotopic to zero (with the identity as homotopy map), i.e. \(f = 0\) in the homotopy category \(\text{K}(\text{Ab})\).

We claim that in the homotopy category \(f\) has no kernel. Recall the categorical definition of the kernel of a morphism \(f : X \to Y\) from Section 2.

Suppose for a contradiction that our morphism \(f\) had a kernel in \(\text{K}(\text{Ab})\). So there is a complex \(\cdots \to K_1 \to K_0 \to K_{-1} \to \cdots\) and a morphism \(k = k_0 : K_0 \to \mathbb{Z}\) of abelian groups (in all other degrees the map \(k\) has to be zero since \(X\) is concentrated in degree 0). The image of \(k\), being a subgroup of \(\mathbb{Z}\), has the form \(r\mathbb{Z}\) for some fixed \(r \in \mathbb{Z}\). Now choose \(K' = X\) and consider the morphisms \(l : K' \to X\) given by multiplication with \(l\) for any \(l \in \mathbb{Z}\). Clearly, \(f \circ l = 0\) in \(\text{K}(\text{Ab})\) since \(f = 0\) in \(\text{K}(\text{Ab})\). According to the universal property of a kernel, there must exist (unique) morphisms \(u_l : \mathbb{Z} \to K_0\) such that \(k \circ u_l = l\) up to homotopy. However, these maps are from \(K' = X\) to \(X\) and this complex is concentrated in degree 0. Thus there are no non-zero homotopy maps and so \(k \circ u_l = l\) as morphism of abelian groups. But the image of \(k \circ u_l\) is contained in the image of \(k\) which is \(r\mathbb{Z}\) for a fixed \(r\), so \(k \circ u_l = l\) can not hold for arbitrary \(l \in \mathbb{Z}\), a contradiction.

Hence axiom (A4) fails and therefore the homotopy category \(\text{K}(\text{Ab})\) is not an abelian category.

### 3. Definition of triangulated categories

We have seen in the previous section that the homotopy category of complexes is not abelian in general. We shall see in Section 6 below that \(\text{K}(A)\) carries the structure of a triangulated category, a concept which we are going to define in this section. Roughly, one should think of the distinguished triangles occurring in this context as a replacement for short exact sequences (which do not exist in general since \(\text{K}(A)\) is not abelian). However, for an additive category to be abelian is purely an inherent property of the category. On the other hand a triangulated structure is an extra piece of data, consisting of a suspension functor and a set of distinguished triangles chosen suitably to satisfy certain axioms. In particular, an additive category can have many different triangulated structures; see [1] for more details and examples.
A functor $\Sigma$ between additive categories is called an \textit{additive functor} if for every pair of objects $X, Y$ the map $\text{Hom}(X, Y) \to \text{Hom}(\Sigma(X), \Sigma(Y))$ is a homomorphism of abelian groups.

Let $\mathcal{T}$ be an additive category and let $\Sigma : \mathcal{T} \to \mathcal{T}$ be an additive functor which is an automorphism (i.e. it is invertible, thus there exists a functor $\Sigma^{-1}$ on $\mathcal{T}$ such that $\Sigma \circ \Sigma^{-1}$ and $\Sigma^{-1} \circ \Sigma$ are the identity functors).

A \textit{triangle} in $\mathcal{T}$ is a sequence of objects and morphisms in $\mathcal{T}$ of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$ 

A \textit{morphism of triangles} is a triple $(f, g, h)$ of morphisms such that the following diagram is commutative in $\mathcal{T}$

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow \Sigma f \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}
\]

If in this situation, the morphisms $f, g$ and $h$ are isomorphisms in $\mathcal{T}$, then the morphism of triangles is called an \textit{isomorphism of triangles}.

\textbf{Definition 3.1.} A \textit{triangulated category} is an additive category $\mathcal{T}$ together with an additive automorphism $\Sigma$, the translation or shift functor, and a collection of distinguished triangles satisfying the following axioms

\begin{enumerate}
\item[(TR0)] Any triangle isomorphic to a distinguished triangle is again a distinguished triangle.
\item[(TR1)] For every object $X$ in $\mathcal{T}$, the triangle $X \xrightarrow{\text{id}} X \to 0 \to \Sigma X$ is a distinguished triangle.
\item[(TR2)] For every morphism $f : X \to Y$ in $\mathcal{T}$ there is a distinguished triangle of the form $X \xrightarrow{\Sigma f} Y \to Z \to \Sigma X$.
\item[(TR3)] If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is a distinguished triangle, then also $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{\Sigma u} \Sigma Y$ is a distinguished triangle, and vice versa.
\item[(TR4)] Given distinguished triangles $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ and $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} \Sigma X'$, then each commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & \Sigma X \\
\downarrow f & & \downarrow g & & \downarrow \Sigma f & & \\
X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & \Sigma X'
\end{array}
\]

\end{enumerate}

\[\text{can be completed to a morphism of triangles (but not necessarily uniquely).}\]

\textbf{(Octahedral axiom)} Given distinguished triangles $X \xrightarrow{u} Y \to Z' \to \Sigma X$, $Y \xrightarrow{v} Z \to X' \to \Sigma Y$ and $X \xrightarrow{w} Z \to Y' \to \Sigma X$, there exists a distinguished triangle $Z' \to Y' \to X' \to \Sigma Z'$ making the following diagram commutative.
Remark 3.2. The above version (TR5) of the octahedral axiom is taken from the book by Kashiwara and Schapira [7, Sec. 1.4]. There are various other versions appearing in the literature which are equivalent to (TR5), see for instance A. Neeman’s article [12] or his book [11]; a short treatment can also be found in A. Hubery’s notes [6] (which are based on the former references).

We shall only mention two variations here. Mainly a reformulation of the axiom (TR5) is the following. Note that in (TR5) the given three distinguished triangles are placed in the first three rows, whereas in (TR5’) below they are placed in the first two rows and the second column.

(TR5’) Given distinguished triangles $X \overset{u}{\to} Y \to Z' \to \Sigma X$, $Y \overset{v}{\to} Z \to X' \to \Sigma Y$ and $X' \overset{s}{\to} Y' \to \Sigma X$, then there exists a distinguished triangle $Z' \to Y' \to \Sigma X'$ making the following diagram commutative and satisfying $(\Sigma u)s = lv$. 

It is not difficult to check that (TR5) and (TR5’) are indeed equivalent; we leave this verification as an exercise to the reader.

The following version (TR5”) of the octahedral axiom can be found in Neeman’s book [11, Prop. 1.4.6]. It is less obvious that it is equivalent to (TR5); for details on this we refer the reader to [11], [12] and [6].

(TR5”) Given distinguished triangles $X \overset{u}{\to} Y \to Z' \to \Sigma X$, $Y \overset{v}{\to} Z \to X' \to \Sigma Y$ and $X' \overset{s}{\to} Y' \to \Sigma X$, then there exists a distinguished triangle $Z' \to Y' \to \Sigma X'$ making the following diagram commutative in which every row and every column is a distinguished triangle.
4. Some formal properties of triangulated categories

We shall draw some first consequences from the definition. Let $T$ be a triangulated category with translation functor $\Sigma$.

**Proposition 4.1.** (Composition of morphisms) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle. Then $v \circ u = 0$ and $w \circ v = 0$, i.e. any composition of two consecutive morphisms in a distinguished triangle vanishes.

**Proof.** By the rotation property (TR3) it suffices to show that $v \circ u = 0$. Also by (TR3) we have a distinguished triangle $Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$. By (TR1) and (TR4) the following diagram can be completed to a morphism of triangles.

$$
\begin{array}{cccccccc}
X & \xrightarrow{u} & Y & \xrightarrow{v} & Z' & \xrightarrow{\Sigma} & \Sigma X \\
\downarrow{id_X} & & \downarrow{v} & \quad & \downarrow{id_{\Sigma X}} \\
X & \xrightarrow{vu} & Z & \xrightarrow{w} & \Sigma Y & \xrightarrow{-\Sigma u} & \Sigma^2 X \\
\downarrow{0} & \quad & \downarrow{id_Y} & \quad & \downarrow{0} & \quad & \\
0 & \xrightarrow{X'} & 0 & \quad & \quad & \quad & \\
\downarrow{\Sigma X} & \quad & \downarrow{\Sigma Y} & \quad & \downarrow{\Sigma Z'} & \quad & \downarrow{\Sigma^2 X}
\end{array}
$$

In particular, $-\Sigma(v \circ u) = -\Sigma v \circ \Sigma u = 0$ which implies $v \circ u = 0$ since $\Sigma$ is an automorphism. \hfill \Box

**Proposition 4.2.** (Long exact sequences) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle. For any object $T \in T$ there is a long exact sequence of abelian groups

$$
\ldots \rightarrow \text{Hom}_T(T, \Sigma^i X) \xrightarrow{\Sigma^i u_*} \text{Hom}_T(T, \Sigma^i Y) \xrightarrow{\Sigma^i v_*} \text{Hom}_T(T, \Sigma^i Z) \xrightarrow{\Sigma^i w_*} \text{Hom}_T(T, \Sigma^{i+1} X) \rightarrow \ldots
$$

**Proof.** For abbreviation we denote by $f_* := \text{Hom}_T(T, f)$ the morphism induced by $f$ under the functor Hom$_T(T, -)$ on the additive category $T$.

By the rotation property, it suffices to show that

$$
\text{Hom}_T(T, \Sigma^i X) \xrightarrow{\Sigma^i u_*} \text{Hom}_T(T, \Sigma^i Y) \xrightarrow{\Sigma^i v_*} \text{Hom}_T(T, \Sigma^i Z)
$$

is an exact sequence of abelian groups.

By Proposition 4.1 we have $\Sigma^i v \circ \Sigma^i u = 0$ and hence also $\Sigma^i v_* \circ \Sigma^i u_* = 0$, i.e. the image of $\Sigma^i u_*$ is contained in the kernel of $\Sigma^i v_*$. Conversely, take $f$ in the kernel of $\Sigma^i v_*$. Consider the following diagram whose rows are distinguished triangles by (TR1) and (TR3).
The left hand square is commutative by assumption on $f$. By (TR4) there exists a morphism $h : \Sigma^{-i+1}T \to \Sigma X$ completing the above diagram to a morphism of triangles. In particular, $\Sigma^{-i+1}f = \Sigma u \circ h$ and hence $f = \Sigma^i u \circ \Sigma^{-1}h$ is in the image of $\Sigma u_*$ as desired. □

**Proposition 4.3.** (Triangulated 5-lemma) Suppose we are given a morphism of distinguished triangles as in the following diagram.

If $f$ and $g$ are isomorphisms then also $h$ is an isomorphism.

**Proof.** We apply the functor $\text{Hom}(Z', -) := \text{Hom}_T(Z', -)$ to the distinguished triangles. By Proposition 4.2 this leads to the following commutative diagram whose rows are exact sequences of abelian groups.

By assumption, $f$ and $g$ are isomorphisms and hence also $f_*, g_*, \Sigma f_*$ and $\Sigma g_*$ are isomorphisms. So we can appeal to the usual 5-lemma in the category of abelian groups to deduce that $h_*$ is an isomorphism. In particular the identity $\text{id}_{Z'}$ has a preimage, i.e. there exists a morphism $q \in \text{Hom}_T(Z', Z)$ such that $h \circ q = \text{id}_{Z'}$. A similar argument using the functor $\text{Hom}_T(-, Z')$ produces a left inverse to $h$, thus $h$ is an isomorphism. □

**Proposition 4.4.** (Split triangles) Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ be a distinguished triangle where $w = 0$ is the zero morphism. Then the triangle splits, i.e. $u$ is a split monomorphism and $v$ is a split epimorphism.

**Remark 4.5.** The notion of split monomorphism is synonymous with that of a section, and a split epimorphism is also known as a retraction.

**Proof.** We first show that $u$ is a split monomorphism, i.e. there exists a morphism $u'$ such that $u' \circ u = \text{id}_X$. We have the following commutative diagram of distinguished triangles.
By (TR3) and (TR4) it can be completed to a morphism of triangles, i.e. there exists $u' : Y \to X$ such that $u' \circ u = \text{id}$.

Similarly, one can show that $v$ is a split epimorphism, i.e. there is a morphism $v' : Z \to Y$ such that $v \circ v' = \text{id}$. □

5. ABELIAN CATEGORIES VS. TRIANGULATED CATEGORIES

As an application of the formal properties in the previous section we shall compare the notions of abelian categories and triangulated categories.

Definition 5.1. An abelian category $\mathcal{A}$ is called semisimple if every short exact sequence in $\mathcal{A}$ splits.

Example 5.2. (i) Let $R$ be a semisimple ring. Then the module categories $\text{R-Mod}$ and $\text{R-mod}$ are semisimple. In particular, the category of vector spaces $\text{Vec}_K$ over a field $K$ is semisimple.

(ii) The category $\text{Ab}$ of abelian groups is not semisimple. For instance, the short exact sequence $0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0$ does not split.

The following result illustrates that the concepts of abelian and triangulated categories overlap only slightly.

Theorem 5.3. Let $\mathcal{T}$ be a category which is triangulated and abelian. Then $\mathcal{T}$ is semisimple.

Proof. Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short exact sequence in $\mathcal{T}$. We have to show that it splits; to this end it suffices to show that $f$ is a section, i.e. there exists a morphism $f' : Y \to X$ such that $f' \circ f = \text{id}_X$.

By (TR2) and (TR3), $f$ can be embedded into a distinguished triangle

$$
\Sigma^{-1}V \xrightarrow{u} X \xrightarrow{f} Y \xrightarrow{v} V.
$$

The composition of consecutive morphisms in a distinguished triangle is always zero by Proposition 4.1, in particular $f \circ u = 0$. But $f$ is a monomorphism in $\mathcal{T}$ since it is the first map in a short exact sequence, hence $u = 0$. Thus we have a distinguished triangle

$$
X \xrightarrow{f} Y \xrightarrow{v} V \xrightarrow{\Sigma u} \Sigma X
$$

where $\Sigma u = 0$. Now the triangle splits by Proposition 4.4. □

We shall see in the next section that the homotopy category $\mathcal{K}(\mathcal{A})$ of complexes over an additive category $\mathcal{A}$ is a triangulated category. This, together with the preceding theorem, will then give a more structural explanation of the earlier observation that $\mathcal{K}(\text{Ab})$ is not abelian in Example 2.6, where we have used an ad-hoc argument to show that morphisms do not necessarily have a kernel.

6. THE HOMOTOPIY CATEGORY OF COMPLEXES IS TRIANGULATED

Let $\mathcal{A}$ be an additive category, with corresponding category of complexes $\mathcal{C}(\mathcal{A})$ and homotopy category $\mathcal{K}(\mathcal{A})$.

As discussed above, the homotopy category $\mathcal{K}(\mathcal{A})$ is in general not abelian, even if $\mathcal{A}$ is abelian. We shall explain in this section how the homotopy category $\mathcal{K}(\mathcal{A})$ becomes a triangulated category.
We first need an additive automorphism on $K(A)$ which serves as translation functor. This functor can already be defined on the level of the category $C(A)$.

**Definition 6.1.** In $C(A)$ we construct a translation functor $\Sigma = [1]$ by shifting any complex one degree to the left. More precisely, for an object $X = (X_n, d^X_n)_{n \in \mathbb{Z}}$ in $C(A)$ we set

$$X[1] := (X[1]_n, d^X[1]_n)_{n \in \mathbb{Z}} \text{ with } X[1]_n = X_{n-1} \text{ and } d^X[1]_n = -d^X_{n-1}.$$ 

For a morphism of complexes $f = (f_n)_{n \in \mathbb{Z}}$ in $C(A)$ we set

$$f[1] := (f[1]_n)_{n \in \mathbb{Z}} \text{ where } f[1]_n = f_{n-1}.$$ 

**Remark 6.2.**

(i) The sign appearing in the differential of $X[1]$ might look auxiliary; it will become clear later when discussing the triangulated structure of the homotopy category why this sign is needed.

(ii) The functor $\Sigma = [1]$ defined above is an additive functor and moreover an automorphism of the category $C(A)$.

(iii) Note that the above definitions are compatible with homotopies so we have a well-defined induced functor $\Sigma = [1]$ on the homotopy category $K(A)$.

The next step for getting a triangulated structure on the homotopy category is to find a suitable set of distinguished triangles. To this end, the following construction of mapping cones is crucial.

**Definition 6.3.** Let $f$ be a morphism between complexes $X = (X_n, d^X_n)$ and $Y = (Y_n, d^Y_n)$. The mapping cone $M(f)$ is the complex in $C(A)$ defined by

$$M(f)_n = X_{n-1} \oplus Y_n \text{ and } d^{M(f)}_n := \begin{pmatrix} -d^X_{n-1} & 0 \\ f_{n-1} & d^Y_n \end{pmatrix}.$$ 

**Remark 6.4.**

(i) There are canonical morphisms in $C(A)$ as follows

$$\alpha(f) : Y \to M(f) , \alpha(f)_n := (0, id_{Y_n})$$

and

$$\beta(f) : M(f) \to X[1] , \beta(f)_n := (id_{X_{n-1}}, 0).$$

Note that $\beta(f)$ is a morphism of complexes because the differential in $X[1]$ carries a sign. From the above definitions we get a short exact sequence of chain complexes

$$0 \to Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \to 0.$$ 

(ii) Let $f : X \to Y$ be a morphism of complexes. The short exact sequence $0 \to Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \to 0$ splits (i.e. there is a morphism of complexes $\sigma : X[1] \to M(f)$ such that $\beta(f) \circ \sigma = id_{X[1]}$) if and only if $f$ is homotopic to zero. In fact, a splitting map is given by $\sigma(x) := (x, -s(x))$ where $s$ is a homotopy map.

**Example 6.5.**

(i) For any complex $X$ consider the zero map $f : X \to 0$ to the zero complex. Then the mapping cone is $M(f) = X[1]$. On the other hand, the mapping cone of $g : 0 \to Y$ is just $M(g) = Y$ itself.
(ii) Let $A$ and $B$ be objects in $\mathcal{A}$ and view them as complexes $X_A$ and $X_B$ concentrated in degree 0. Any morphism $f : A \rightarrow B$ in $\mathcal{A}$ induces a morphism of complexes $f : X_A \rightarrow X_B$. Its mapping cone is the complex

$$\cdots \rightarrow 0 \rightarrow A \xrightarrow{f} B \rightarrow 0 \rightarrow \cdots$$

where $A$ is in degree 1 and $B$ in degree 0.

(iii) Let $X = (X_n, d^X_n)$ be any complex in $\mathcal{C}(\mathcal{A})$. The mapping cone of the identity morphism $\text{id}_X$ has degree $n$ term equal to $X_{n-1} \oplus X_n$ and differential

$$\left( \begin{array}{cc} -d^X_{n-1} & 0 \\ \text{id}_{X_{n-1}} & d^X_n \end{array} \right) : X_{n-1} \oplus X_n \rightarrow X_{n-2} \oplus X_{n-1}.$$ 

The identity morphism on the mapping cone $M(\text{id}_X)$ is homotopic to zero, via the map $s = (s_n)_{n \in \mathbb{Z}}$ where $s_n = \left( \begin{array}{cc} 0 & \text{id}_{X_n} \\ 0 & 0 \end{array} \right)$. Thus, in the homotopy category $\mathcal{K}(\mathcal{A})$ the identity $\text{id}_{M(\text{id}_X)}$ is equal to the zero map. As a consequence, in the homotopy category, the mapping cone $M(\text{id}_X)$ is isomorphic to the zero complex.

It is easy to check that the morphisms $\alpha(f)$ and $\beta(f)$ are also well-defined on the homotopy category $\mathcal{K}(\mathcal{A})$ (i.e. independent on the choice of representatives of the equivalence class of morphisms). This leads to the following definition.

**Definition 6.6.** A sequence of objects and morphisms in the homotopy category $\mathcal{K}(\mathcal{A})$ of the form

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$$

is called a standard triangle.

A distinguished triangle in $\mathcal{K}(\mathcal{A})$ is a triangle which is isomorphic (in $\mathcal{K}(\mathcal{A})$!) to a standard triangle.

With this class of distinguished triangles the homotopy category obtains a triangulated structure as we shall show next. Due to the technical nature of the axioms of a triangulated category, the proof that a certain additive category is indeed triangulated is usually rather long, can be partly tedious and can still be quite involved. In this introductory chapter we want to present such a proof at least once in detail.

**Theorem 6.7.** Let $\mathcal{A}$ be an additive category. Then the homotopy category of complexes $\mathcal{K}(\mathcal{A})$ is a triangulated category.

**Proof.** We have to show that with the above translation functor [1] and the set of distinguished triangles just defined, the axioms (TR0)-(TR5) are satisfied.

The axioms (TR0) and (TR2) hold by Definition 6.6.

(TR1) From the mapping cone construction there is a standard triangle

$$X \xrightarrow{\text{id}_X} X \rightarrow M(\text{id}_X) \rightarrow X[1].$$

By Example 6.5 above, $M(\text{id}_X)$ is isomorphic to the zero complex in the homotopy category. Hence we indeed have a distinguished triangle

$$X \xrightarrow{\text{id}_X} X \rightarrow 0 \rightarrow X[1].$$

(TR3) Because the rotation property is compatible with isomorphisms of triangles, it suffices to prove (TR3) for a standard triangle

$$X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1].$$
We shall show that the rotated triangle
\[ Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \xrightarrow{-f[1]} Y[1] \]
is isomorphic in \( K(A) \) to the following standard triangle for \( \alpha(f) \),
\[ Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\alpha(\alpha(f))} M(\alpha(f)) \xrightarrow{\beta(\alpha(f))} Y[1]. \]

For constructing an isomorphism between the latter two triangles we take the identity maps for the first, second and fourth entries. Moreover, we define morphisms
\[ \phi = (\phi_n) : X[1] \rightarrow M(\alpha(f)) \]by setting \( \phi_n = (-f_{n-1}, \text{id}_{X_{n-1}}, 0) \)
and conversely
\[ \psi = (\psi_n) : M(\alpha(f)) \rightarrow X[1] \]by setting \( \psi_n = (0, \text{id}_{X_{n-1}}, 0). \)
These yield morphisms of triangles since by definition \( \beta(\alpha(f)) \circ \phi = -f[1] \), and \( \phi \circ \beta(f) \sim \alpha(\alpha(f)) \) via the homotopy given by
\[ \begin{pmatrix} 0 & 0 & -\text{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : M(f)_n = X_{n-1} \oplus Y_n \rightarrow M(\alpha(f))_{n+1} = Y_n \oplus X_n \oplus Y_{n+1}. \]

Similarly, \( \psi \) is a morphism of triangles since \( \beta(f) = \psi \circ \alpha(\alpha(f)) \) by definition and \( -f[1] \circ \psi \sim \beta(\alpha(f)) \) via the homotopy \( (0, 0, -\text{id}) : M(\alpha(f))_n \rightarrow Y[1]_n. \)

Finally, and most importantly for proving (TR3), the above morphisms are isomorphisms in \( K(A) \) because we have \( \psi \circ \phi = \text{id}_{X[1]} \) (by definition) and \( \phi \circ \psi \sim \text{id}_{M(\alpha(f))} \) via the homotopy map
\[ \begin{pmatrix} 0 & 0 & -\text{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : M(\alpha(f))_n \rightarrow M(\alpha(f))_{n+1} \]
(recall that \( M(\alpha(f))_n = Y_{n-1} \oplus X_{n-1} \oplus Y_n \)).

(Tr4) Again it suffices to prove the axiom for standard triangles. By assumption we have a diagram
\[ \begin{array}{ccc}
X & \xrightarrow{u} & Y \\
\downarrow{f} & & \downarrow{g} \\
X' & \xrightarrow{u'} & Y'
\end{array} \xrightarrow{\alpha(u)} M(u) \xrightarrow{\beta(u)} X[1] \xrightarrow{-f[1]} Y[1]
\]
where the left square commutes in \( K(A) \), i.e. there exist homotopy maps \( s_n : X_n \rightarrow Y_{n+1} \) such that \( g_n u_n - u'_n f_n = d_{n+1} u'_n s_n + s_{n-1} d_n X \) for all \( n \in \mathbb{Z} \). For completing the diagram to a morphism of triangles we define \( h = (h_n)_{n \in \mathbb{Z}} : M(u) \rightarrow M(u') \) by setting
\[ h_n = \begin{pmatrix} f_{n-1} \\ s_n \\ g_n \end{pmatrix} : M(u)_n = X_{n-1} \oplus Y_n \rightarrow M(u')_n = X'_{n-1} \oplus Y'_{n+1}. \]
This is indeed a morphism of complexes because of the homotopy property of \( s \)
given above. Moreover, the completed diagram commutes since by definition we have that \( h \circ \alpha(u) = \alpha(u') \circ g \) and \( \beta(u') \circ h = f[1] \circ \beta(u) \); note that these are proper equalities, not only up to homotopy.

(Tr5) Again it suffices to prove the octahedral axiom for standard triangles. From the assumptions we already have the following part of the relevant diagram
We now define the missing morphisms as follows. Let \( f = (f_n) : M(u) \to M(vu) \) be given in degree \( n \) by \( f_n = \begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & v_n \end{pmatrix} \) and set \( g = (g_n) : M(vu) \to M(v) \) to be given by \( g_n = \begin{pmatrix} u_{n-1} & 0 \\ 0 & \mathrm{id}_{Z_n} \end{pmatrix} \). Finally define \( h : M(v) \to M(u)[1] \) as the composition \( \alpha(f)[1] \circ \beta(v) \), i.e. it is given by the matrix \( \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y_{n-1}} & 0 \end{pmatrix} \). Then it is easy to check from the definitions that all squares in the completed diagram commute (not only up to homotopy).

For proving (TR5) it now remains to show that the bottom line
\[
M(u) \xrightarrow{f} M(vu) \xrightarrow{g} M(v) \xrightarrow{h} M(u)[1]
\]
is a distinguished triangle in \( \mathbf{K}(A) \). To this end we construct an isomorphism to the standard triangle
\[
M(u) \xrightarrow{f} M(vu) \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} M(u)[1].
\]
Note that only the third entries in the triangles are different. So it suffices to find morphisms \( \sigma = (\sigma_n) : M(v) \to M(f) \) and \( \tau = (\tau_n) : M(f) \to M(v) \) leading to commutative diagrams (in \( \mathbf{K}(A)! \)), i.e. we need that \( \beta(f) \circ \sigma = h \), \( h \circ \tau = \beta(f) \), \( \sigma \circ g = \alpha(f) \) and \( \tau \circ \alpha(f) = g \), up to homotopy. Moreover, we have to show that they are isomorphisms in the homotopy category. We set
\[
\sigma_n := \begin{pmatrix} 0 & 0 \\ \mathrm{id}_{Y_{n-1}} & 0 \\ 0 & \mathrm{id}_{Z_n} \\ 0 & 0 \end{pmatrix}
\]
and
\[
\tau_n := \begin{pmatrix} 0 & \mathrm{id}_{Y_{n-1}} & u_{n-1} & 0 \\ 0 & 0 & 0 & \mathrm{id}_{Z_n} \end{pmatrix}.
\]
First, let us check that \( \sigma \) and \( \tau \) give commutative diagrams. Directly from the definitions we get that \( \tau \circ \alpha(f) = g \); in fact both are given in degree \( n \) by the map \( \begin{pmatrix} u_{n-1} & 0 \\ 0 & \mathrm{id}_{Z_n} \end{pmatrix} : X_{n-1} \oplus Z_n \to Y_{n-1} \oplus Z_n \). Also by definition we see that \( \beta(f) \circ \sigma = h \), both given by \( \begin{pmatrix} 0 & \mathrm{id}_{Y_{n-1}} & 0 \\ 0 & 0 & 0 \end{pmatrix} : Y_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1} \).

The remaining commutativities will now only hold up to homotopy. Note that
\[ \alpha(f) - \sigma \circ g : M(vu) \to M(f) \] is given in degree \( n \) by
\[
\begin{pmatrix}
0 & 0 \\
-u_{n-1} & 0 \\
\text{id}_{X_{n-1}} & 0 \\
0 & 0
\end{pmatrix} : X_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n.
\]

We claim that \( \alpha(f) - \sigma \circ g \) is homotopic to zero, i.e. \( \alpha(f) = \sigma \circ g \) in \( K(A) \). In fact, a homotopy map \( s = (s_n) \) where \( s_n : M(vu)_n \to M(f)_{n+1} \) is given by
\[
\begin{pmatrix}
\text{id}_{X_{n-1}} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix} : X_{n-1} \oplus Z_n \to X_{n-1} \oplus Y_n \oplus X_n \oplus Z_{n+1}.
\]

For verifying the details recall that the differential of the mapping cone \( M(f) \) is given by
\[
d_n^{M(f)} = \begin{pmatrix}
d_{n-2}^X & 0 & 0 & 0 \\
-u_{n-2} & -d_{n-1}^Y & 0 & 0 \\
\text{id}_{X_{n-2}} & 0 & -d_{n-1}^X & 0 \\
0 & v_{n-1} & (vu)_{n-1} & d_n^Z
\end{pmatrix}.
\]

Finally, consider \( \beta(f) - h \circ \tau : M(f) \to M(u)[1] \) which in degree \( n \) is given by
\[
\begin{pmatrix}
\text{id}_{X_{n-2}} & 0 & 0 & 0 \\
0 & 0 & -u_{n-1} & 0
\end{pmatrix} : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \to X_{n-2} \oplus Y_{n-1}.
\]

This can be seen to be homotopic to zero by using the homotopy map \( s = (s_n) \) where
\[
s_n = \begin{pmatrix}
0 & 0 & \text{id}_{X_{n-1}} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} : X_{n-2} \oplus Y_{n-1} \oplus X_{n-1} \oplus Z_n \to X_{n-1} \oplus Y_n.
\]

For the straightforward verification again use the differential of \( M(f) \) as given above.

For completing the proof it now remains to show that \( \sigma \) and \( \tau \) are isomorphisms in the homotopy category. We have \( \tau \circ \sigma = \text{id}_{M(vu)} \) by definition. Conversely, the composition \( \sigma \circ \tau \) is in degree \( n \) given by
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \text{id}_{Y_{n-1}} & u_{n-1} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \text{id}_{Z_n}
\end{pmatrix}
\]

If we then define homotopy maps \( s_n : M(f)_n \to M(f)_{n+1} \) by setting
\[
s_n := \begin{pmatrix}
0 & 0 & -\text{id}_{X_{n-1}} & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

then we have \( \sigma \circ \tau - \text{id}_{M(f)} = d_{n+1}^{M(f)} \circ s_n + s_{n-1} \circ d_n^{M(f)} \) which is easily checked using the differential of \( M(f) \) as given above.

Thus \( \sigma \circ \tau = \text{id}_{M(f)} \) in the homotopy category \( K(A) \) and we have proved the octahedral axiom for \( K(A) \). \( \square \)
Remark 6.8. We have seen that for every standard triangle
\[ X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \]
in \( \mathbf{K}(\mathcal{A}) \) there is a corresponding short exact sequence
\[ 0 \rightarrow Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \rightarrow 0 \]
in \( \mathbf{C}(\mathcal{A}) \). On the other hand, it is not true that any short exact sequence in \( \mathbf{C}(\mathcal{A}) \) would lead to a distinguished triangle in the homotopy category \( \mathbf{K}(\mathcal{A}) \).

As an example, consider the short exact sequence of abelian groups
\[ 0 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow 0, \]
and consider the abelian groups as complexes concentrated in degree 0. There is no corresponding distinguished triangle
\[ \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}[1]. \]
in \( \mathbf{K}(\mathbf{Ab}) \). In fact, suppose for a contradiction that such a distinguished triangle existed. The morphisms in \( \mathbf{K}(\mathbf{Ab}) \) are just equivalence classes of morphisms of complexes modulo homotopy. But since \( \mathbb{Z}/2\mathbb{Z} \) is a complex concentrated in a single degree, there are no nonzero morphisms \( \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}[1] \) to its shifted version. Thus we must have \( w = 0 \).

By Proposition 4.4 a triangle with a zero map is a split triangle. Hence \( \mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) in \( \mathbf{K}(\mathbf{Ab}) \), i.e. there must exist a homotopy equivalence between these complexes. However, all these complexes are complexes concentrated in a single degree, hence there are no nonzero homotopy maps. So the above isomorphism would have to be an isomorphism already in \( \mathbf{Ab} \) which is impossible, a contradiction.

We shall later see that this phenomenon disappears when passing from \( \mathbf{K}(\mathcal{A}) \) to the derived category. There every short exact sequence does lead to a distinguished triangle; see Section 7.6 below for details.

7. Derived categories

A very important class of triangulated categories is formed by derived categories. They occur frequently in many different areas of mathematics and have found numerous applications. In this section we shall provide the relevant constructions leading from the homotopy category to the derived category.

7.1. Homology and quasi-isomorphisms. In this short section we shall introduce the notion of quasi-isomorphism which is fundamental for derived categories.

Although one could set up a homology theory in a categorical manner in every abelian category we shall restrict from now on to categories of modules and to complexes over them. This considerably simplifies the presentation in certain parts of this section since we can then use element-wise arguments and hence avoid technical overload which might obscure the fundamental ideas underlying the definition of a derived category.

For the remainder of this section we let \( \mathcal{A} \) be a category of modules over a ring.

Definition 7.1. (i) Let \( X = (X_n, d^X_n) \) be a complex in \( \mathbf{C}(\mathcal{A}) \). The \( n \)-th homology of the complex \( X \) is defined as the following object from \( \mathcal{A} \),
\[ H_n(X) := \ker d^X_n / \text{im} d^X_{n+1} \]
Then they induce the same map in homology, i.e. \( H_n(K) \) functors on the homotopy category.

Proof. By assumption there is a homotopy map \( f, g : X \to Y \) be a morphism of complexes. We define an induced map on the level of homology by setting

\[
H_n(f) : H_n(X) \to H_n(Y), \quad x + \text{im} \, d_{n+1}^X \mapsto f_n(x) + \text{im} \, d_{n+1}^Y.
\]

In this way we get homology functors \( H_n : \mathcal{C}(\mathcal{A}) \to \mathcal{A} \) where \( n \in \mathbb{Z} \).

Remark 7.2. Note that the induced map on homology is well-defined; in fact let \( x' = d_{n+1}^X(x'') \in \text{im} \, d_{n+1}^X; \) then

\[
f_n(x') = f_n(d_{n+1}^X(x'')) = d_{n+1}^Y(f_{n+1}(x'')) \in \text{im} \, d_{n+1}^Y.
\]

Proposition 7.3. Let \( f, g : X \to Y \) be morphisms in \( \mathcal{C}(\mathcal{A}) \) which are homotopic. Then they induce the same map in homology, i.e. \( H_n(f) = H_n(g) \) for all \( n \in \mathbb{Z} \).

As a consequence, the homology functors on \( \mathcal{C}(\mathcal{A}) \) induce well-defined homology functors on the homotopy category \( \mathcal{K}(\mathcal{A}) \).

Proof. By assumption there is a homotopy map \( s = (s_n)_{n \in \mathbb{Z}} \) such that \( f_n - g_n = d_{n+1}^Y s_n + s_{n-1} d_n^X \) for all \( n \in \mathbb{Z} \). Let \( x + \text{im} \, d_{n+1}^X \in H_n(X) \), in particular \( x \in \ker d_{n+1}^X \). Then it follows that

\[
H_n(f)(x + \text{im} \, d_{n+1}^X) = f_n(x) + \text{im} \, d_{n+1}^Y \\
= (d_{n+1}^Y s_n + s_{n-1} d_n^X + g_n)(x) + \text{im} \, d_{n+1}^Y \\
= g_n(x) + \text{im} \, d_{n+1}^Y = H_n(g)(x + \text{im} \, d_{n+1}^X).
\]

Hence \( H(f) = H(g) \). \hfill \Box

Definition 7.4. A morphism \( f : X \to Y \) of complexes in \( \mathcal{C}(\mathcal{A}) \) is called a quasi-isomorphism if it induces isomorphisms in homology, i.e. \( H_n(f) : H_n(X) \to H_n(Y) \) are isomorphisms for all \( n \in \mathbb{Z} \).

Example 7.5. (i) (Projective resolutions) As we are restricting in this section to categories \( \textbf{R-Mod} \) of modules over a ring any object \( X \) in \( \mathcal{A} \) has a projective resolution, i.e. a sequence

\[
\ldots \to P_2 \to P_1 \to P_0 \to 0
\]

where all \( P_i \) are projective objects, together with a morphism \( \epsilon : P_0 \to X \) such that the following augmented sequence is exact

\[
\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} X \to 0.
\]

This gives rise to a morphism of complexes, also denoted \( \epsilon : P \to X \),

\[
\ldots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} X \to 0
\]

where \( X \) is supposed to be in degree 0. Then \( \epsilon \) is a quasi-isomorphism. In fact, in non-zero degrees both complexes have zero homology, and in degree 0 we have isomorphisms \( H_0(P) \cong X \cong H_0(X) \) induced by \( \epsilon \).
(ii) (Injective resolutions) Dually, every object has an injective resolution, i.e. there is a sequence

\[ 0 \to I_0 \to I_{-1} \to I_{-2} \to \ldots \]

where all \( I_j \) are injective objects, and a morphism \( \iota : X \to I_0 \) such that the following augmented sequence is exact

\[ 0 \to X \xrightarrow{\iota} I_0 \to I_{-1} \to I_{-2} \to \ldots \]

This also induces a morphism of complexes \( \iota \) which is a quasi-isomorphism.

Our next goal is to characterize quasi-isomorphisms in terms of mapping cones. To this end we shall use the following standard result on long exact sequences; for a proof we refer for instance to Weibel’s book [16, section 1.3].

**Proposition 7.6.** Let \( 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \) be a short exact sequence of complexes in \( \mathbf{C}(\mathcal{A}) \). Then there are connecting morphisms \( \delta_n : H_n(Z) \to H_{n-1}(X) \) giving rise to the following long exact homology sequence

\[ \ldots \xrightarrow{H_n(g)} H_{n+1}(Z) \xrightarrow{\delta_{n+1}} H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{\delta_n} H_{n-1}(X) \xrightarrow{H_{n-1}(g)} \ldots \]

For the definition of mapping cones recall Definition 6.3.

**Proposition 7.7.** In \( \mathbf{C}(\mathcal{A}) \) a morphism \( f : X \to Y \) is a quasi-isomorphism if and only if the mapping cone complex \( M(f) \) is exact.

**Proof.** By Remark 6.4 we have an exact sequence of complexes

\[ 0 \to Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1] \to 0. \]

The corresponding long exact homology sequence has the form

\[ \ldots \xrightarrow{H_n(g)} H_{n+1}(Z) \xrightarrow{\delta_{n+1}} H_n(X) \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{\delta_n} H_{n-1}(X) \xrightarrow{H_{n-1}(g)} \ldots \]

But \( H_n(X[1]) \) can be identified with \( H_{n-1}(X) \) for all \( n \in \mathbb{Z} \) and it can be checked that then in our situation \( \delta_n = H_{n-1}(f) \), so the above long exact sequence takes the form

\[ \ldots \xrightarrow{H_n(f)} H_n(Y) \xrightarrow{H_n(\alpha(f))} H_n(M(f)) \xrightarrow{H_n(\beta(f))} H_n(X) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y) \ldots \]

For necessity, suppose that \( f \) is a quasi-isomorphism. Then \( H_n(f) \) are isomorphisms by assumption, hence by exactness we have \( H_n(\alpha(f)) = 0 \) and \( H_n(\beta(f)) = 0 \) for all \( n \in \mathbb{Z} \). But then again by exactness we deduce that \( H_n(M(f)) = 0 \) for all \( n \in \mathbb{Z} \), i.e. the mapping cone \( M(f) \) is exact.

Conversely, suppose that \( M(f) \) is exact. Then the long exact sequence takes the form

\[ \ldots 0 \to H_n(X) \xrightarrow{H_n(f)} H_n(Y) \to 0 \to H_{n-1}(X) \xrightarrow{H_{n-1}(f)} H_{n-1}(Y) \to 0 \ldots \]

from which it immediately follows by exactness that \( H_n(f) \) is an isomorphism for all \( n \in \mathbb{Z} \), i.e. \( f \) is a quasi-isomorphism. \( \square \)

**Remark 7.8.** Let \( X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \) be a distinguished triangle in \( \mathbf{K}(\mathcal{A}) \). Then it follows from the previous proposition that \( f \) is a quasi-isomorphism if and only if \( Z \) is exact (i.e. \( H_n(Z) = 0 \) for all \( n \in \mathbb{Z} \)).
7.2. Localisation of categories. Derived categories of abelian categories are obtained from the homotopy categories of complexes by localising with respect to quasi-isomorphisms, i.e. by a formal process inverting all quasi-isomorphisms.

We shall not aim in this introductory chapter to provide a general account of localisation of categories. For a thorough treatment of this topic see H. Krause’s article in this volume [9].

Instead we shall concentrate here on the special case leading from the homotopy category $\mathbf{K}(\mathcal{A})$ of complexes to the derived category $\mathbf{D}(\mathcal{A})$. We first want to give an elementary construction of a derived category, following the approach in the book by Gelfand and Manin [4, III.2]. In the following sections we shall also give alternative equivalent descriptions of the derived category which are perhaps more common and more suitable for explicit computations.

Remark 7.9. From now on we are following a time-honoured tradition by ignoring some set-theoretical issues.

Theorem 7.10. Let $\mathcal{A}$ be an abelian category. Then there exists a category $\mathbf{D}(\mathcal{A})$, called the derived category of $\mathcal{A}$, and a functor $L : \mathbf{K}(\mathcal{A}) \to \mathbf{D}(\mathcal{A})$ satisfying the following properties:

(L1) For every quasi-isomorphism $q$ in $\mathbf{K}(\mathcal{A})$, $L(q)$ is an isomorphism in $\mathbf{D}(\mathcal{A})$.

(L2) Every functor $F : \mathbf{K}(\mathcal{A}) \to \mathbf{D}$ (where $\mathbf{D}$ is any category), having the property that quasi-isomorphisms are mapped to isomorphisms, factors uniquely through $L$.

Property (L2) implies in particular that the category $\mathbf{D}(\mathcal{A})$, if it exists, is unique up to equivalence of categories.

Proof. The objects in $\mathbf{D}(\mathcal{A})$ are defined to be the same as in $\mathbf{K}(\mathcal{A})$, i.e. complexes over $\mathcal{A}$. But the morphisms have to be changed in order for quasi-isomorphisms to become isomorphisms. For each quasi-isomorphism $q$ in $\mathbf{K}(\mathcal{A})$ we introduce a formal variable $q^{-1}$. We then consider ‘words’ in $f$’s and $q^{-1}$’s, i.e. formal compositions of the form

\[(*) = f_1 \circ q_1^{-1} \circ f_2 \circ q_2^{-1} \circ \ldots \circ f_r \circ q_r^{-1}\]

where $r \in \mathbb{N}_0$, the $f_i$ are morphisms and the $q_j$ are quasi-isomorphisms in $\mathbf{K}(\mathcal{A})$. This has to be read so that some $f_i$ or some $q_j^{-1}$ can be the identity and then can be deleted, i.e. consecutive subexpressions $f_i \circ f_{i+1}$ or $q_j^{-1} \circ q_{j+1}^{-1}$ are also allowed in (*).

As usual we read compositions from right to left; so if $f_1 : X_1 \to Y_1$ then $Y_1$ is called the end point of (*) and if $q_r : X_r \to Y_r$ then $Y_r$ is called the starting point of (*). The length of (*) is the total number of $f_i$’s and $q_j^{-1}$’s occurring. For each object $X$ there is an empty expression of length 0 representing the identity on $X$.

We call two such expressions equivalent if they have the same starting and end point and if one can be obtained from the other by a sequence of the following operations

(i) for any composable morphisms $f, g$ in $\mathbf{K}(\mathcal{A})$ replace $f \circ g$ by their composition $(f \circ g)$;

(i') for any composable quasi-isomorphisms $q, r$ in $\mathbf{K}(\mathcal{A})$ replace $q^{-1} \circ r^{-1}$ by $(r \circ q)^{-1}$;

(ii) for any quasi-isomorphism $q$ in $\mathbf{K}(\mathcal{A})$ replace $q \circ q^{-1}$ or $q^{-1} \circ q$ by id.
The morphisms in $\mathbf{D}(\mathcal{A})$ are defined as equivalence classes of expressions of the form $(\ast)$. The composition of morphisms is induced by concatenating expressions of the form $(\ast)$ if the starting point of the second matches the end point of the first, and zero otherwise.

The crucial localisation functor $L$ is now defined as follows: on objects, $L$ is just the identity; a morphism $f$ in $\mathbf{K}(\mathcal{A})$ is sent by $L$ to its equivalence class in $\mathbf{D}(\mathcal{A})$.

In particular, if $q$ is a quasi-isomorphism in $\mathbf{K}(\mathcal{A})$ then $L(q)$ becomes invertible in $\mathbf{D}(\mathcal{A})$ with inverse $q^{-1}$ (because $q \circ q^{-1}$ is equivalent to the empty expression of length 0, representing the identity). Thus, axiom (L1) is satisfied.

For proving axiom (L2) let a functor $F: \mathbf{K}(\mathcal{A}) \to \mathcal{D}$ be given (where $\mathcal{D}$ is any category) which sends quasi-isomorphisms to isomorphisms. We need to define a functor $G: \mathbf{D}(\mathcal{A}) \to \mathcal{D}$ such that $G \circ L = F$. First we note that there is at most one possibility to define such a functor, namely setting $G(X) = F(X)$ on objects, defining $G(f) = F(f)$ for morphisms $f$ in $\mathbf{K}(\mathcal{A})$ and $G(q^{-1}) = F(q)^{-1}$ for quasi-isomorphisms $q$ in $\mathbf{K}(\mathcal{A})$ (and then extending $G$ to arbitrary compositions, in particular $G(id) = id$ for the empty composition). Note that the latter makes sense since $F(q)$ is an isomorphism by assumption.

It only remains to check that this functor is well-defined, i.e. compatible with the equivalence relation defining morphisms in $\mathbf{D}(\mathcal{A})$. For instance, for part (ii) of the above equivalence relation we have in $\mathbf{D}(\mathcal{A})$ that

$$G(q \circ q^{-1}) = G(q) \circ G(q^{-1}) = F(q) \circ F(q)^{-1} = id = G(id)$$

showing well-definedness. The other parts also follow easily from the definition. □

7.3. **Morphisms in the derived category.** The above description of morphisms in the derived category as equivalence classes of expressions of the form

$$(\ast) = f_1 \circ q_1^{-1} \circ f_2 \circ q_2^{-1} \circ \ldots \circ f_r \circ q_r^{-1}$$

is pretty inconvenient. We shall describe in this section a 'calculus of fractions' which will lead to a simpler description of the morphisms in the derived category. To this end we shall make use of certain useful properties of the class of quasi-isomorphisms in the homotopy category.

**Lemma 7.11.** Let $\mathcal{A}$ be an abelian category. The class $Q$ of quasi-isomorphisms in the homotopy category $\mathbf{K}(\mathcal{A})$ satisfies the following properties:

(Q1) For every object $X$ in $\mathbf{K}(\mathcal{A})$ the identity $id_X$ is in $Q$.

(Q2) $Q$ is closed under composition.

(Q3) (Ore condition) Given a quasi-isomorphism $q \in Q$ and a morphism $f$ in $\mathbf{K}(\mathcal{A})$ (with same target) then there exist an object $W$, a morphism $g$ and a quasi-isomorphism $t \in Q$ such that the following diagram is commutative.

$$\begin{align*}
W & \xrightarrow{g} Z \\
\downarrow t & \quad \downarrow q \\
X & \xrightarrow{f} Y
\end{align*}$$
Similarly, given a quasi-isomorphism \( q \in Q \) and a morphism \( f \) in \( K(A) \) (with same range) then there exist an object \( V \), a morphism \( h \) and a quasi-isomorphism \( r \in Q \) such that the following diagram is commutative.

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow q & & \downarrow r \\
X & \xrightarrow{h} & V
\end{array}
\]

(Q4) For any morphisms \( f, g : X \to Y \) in \( K(A) \) the following are equivalent:

(i) There exists a quasi-isomorphism \( q : Y \to Y' \) in \( Q \) (for some \( Y' \)) such that \( q \circ f = q \circ g \).

(ii) There exists a quasi-isomorphism \( t : X' \to X \) in \( Q \) (for some \( X' \)) such that \( f \circ t = g \circ t \).

**Remark 7.12.** The class of quasi-isomorphisms does not in general satisfy the conditions of the preceding lemma already in \( C(A) \); it is crucial first to pass to the homotopy category. In fact, in the proof below we shall make heavy use of the triangulated structure of the homotopy category (which has been proven in Theorem 6.7).

**Proof.** (Q1) and (Q2) are clear.

For (Q3) we have given a morphism \( f \) and a quasi-isomorphism \( q \). By axiom (TR2) for the triangulated category \( K(A) \) there exists a distinguished triangle \( Z \xrightarrow{q} Y \xrightarrow{u} U \xrightarrow{v} Z[1] \). Similarly, considering \( uf : X \to U \) there is a distinguished triangle \( W \xrightarrow{t} X \xrightarrow{uf} U \xrightarrow{w} W[1] \). Applying axioms (TR4) and (TR3) we can deduce the existence of the morphism \( g \) (and \( g[1] \)) in the following commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{t} & X \\
\downarrow g & & \downarrow f \\
Z & \xrightarrow{q} & Y \\
\downarrow \text{id} & & \downarrow \text{id} \\
U & \xrightarrow{u} & U \\
\downarrow v & & \downarrow v \\
Z[1]
\end{array}
\]

Since \( q \) is a quasi-isomorphism by assumption, the long exact homology sequence applied to the bottom row yields that \( H_n(U) = 0 \) for all \( n \in \mathbb{Z} \). And then the long exact homology sequence for the top row implies that \( t \) must be a quasi-isomorphism, as desired (cf. Remark 7.8).

The symmetrical second claim in (Q3) is shown similarly.

Finally, let us prove (Q4). We will prove the direction \((i)\Rightarrow(ii)\), the converse is proved similarly. For simplicity, set \( h := f - g \), thus \( q \circ h = 0 \) by assumption. By (TR2) and (TR3) there exists a distinguished triangle \( Z \xrightarrow{u} Y \xrightarrow{q} Y' \xrightarrow{w} Z[1] \). Since \( q \) is a quasi-isomorphism, \( Z[1] \) and hence \( Z \) is exact (cf. Remark 7.8). By (TR1), (TR3) and (TR4) there exists a morphism \( v \) making the following diagram commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\text{id}} & X \\
\downarrow v & & \downarrow h \\
Z & \xrightarrow{u} & Y \\
\downarrow \text{id} & & \downarrow \text{id} \\
X[1] & \xrightarrow{v[1]} & X'[1]
\end{array}
\]

Now again by (TR2) and (TR3) \( v \) can be embedded in a distinguished triangle \( X' \xrightarrow{t} X \xrightarrow{v} Z \xrightarrow{v[1]} X'[1] \). Here \( t \) is a quasi-isomorphism since \( Z \) is exact (cf.
Remark 7.8). Moreover, \( v \circ t = 0 \) since the composition of any consecutive maps in a distinguished triangle vanishes. It follows that \( h \circ t = u \circ v \circ t = 0 \), i.e. \( f \circ t = g \circ t \), as desired. □

The properties satisfied by the family \( Q \) of quasi-isomorphisms in \( K(A) \) has useful consequences for the description of morphisms in the derived category \( D(A) \). As described above, a morphism in \( D(A) \) is an equivalence class of an expression of the form

\[
(*) = f_1 \circ q_1^{-1} \circ f_2 \circ q_2^{-1} \circ \ldots \circ f_r \circ q_r^{-1}
\]

where \( f_i \) are morphisms in \( K(A) \) and \( q_i \) are quasi-isomorphisms in \( Q \).

Property (Q3) above states that \( q_i^{-1} \circ f = g \circ t_i^{-1} \) for some morphism \( g \) and quasi-isomorphism \( t_i \in Q \), and \( f \circ q_i^{-1} = r_i^{-1} \circ h \) with \( r_i \in Q \), respectively. This means that in the above expression (*) we can move all 'denominators' \( q_i \) to the right (or to the left). This means that any morphism in the derived category can be represented by an expression of the form \( f \circ q_i^{-1} \) with a quasi-isomorphism \( q_i \) and a morphism \( f \). This can be conveniently visualised as a 'roof'

\[
\begin{array}{c}
X \\
\downarrow q \\
X' \\
\downarrow f \\
Y
\end{array}
\]

We shall use this description frequently in the sequel and hence want to make this more precise. Again, we follow the approach in the book by Gelfand and Manin [4, section III.2]. We shall define a category \( \tilde{D}(A) \) where morphisms are represented by such roofs, and then show that this category is indeed equivalent to the derived category \( D(A) \) introduced in Theorem 7.10.

For computing with these roofs we need to introduce a suitable notion of equivalence for roofs. Two roofs \( (q,f) \) and \( (t,g) \) are called equivalent if there exists another roof \( (r,h) \) making the following diagram commutative

\[
\begin{array}{c}
X' \\
\downarrow r \\
X'' \\
\downarrow h \\
X'''
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow q \\
X' \\
\downarrow t \\
X'' \\
\downarrow f \\
Y
\end{array}
\]

We leave it to the reader to verify that this indeed defines an equivalence relation. The non-obvious property is transitivity, see [4, Lemma III.2.8] for a detailed proof; actually, in the proof of transitivity the property (Q4) of Lemma 7.11 is used.

For the composition of roofs one makes use of the Ore condition (Q3) above. Namely, given roofs \( (q,f) : X \to Y \) and \( (t,g) : Y \to Z \) we can find by (Q3) an object \( W \) and a roof \( (\ell,\rho) \) making the following diagram commutative

\[
\begin{array}{c}
X' \\
\downarrow r \\
X'' \\
\downarrow h \\
X'''
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow q \\
X' \\
\downarrow t \\
X'' \\
\downarrow f \\
Y
\end{array}
\]

We leave it to the reader to verify that this indeed defines an equivalence relation. The non-obvious property is transitivity, see [4, Lemma III.2.8] for a detailed proof; actually, in the proof of transitivity the property (Q4) of Lemma 7.11 is used.

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\[
\begin{array}{c}
X' \\
\downarrow r \\
X'' \\
\downarrow h \\
X'''
\end{array}
\]

\[
\begin{array}{c}
X \\
\downarrow q \\
X' \\
\downarrow t \\
X'' \\
\downarrow f \\
Y
\end{array}
\]
The composition of roofs \((t, g) \circ (q, f)\) is then defined to be the equivalence class represented by the roof \((q \circ t', g \circ g') : X \to Z\). It is not difficult to verify that this is well-defined, i.e. independent of the representatives of the roofs involved.

Note that the identity morphism for an object \(X\) is represented by the roof \((\text{id}_X, \text{id}_X)\).

The category \(\tilde{D}(A)\) is defined as having the same objects as \(D(A)\) (and hence as the homotopy category \(K(A)\)), namely complexes over \(A\).

The morphisms in \(\tilde{D}(A)\) are defined to be the equivalence classes of roofs, with the above composition.

**Proposition 7.13.** Let \(A\) be an abelian category. Then the category \(\tilde{D}(A)\) satisfies the universal property of the derived category \(D(A)\) given in Theorem 7.10. In particular, the categories \(D(A)\) and \(\tilde{D}(A)\) are equivalent.

**Proof.** We first define a functor \(\tilde{L} : K(A) \to \tilde{D}(A)\) as the identity on objects and on morphisms by sending \(f\) to the roof \((\text{id}, f)\). Clearly, \(\tilde{L}\) maps the identity to the identity. As for composition of morphisms, a composition \(g \circ f\) is on the one hand sent by \(\tilde{L}\) to the roof \((\text{id}, g \circ f)\); on the other hand the composition \(\tilde{L}(g) \circ \tilde{L}(f)\) is given by the roof obtained from the commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{id} & X \\
\downarrow{f} & & \downarrow{id} \\
Y & & Y \\
\downarrow{g} & & \downarrow{\text{id}} \\
Z & & Z
\end{array}
\]

Thus, \(\tilde{L}(g \circ f) = \tilde{L}(g) \circ \tilde{L}(f)\) and \(\tilde{L}\) is indeed a functor.

Now it remains to prove that the category \(\tilde{D}(A)\), together with the functor \(\tilde{L}\), satisfies the properties (L1) and (L2) from Theorem 7.10.

For (L1), any quasi-isomorphism \(q\) in \(K(A)\) is mapped to the roof \((\text{id}, q)\). It is immediate from the above composition of morphisms that \((q, \text{id}) \circ (\text{id}, q) = (\text{id}, \text{id})\), and that \((\text{id}, q) \circ (q, \text{id}) = (q, q)\); but the latter roof is equivalent to \((\text{id}, \text{id})\). Thus, \(\tilde{L}\) maps quasi-isomorphisms to isomorphisms and (L1) is satisfied.

For proving (L2), let \(F : K(A) \to D\) (\(D\) any category) be a functor which maps quasi-isomorphisms to isomorphisms. We have to show that there is a unique functor \(\tilde{F} : \tilde{D}(A) \to D\) such that \(\tilde{F} \circ \tilde{L} = F\).

We first deal with uniqueness. On objects \(X\), the only choice is \(\tilde{F}(X) = F(X)\) since \(\tilde{L}\) is the identity on objects. Now consider a morphism in \(\tilde{D}(A)\), represented by a roof \((q, f)\). In \(\tilde{D}(A)\) we have that

\[(q, f) \circ \tilde{L}(q) = (q, f) \circ (\text{id}, q) = (\text{id}, f) = \tilde{L}(f).\]
Since $\tilde{F}$ has to be a functor with $\tilde{F} \circ \tilde{L} = \tilde{F}$ we can deduce that
\[ F(f) = \tilde{F}(\tilde{L}(f)) = \tilde{F}((q, f) \circ \tilde{L}(q)) = \tilde{F}((q, f)) \circ \tilde{F}(\tilde{L}(q)) = \tilde{F}((q, f)) \circ F(q). \]

By assumption, $F(q)$ is an isomorphism, so the only possibility to define $\tilde{F}$ on morphisms is to set
\[ \tilde{F}((q, f)) = F(f) \circ F(q)^{-1}. \]

Hence, the functor $\tilde{F}$, if it exists, is unique.

For existence, we actually define $\tilde{F}$ by the properties just exhibited, i.e. $\tilde{F}(X) = F(X)$ on objects and $\tilde{F}((q, f)) = F(f) \circ F(q)^{-1}$ on morphisms. Of course, we now have to prove that this indeed defines a functor.

We claim that the definition is well-defined, i.e. independent of the choice of the representative. In fact, let $(q, f)$ and $(t, g)$ be equivalent roofs, i.e. we have a commutative diagram of the form

```
X  X''  X'''
|    |    |
h  r   |
|    |    |
Y  X   |
|    |    |
t  f   |
|    |    |
Y'  Y  |
```

where $r$ is a quasi-isomorphism. Note that since $r, q, t$ are quasi-isomorphisms and $q \circ r = t \circ h$, also $h$ must be a quasi-isomorphism. Then we get from the functoriality of $F$ and the assumption that $F$ sends quasi-isomorphisms to isomorphisms that
\[ \tilde{F}((q, f)) = F(f) \circ F(q)^{-1} = F(f) \circ F(r) \circ F(r)^{-1} \circ F(q)^{-1} \]
\[ = F(f \circ r) \circ (F(q \circ r))^{-1} = F(g \circ h) \circ (F(t \circ h))^{-1} \]
\[ = F(g) \circ F(h) \circ F(t)^{-1} \circ F(t)^{-1} = F(g) \circ F(t)^{-1} = \tilde{F}((t, g)). \]

By definition, $\tilde{F}$ maps identity morphisms $(\text{id}, \text{id})$ in $\tilde{D}(A)$ to identity morphisms in $D$. Finally, consider a composition $(t, g) \circ (q, f)$ in $\tilde{D}(A)$; this is represented by a roof $(q \circ t', g \circ g')$ coming from a commutative diagram

```
W
|   |
t' g' |
|    |
X'  |
|    |
g  f  t
|    |
X  Y  Z
```

Since $f \circ t' = t \circ g'$ we get $F(f) \circ F(t') = F(t) \circ F(g')$ and since $F$ sends quasi-isomorphisms to isomorphisms $F(t)^{-1} \circ F(f) = F(g') \circ F(t')^{-1}$. This implies that
\[ \tilde{F}((t, g) \circ (q, f)) = \tilde{F}((q \circ t', g \circ g')) = F(g \circ g') \circ F(q \circ t')^{-1} \]
\[ = F(g) \circ F(g') \circ F(t')^{-1} \circ F(q)^{-1} \]
\[ = F(g) \circ F(t)^{-1} \circ F(f) \circ F(q)^{-1} \]
\[ = \tilde{F}((t, g)) \circ \tilde{F}((q, f)). \]
Thus $\overline{F}$ is indeed a functor and this completes the proof of the universal property (L2).

**Remark 7.14.** In the sequel we shall denote the derived category exclusively by $D(\mathcal{A})$ even if we usually use the more convenient equivalent version $\overline{D}(\mathcal{A})$ just described.

**Proposition 7.15.** Let $\mathcal{A}$ be an abelian category. Then the derived category $D(\mathcal{A})$ is an additive category.

**Proof.** Following Definition 1.1 we have to show the properties (A1), (A2) and (A3).

(A1) We first describe addition of morphisms. Let two morphisms $F, G$ from $X$ to $Y$ be represented by roofs $(q, f)$ and $(q', f')$. By the Ore condition (Q3) there exists an object $W$, a morphism $g$ and a quasi-isomorphism $t$ making the following diagram commutative:

\[
\begin{array}{ccc}
W & \overset{g}{\longrightarrow} & X' \\
\downarrow t & & \downarrow q' \\
X'' & \overset{q'}{\longrightarrow} & X
\end{array}
\]

Since $q, q'$ and $t$ are quasi-isomorphisms, also $g$ must be a quasi-isomorphism. From the definition of equivalence it is easy to check that the roof $(q, f)$ is equivalent to the roof $(q \circ g, f \circ g)$, and that $(q', f')$ is equivalent to the roof $(q' \circ t, f' \circ t) = (q \circ g, f' \circ t)$.

Thus we have found a 'common denominator' and can set $F + G$ to be the roof represented by $(q \circ g, f \circ g + f' \circ t)$.

We leave it to the reader to verify that this addition is well-defined (i.e. independent of the representatives) and that the addition of morphisms is bilinear.

Note that in the derived category there is for any objects $X, Y$ a zero morphism in $D(\mathcal{A})$ which is represented by the roof $(\text{id}_X, 0)$ where $0$ is the zero morphism of complexes from $X$ to $Y$.

(A2) The zero object in $D(\mathcal{A})$ is the zero complex (i.e. it is the same zero object as in the homotopy category). We have to show that for every object $X$ the morphism sets $\text{Hom}_{D(\mathcal{A})}(X, 0)$ and $\text{Hom}_{D(\mathcal{A})}(0, X)$ contain only the morphism represented by the roof $(\text{id}_X, 0)$ and $(0, \text{id}_X)$, respectively. In fact, any morphism from $X$ to the zero complex is represented by a roof $(q, 0)$ where $q : X \rightarrow Y$ is a quasi-isomorphism. But it easily follows from the definition that the roof $(q, 0)$ is equivalent to $(\text{id}_X, 0)$, thus $\text{Hom}_{D(\mathcal{A})}(X, 0)$ contains precisely one element. The assertion for $\text{Hom}_{D(\mathcal{A})}(0, X)$ is shown similarly.

(A3) For the coproduct of two objects $X$ and $Y$ in $D(\mathcal{A})$ one uses the image of the coproduct $X \oplus Y$ in $K(\mathcal{A})$ under the localisation functor $L$ (which is the identity on objects and maps a morphism $f$ in $K(\mathcal{A})$ to the roof $(\text{id}, f)$ in $D(\mathcal{A})$). The corresponding maps $L(\iota_X) : X \rightarrow X \oplus Y$ and $L(\iota_Y) : X \rightarrow X \oplus Y$ are given by the roofs $(\text{id}, \iota_X)$ and $(\text{id}, \iota_Y)$, respectively, where $\iota_X$ and $\iota_Y$ are the embeddings (or more precisely, their equivalence classes in $K(\mathcal{A})$).

We have to show that the universal property (A3) is satisfied. So let $\tilde{f}_X$ and $\tilde{f}_Y$ be arbitrary morphisms in $D(\mathcal{A})$ from $X$ and $Y$ to some object $Z$. They are represented by roofs of the form:

\[
\begin{array}{ccc}
X & \overset{\tilde{f}_X}{\longrightarrow} & Z \\
\downarrow \tilde{f}_Y & & \downarrow \tilde{f}_Y \\
Y & \overset{\tilde{f}_Y}{\longrightarrow} & Z
\end{array}
\]
The required morphism $\tilde{f}$ from $X \oplus Y$ to $Z$ can then be defined as being represented by the roof $(q \oplus t, (f_X, f_Y))$ (where the notation $\oplus$ on maps between direct sums of complexes means componentwise application). Then indeed we have that the composition $\tilde{f} \circ L(\iota_X)$ in $D(A)$ is the roof $(q, f_X)$, as can be seen from the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\iota_X} & \mathbb{X} \\
\downarrow{\text{id}} & & \downarrow{\text{id}} \\
X & \xrightarrow{\iota_X} & \mathbb{X} \oplus \mathbb{Y} \\
\downarrow{q \oplus t} & & \downarrow{\tilde{X}} \\
X \oplus Y & \xrightarrow{(f_X, f_Y)} & Z
\end{array}
$$

A similar diagram shows that $\tilde{f} \circ L(\iota_Y) = (t, f_Y)$ in $D(A)$.

We leave it as an exercise to show that the morphism $\tilde{f}$ with these properties is actually unique, as required in (A3). \hfill \Box

7.4. **Derived categories are triangulated.** Recall that the derived category has been obtained by localising the homotopy category with respect to the class of quasi-isomorphisms. In particular, there is a functor $L : K(A) \rightarrow D(A)$ sending quasi-isomorphisms in $K(A)$ to isomorphisms in $D(A)$ (and satisfying a universal property). We have seen earlier that the homotopy category is triangulated, with distinguished triangles being the triangles isomorphic in $K(A)$ to the standard triangles coming from mapping cones $X \xrightarrow{f} Y \xrightarrow{\alpha(f)} M(f) \xrightarrow{\beta(f)} X[1]$.

For obtaining a triangulated structure on the derived category the idea is to transport the triangulated structure on the homotopy category via the localisation functor $L$.

**Definition 7.16.** The translation functor on $D(A)$ is defined as the shift $[1]$ on objects, and for a morphism $F$ in $D(A)$ represented by a roof $(q, f)$ we set $F[1]$ to be the equivalence class of the roof $(q[1], f[1])$.

A triangle in $D(A)$ is a distinguished triangle if it is isomorphic (in $D(A)$!) to the image of a distinguished triangle from $K(A)$ under the localisation functor $L$.

**Remark 7.17.** When passing from $K(A)$ to $D(A)$ all quasi-isomorphisms become isomorphisms, i.e. there are ‘more’ isomorphisms in $D(A)$ than in $K(A)$. This in turn means that in $D(A)$ it is easier for a triangle to become isomorphic to a standard triangle than in $K(A)$, i.e. the derived category contains ‘more’ distinguished triangles than the homotopy category.

As a crucial observation we shall see in the next section that the derived category has the property that every short exact sequence of complexes in $C(A)$ gives rise to a corresponding distinguished triangle in the derived category. This is not yet the case in the homotopy category, see Remark 6.8 above for an example.
With the above definitions one can then show the main structural property of derived categories. Unfortunately, the proof of the axioms for a triangulated category will become very technical so that we shall refrain from providing a proof in this introductory chapter.

**Theorem 7.18.** Let \( A \) be an abelian category. Then the derived category \( D(A) \) is triangulated.

7.5. **Comparing morphisms.** Let \( A \) be an abelian category. We have now constructed various categories from \( A \):

\[
\begin{array}{ccc}
A & \hookrightarrow & C(A) & \hookrightarrow & K(A) & \hookrightarrow & D(A) \\
\text{abelian} & & \text{abelian} & & \text{triangulated} & & \text{triangulated}
\end{array}
\]

**Proposition 7.19.** We have the following implications for a morphism \( f \) in \( C(A) \).

\[
f = 0 \text{ in } C(A) \Rightarrow f = 0 \text{ in } K(A) \Rightarrow f = 0 \text{ in } D(A) \Rightarrow H_n(f) = 0 \text{ for all } n \in \mathbb{Z}
\]

**Proof.** The first two implications are obvious. For the third, \( f : X \to Y \) as a morphism in \( D(A) \) is represented by the roof \((\text{id}_X, f)\). For \( f \) being 0 in \( D(A) \) means being equivalent to the roof \((\text{id}_X, 0)\), i.e. in \( K(A) \) there is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

with a quasi-isomorphism \( r \). By commutativity, \( f \circ r = 0 \) (in \( K(A) \)) and passing to homology we get \( H_n(f) \circ H_n(r) = H_n(f \circ r) = H_n(0) = 0 \) for all \( n \in \mathbb{Z} \). But \( r \) is a quasi-isomorphism, i.e. all \( H_n(r) \) are isomorphisms, thus we conclude that \( H_n(f) = 0 \) for all \( n \in \mathbb{Z} \). \( \square \)

**Remark 7.20.**

1. Note that a morphism \( f \) in \( K(A) \) becomes zero in \( D(A) \) if and only if there exists a quasi-isomorphism \( r \) such that \( f \circ r \) is homotopic to zero.

2. All implications given in Proposition 7.19 are strict. Let us give examples for each case. We consider the category \( A = \text{Ab} \).

   For the first implication, consider the following morphism of complexes

   \[
   \begin{array}{ccc}
   \cdots & \to & 0 & \to & 0 & \to & Z & \to & 0 & \to & \cdots \\
   \downarrow & & & & \downarrow & & & & \downarrow & & \\
   \cdots & \to & 0 & \to & Z & \xrightarrow{\text{id}} & Z & \to & 0 & \to & \cdots
   \end{array}
   \]

   Clearly, this morphism is zero in \( K(A) \), but nonzero in \( C(A) \).

   For the second implication, consider the identity map on the (exact) complex

   \[
   0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot \pi} \mathbb{Z}/2\mathbb{Z} \to 0.
   \]

   This morphism is not homotopic to zero (i.e. nonzero in \( K(A) \)) because \( \text{Hom}_\mathbb{Z}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0 \) and hence the identity on \( \mathbb{Z}/2\mathbb{Z} \) can not factor through
a homotopy. However, it is zero in $\mathbf{D}(\mathcal{A})$ because we can find a quasi-isomorphism $r$ such that $f \circ r$ is homotopic to zero. In fact, let $r$ be the morphism of complexes

$$
\begin{array}{ccccccccc}
0 & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \overset{\text{id}}\rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \pi \\
0 & \rightarrow & \mathbb{Z} & \overset{2}\rightarrow & \mathbb{Z} & \overset{\pi}\rightarrow & \mathbb{Z}/2\mathbb{Z} & \rightarrow & 0
\end{array}
$$

Since both complexes are exact, $r$ is a quasi-isomorphism. Moreover, $f \circ r$ is homotopic to zero (a homotopy map is given by $0$ and $\text{id}$ in the two relevant degrees). This implies that $f$ is zero when considered as a morphism in $\mathbf{D}(\mathcal{A})$.

For the third implication we consider the morphism $f : X \rightarrow Y$ of complexes given as follows

$$
\begin{array}{ccccccccc}
0 & \rightarrow & \mathbb{Z} & \overset{2}\rightarrow & \mathbb{Z} & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \pi \\
0 & \rightarrow & \mathbb{Z} & \overset{2}\rightarrow & \mathbb{Z}/3\mathbb{Z} & \rightarrow & 0
\end{array}
$$

The homology of $X$ is given by $H_0(X) = \mathbb{Z}/2\mathbb{Z}$ and $H_1(X) = 0$, whereas $H_0(Y) = 0$ and $H_1(Y) = 3\mathbb{Z}$ (and all other being zero). In particular, $H_n(f) = 0$ for all $n \in \mathbb{Z}$.

However, we claim that $f$ is nonzero in the derived category. Suppose for a contradiction that $f = 0$ in $\mathbf{D}(\mathcal{A})$, i.e. there exist a complex $R = (R_0, d_0^R)$ and a quasi-isomorphism $r : R \rightarrow X$ such that $f \circ r : R \rightarrow Y$ is homotopic to zero. Since $r$ is a quasi-isomorphism, we have that $H_n(R) \cong H_n(X) = 0$ for $n \neq 0$ and $H_0(R) = H_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Choose a generator of $H_0(R)$, i.e. $z_0 \in \ker d_0^R \setminus \text{im } d_1^R$. Since $r$ is a quasi-isomorphism, $r_0(z_0)$ must not be in the image of $d_1^X$, i.e. $r_0(z_0) \notin 2\mathbb{Z}$. On the other hand, $f \circ r$ is homotopic to zero, thus there exist homotopy maps $s_0 : R_0 \rightarrow \mathbb{Z}$ and $s_{-1} : R_{-1} \rightarrow \mathbb{Z}/3\mathbb{Z}$ such that $(f \circ r)_0 = \pi \circ r_0 = 2s_0 + s_{-1} \circ d_0^R$. Applied to the generator $z_0$ (which is in the kernel of $d_0^R$) this yields

$$(\pi \circ r_0)(z_0) = (2s_0 + s_{-1} \circ d_0^R)(z_0) = 2s_0(z_0) \in 2(\mathbb{Z}/3\mathbb{Z}).$$

But then also $r_0(z_0) \in 2\mathbb{Z}$, a contradiction to the earlier conclusion.

Hence there is no such quasi-isomorphism $r$, i.e. $f \neq 0$ in $\mathbf{D}(\mathcal{A})$.

7.6. Short exact sequences vs. triangles. In this section we shall explain the crucial observation that a short exact sequence in $\mathbf{C}(\mathcal{A})$ induces a distinguished triangle in $\mathbf{D}(\mathcal{A})$. Recall that we have seen earlier that this does not yet happen in the homotopy category $\mathbf{K}(\mathcal{A})$ (cf. Remark 6.8).

In this subsection we will again use our assumption that the abelian category $\mathcal{A}$ is an abelian subcategory of the category of modules over a ring, which allows us to define maps on elements.

7.6.1. Mapping cylinders. Let $f : X \rightarrow Y$ be a morphism of complexes in $\mathbf{C}(\mathcal{A})$. The mapping cylinder of $f$ is the complex $\text{Cyl}(f)$ having degree $n$ part equal to $X_n \oplus X_{n-1} \oplus Y_n$ and the differential is given by

$$d_n^{\text{Cyl}(f)}(x, x', y) := (d_n^X x - x', -d_{n-1}^X x', f_{n-1}(x') + d_n^Y y).$$
In perhaps more convenient matrix notation,
\[
d^Cyl_n(f) = \begin{pmatrix}
  d^X_n & -id_{X_{n-1}} & 0 \\
  0 & -d^X_{n-1} & 0 \\
  0 & f_{n-1} & d^Y_n
\end{pmatrix}.
\]

It is now easily checked that \( Cyl(f) \) is indeed a complex, i.e. \( d^{Cyl}_{n-1} \circ d^{Cyl}_n = 0 \).

We next aim at comparing the mapping cylinder with the mapping cone, as defined in Definition 6.3. We consider the following morphisms of complexes
\[
i : X \rightarrow Cyl(f) \text{ given in degree } n \text{ by } i_n = (\text{id}_{X_n}, 0, 0),
\]
\[
\pi : Cyl(f) \rightarrow M(f) \text{ given in degree } n \text{ by } \pi_n = \begin{pmatrix}
  0 & \text{id}_{X_{n-1}} & 0 \\
  0 & 0 & \text{id}_{Y_n}
\end{pmatrix}.
\]

We leave the straightforward verification to the reader that these maps indeed commute with the differentials. Clearly, the resulting sequence
\[
0 \rightarrow X \xrightarrow{i} Cyl(f) \xrightarrow{\pi} M(f) \rightarrow 0
\]
is a short exact sequence in \( \text{C}(A) \).

Lemma 7.21. (Mapping cylinder vs. mapping cone) Let \( f : X \rightarrow Y \) be a morphism of complexes. Then there are morphisms of complexes \( \sigma : Y \rightarrow Cyl(f) \) with \( \sigma_n = (0, 0, \text{id}_{Y_n}) \) and \( \tau : Cyl(f) \rightarrow Y \) with \( \tau_n = (f_n, 0, \text{id}_{Y_n}) \) such that the following holds:

(i) The following diagram with exact rows is commutative in the category \( \text{C}(A) \) of complexes:
\[
\begin{array}{ccccccccc}
0 & \rightarrow & Y & \xrightarrow{\alpha(f)} & M(f) & \xrightarrow{\beta(f)} & X[1] & \rightarrow & 0 \\
\downarrow{\sigma} & & \downarrow{id} & & \downarrow{id} & & \downarrow{id} & & \\
0 & \rightarrow & X & \xrightarrow{i} & Cyl(f) & \xrightarrow{\pi} & M(f) & \rightarrow & 0
\end{array}
\]

(ii) \( \tau \circ \sigma = \text{id}_Y \) and \( \sigma \circ \tau \) is homotopic to the identity \( \text{id}_{Cyl(f)} \), i.e. \( Y \) and \( Cyl(f) \) are isomorphic in the homotopy category, and hence also in the derived category \( \text{D}(A) \).

(iii) \( \sigma \) and \( \tau \) are quasi-isomorphisms.

Proof. (i) It is immediately checked from the definitions (for \( \alpha(f) \) see Remark 6.4) that \( \sigma \) and \( \tau \) are indeed morphisms of complexes and that all squares in the diagram commute (in \( \text{C}(A) \)), not only up to homotopy).

(ii) By definition we have \( \tau \circ \sigma = \text{id}_Y \). On the other hand, \( \sigma \circ \tau \) is homotopic to the identity via the homotopy map \( s = (s_n) \) with
\[
s_n = \begin{pmatrix}
  0 & 0 & 0 \\
  \text{id}_{X_n} & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]

(iii) By (ii) the compositions \( \tau \circ \sigma \) and \( \sigma \circ \tau \) are homotopy equivalences, in particular they induce the identity in homology. Thus,
\[
H_n(\sigma) \circ H_n(\tau) = H_n(\sigma \circ \tau) = H_n(\text{id}_{Cyl(f)}) = \text{id}_{H_n(Cyl(f))}
\]
and similarly $H_n(\tau) \circ H_n(\sigma) = \text{id}_{H_n(Y)}$ which implies that $H_n(\sigma)$ and $H_n(\tau)$ are isomorphisms, i.e. $\sigma$ and $\tau$ are quasi-isomorphisms.

As a consequence we can now state the main result of this section.

**Corollary 7.22.** To any short exact sequence $0 \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0$ in $\mathbf{C}(\mathbf{A})$ there exists a corresponding distinguished triangle in $\mathbf{D}(\mathbf{A})$ of the form

$$X \overset{f}{\to} Y \overset{g}{\to} Z \to X[1].$$

**Proof.** Using the notations of the previous lemma we consider the following diagram with exact rows

$$
\begin{align*}
0 & \to X \overset{i}{\to} Cyl(f) \overset{\pi}{\to} M(f) \to 0 \\
0 & \to X \overset{f}{\to} Y \overset{g}{\to} Z \to 0
\end{align*}
$$

where $\gamma = (\gamma_n)$ is defined by setting $\gamma_n(x, y) := g_n(y)$; this is easily checked to be a morphism of complexes (using that $g$ is a morphism of complexes and that $g \circ f = 0$ since they are consecutive maps in a short exact sequence). Also one immediately deduces from the definitions that the above diagram is commutative (where the left hand square already appeared in the previous lemma).

Since $\text{id}$ and $\tau$ (by the previous lemma) are both quasi-isomorphisms it follows from the long exact homology sequences and the usual 5-lemma that also $\gamma$ must be a quasi-isomorphism, hence an isomorphism in the derived category (but not necessarily in $\mathbf{K}(\mathbf{A})$, see the following remark). So we have a morphism of triangles in $\mathbf{D}(\mathbf{A})$

$$
\begin{array}{ccc}
X & \overset{f}{\to} & Y \\
\downarrow & & \downarrow \pi \\
M(f) & \overset{\beta(f)}{\to} & X[1]
\end{array}
$$

where the inverse $\gamma^{-1}$ exists in the derived category. For the commutativity of the second square note that by the previous lemma and the above definition of $\gamma$ we have

$$
\gamma \circ \alpha(f) = \gamma \circ \pi \circ \sigma = g \circ \tau \circ \sigma = g \circ \text{id} = g.
$$

Moreover, since $\gamma$ is an isomorphism in $\mathbf{D}(\mathbf{A})$ this morphism of triangles is indeed an isomorphism of triangles, i.e. the bottom line

$$
\begin{array}{ccc}
X & \overset{f}{\to} & Y \\
\downarrow & & \downarrow \beta(f) \circ \gamma^{-1} \\
Z & \overset{\gamma^{-1}}{\to} & X[1]
\end{array}
$$

is isomorphic to the image of a standard triangle from $\mathbf{K}(\mathbf{A})$ under the localisation functor, hence a distinguished triangle in $\mathbf{D}(\mathbf{A})$.

**Remark 7.23.** The crucial quasi-isomorphism $\gamma$ occurring in the previous proof is in general not a homotopy equivalence (i.e. not an isomorphism in $\mathbf{K}(\mathbf{A})$) and hence the isomorphism of triangles exists only in the derived category but not yet in the homotopy category.

As an example, consider $\mathbf{A} = \text{R-mod}$ for a ring $R$ and consider $R$-modules $X, Y$ as complexes concentrated in degree 0. The mapping cone of a (module) morphism $f : X \to Y$ is just $0 \to X \overset{f}{\to} Y \to 0$, and the morphism $\gamma : M(f) \to Y/X$ is just
the natural projection. If this $\gamma$ is a homotopy equivalence then $f$ must be a split monomorphism. In fact, up to homotopy there exists an inverse $\rho : Y/X \to Y$; for $\rho \circ \gamma$ to be homotopic to the identity on $M(f)$ there must be a homotopy map $s : Y \to X$ which (when looking in degree 1) in particular satisfies $s \circ f = \text{id}_X$, i.e. $f$ splits.

This explains again the earlier example (cf. Remark 6.8) of the short exact sequence of abelian groups

$$0 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{1} \mathbb{Z}/2\mathbb{Z} \to 0$$

which does not have a corresponding distinguished triangle in $K(A)$, because clearly the map $\mathbb{Z}/2\mathbb{Z} \xrightarrow{2} \mathbb{Z}/4\mathbb{Z}$ does not split.

8. Frobenius categories and stable categories

In the earlier sections we have considered categories of complexes leading to derived categories which form an important source of examples of triangulated categories.

In this section we shall briefly describe another source for triangulated categories, namely stable categories of Frobenius algebras. The aim is to give the relevant definitions and constructions and to provide some examples, in order to prepare the ground for the later articles in this book.

For more details, in particular for a complete proof of the triangulated structure of the stable category of a Frobenius algebra we refer the reader for instance to the well-written chapter on Frobenius categories in D. Happel’s book [5].

We start by defining an exact category, a concept introduced by D. Quillen, which generalizes abelian categories, in the sense that an exact category has a certain class of 'exact triples' as a replacement for short exact sequences without having to be abelian itself.

**Definition 8.1. (Exact category)** Let $A$ be an abelian category, and let $B$ be an additive subcategory of $A$ which is full and closed under extensions (i.e. if $0 \to X \to Y \to Z \to 0$ is an exact sequence in $A$ where $X$ and $Z$ are objects in $B$ then $Y$ is isomorphic to an object of $B$). Take $E$ to be the class of all triples $X \to Y \to Z$ in $B$ whose corresponding sequences $0 \to X \to Y \to Z \to 0$ in $A$ are exact.

Then the pair $(B, E)$ is called an exact category.

**Example 8.2.** (i) Every abelian category is an exact category; in fact, take for $E$ the class of all short exact sequences in $A$.

(ii) Let $\mathcal{A} = \text{Ab}$ be the category of abelian groups and $\mathcal{B} := \text{tf-Ab}$ the full subcategory of torsionfree abelian groups. Then $\mathcal{B}$ is closed under extensions; in fact, let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a short exact sequence with $X$ and $Z$ torsionfree. Suppose $ny = 0$ for some $y \in Y$ and $n \in \mathbb{Z}$. Then $n\beta(y) = \beta(ny) = 0$ and hence $\beta(y) = 0$ since $Z$ is torsionfree. By exactness of the sequence it follows that $y$ is in the image of $\alpha$, say $y = \alpha(x)$. But then $\alpha(nx) = n\alpha(x) = ny = 0$ which implies $nx = 0$ since $\alpha$ is a monomorphism. From the torsionfreeness of $X$ we deduce that $x = 0$ and thus also $y = 0$, i.e. $Y$ is also torsionfree. So the pair $(\text{tf-Ab}, E)$ is an exact category.

However, note that $\text{tf-Ab}$ is not an abelian category (e.g. the morphism $\mathbb{Z} \xrightarrow{2} \mathbb{Z}$ does not have a cokernel in $\text{tf-Ab}$).
(iii) Similarly to the preceding example, consider the category \( t-\text{Ab} \) with objects the abelian groups containing torsion elements and the trivial group. This is a full subcategory of \( \text{Ab} \) which is closed under extensions; in fact, if 
\[
0 \to X \to Y \to Z \to 0
\]
is a short exact sequence of abelian groups and \( X \) and \( Z \) contain torsion elements, then \( Y \) contains the torsion elements of \( X \) if \( X \) is nonzero, and otherwise \( Y \) has the torsion elements of \( Z \). So, \( (t-\text{Ab}, \mathcal{E}) \) is also an exact category.

**Definition 8.3.** An exact category \( (\mathcal{B}, \mathcal{E}) \) is called a Frobenius category if the following holds:

(i) Projective and injective objects coincide (where an object \( I \) of \( \mathcal{B} \) is called injective if every exact triple in \( \mathcal{E} \) of the form \( I \to Y \to Z \) splits; and an object \( P \) of \( \mathcal{B} \) is called projective if every exact triple in \( \mathcal{E} \) of the form \( X \to Y \to P \) splits).

(ii) The category has enough projective objects and enough injective objects, i.e. for every object \( X \) in \( \mathcal{B} \) there exist triples from \( \mathcal{E} \) of the form \( X' \to I \to X \) and \( X \to I' \to X'' \) where \( I \) and \( I' \) are injective.

**Example 8.4.**

(i) A (finite-dimensional) algebra \( \Lambda \) over a field \( K \) is called a Frobenius algebra if there exists a non-degenerate associative bilinear form on \( \Lambda \). Equivalently, if there exists a linear form \( \pi : \Lambda \to K \) such that the kernel of \( \pi \) does not contain any nonzero left ideal of \( \Lambda \).

The notion of Frobenius algebra is closely related to that of selfinjective algebras (for which by definition projective and injective modules coincide). In fact, every Frobenius algebra is selfinjective, and every basic selfinjective algebra is Frobenius. For more details we refer to the notes by R. Farnsteiner [3].

Then, for any Frobenius algebra \( \Lambda \) the category of finite-dimensional \( \Lambda \)-modules is a Frobenius category. For instance, for a finite group \( G \), the category of finitely generated modules over the group algebra \( KG \) is a Frobenius category.

(ii) Consider the abelian category \( \mathcal{B} = \text{Ab} \) of abelian groups.

An object \( A \) in \( \text{Ab} \) is injective if and only if it is divisible (i.e. for every \( n \in \mathbb{Z} \setminus \{0\} \) the multiplication map \( A \to A \) is surjective; e.g. \( \mathbb{Q}, \mathbb{R} \) or \( \mathbb{Q}/\mathbb{Z} \) are injective abelian groups. On the other hand, abelian groups are nothing but modules over \( \mathbb{Z} \) and since \( \mathbb{Z} \) is a principal ideal domain, the projective objects are precisely the free objects, i.e. direct sums of copies of \( \mathbb{Z} \). In particular, in \( \text{Ab} \), projective and injective objects do not coincide and hence \( \text{Ab} \) can not be a Frobenius category.

**Definition 8.5.** Let \( (\mathcal{B}, \mathcal{E}) \) be a Frobenius category. For objects \( X \) and \( Y \) in \( \mathcal{B} \) let \( \text{Inj}(X,Y) \) denote those morphisms from \( X \) to \( Y \) which factor through some injective object.

The stable category \( \mathcal{B} \) of the Frobenius category \( (\mathcal{B}, \mathcal{E}) \) has the same objects as \( \mathcal{B} \); the morphisms are equivalence classes of morphisms modulo those factoring through injective objects, i.e.

\[
\text{Hom}_\mathcal{B}(X,Y) := \text{Hom}(X,Y) = \text{Hom}_\mathcal{B}(X,Y)/\text{Inj}(X,Y).
\]

**Example 8.6.** (Dual numbers) Let \( K \) be a field and consider the algebra \( \Lambda = K[X]/(X^2) \). This 2-dimensional algebra is a Frobenius algebra; in fact, \( \pi(a+bX) := \)
$b$ defines a linear form on $\Lambda$ such that its kernel does not contain any nonzero left ideal.

The category of finitely generated $\Lambda$-modules is then a Frobenius category. It has only two indecomposable objects, $\Lambda$ itself and the one-dimensional simple module $K$, where $\Lambda$ is the only injective (and projective) module.

In the corresponding stable module category several module homomorphisms vanish, e.g. we have that $\text{Hom}(\Lambda, \Lambda) = 0$, $\text{Hom}(\Lambda, K) = 0$ and $\text{Hom}(K, \Lambda) = 0$; on the other hand $\text{Hom}(K, K)$ remains 1-dimensional since the isomorphism can not factor through the injective module $\Lambda$.

Our main aim is to describe how the stable category of a Frobenius category $(\mathcal{B}, \mathcal{E})$ carries the structure of a triangulated category. To this end we shall briefly describe the construction of a suspension functor and then of the distinguished triangles.

For every object $X$ in $\mathcal{B}$ we choose an exact triple $X \xrightarrow{i_X} I(X) \xrightarrow{\pi_X} \Sigma X$ with entries from $\mathcal{B}$ where $I(X)$ is an injective object, i.e. in the ambient abelian category $\mathcal{A}$ there is a short exact sequence $0 \to X \to I(X) \to \Sigma X \to 0$. Thus, on objects of $\mathcal{B}$ (and hence also on objects of $\mathcal{B}$) we have a map $X \to \Sigma X$ and this will be the candidate for the suspension functor on objects.

For defining $\Sigma$ on morphisms, let $u : X \to Y$ be a morphism and consider the chosen exact triples $X \xrightarrow{j_X} I(X) \xrightarrow{\pi_X} \Sigma X$ and $Y \xrightarrow{j_Y} I(Y) \xrightarrow{\pi_Y} \Sigma Y$. Since $I(Y)$ is injective there exists a morphism $I(u) : I(X) \to I(Y)$ such that $I(u)j_X = j_Y u$. By considering the corresponding short exact sequences in the ambient abelian category the morphism $I(u)$ induces a morphism $\Sigma(u) : \Sigma X \to \Sigma Y$ satisfying $\Sigma(u)\pi_X = \pi_Y I(u)$.

**Lemma 8.7.** The morphisms $\Sigma(u)$ are well-defined in the stable category $\mathcal{B}$, i.e. they are independent of the choice of the lifting morphisms $I(u)$ and of the representatives of the morphisms $u$.

**Proof.** For a morphism $u$ the morphism $\Sigma(u)$ has been defined above as indicated in the following diagram

$$
\begin{array}{ccc}
X & \xrightarrow{i_X} & I(X) & \xrightarrow{\pi_X} & \Sigma X \\
\downarrow{u} & & \downarrow{I(u)} & & \downarrow{\Sigma(u)} \\
Y & \xrightarrow{i_Y} & I(Y) & \xrightarrow{\pi_Y} & \Sigma Y
\end{array}
$$

Now let $I(u)$ and $\tilde{I}(u)$ be two liftings for $u$, with corresponding induced morphisms $\Sigma(u)$ and $\tilde{\Sigma}(u)$ from $\Sigma X$ to $\Sigma Y$. Then

$$(I(u) - \tilde{I}(u))j_X = I(u)j_X - \tilde{I}(u)j_X = j_Y u - j_Y u = 0.$$ 

By exactness of the top row and by injectivity of $I(Y)$ there exists a morphism $\sigma : \Sigma X \to I(Y)$ such that $\sigma j_X = I(u) - \tilde{I}(u)$. This implies that

$$(\Sigma(u) - \tilde{\Sigma}(u))j_X = \pi_Y I(u) - \pi_Y \tilde{I}(u) = \pi_Y \sigma j_X.$$ 

Since $\pi_X$ is an epimorphism we can deduce that $\Sigma(u) - \tilde{\Sigma}(u) = \pi_Y \sigma$. Hence $\Sigma(u) - \tilde{\Sigma}(u)$ factors through the injective object $I(Y)$, i.e. $\Sigma(u) = \tilde{\Sigma}(u)$ in the stable category $\mathcal{B}$. 

A similar argument shows that if $u$ factors through an injective object then also $\Sigma(u)$ factors through an injective object, i.e. $\Sigma(u)$ is independent of the representative of the morphism $u$ in $\mathcal{B}$. We leave the details to the reader. □

Our above construction of the objects $\Sigma X$ and of the morphisms $\Sigma(u)$ used a fixed choice of exact triples $X \to I(X) \to \Sigma X$. The following lemma shows that this construction does not depend on the choice of the exact triples, more precisely, a different choice leads to naturally isomorphic functors. In particular, the object $\Sigma X$ is uniquely defined up to isomorphism in the stable category $\mathcal{B}$. 

**Lemma 8.8.** For any object $X$ let $X \to I(X) \to \Sigma X$ and $X \to I'(X) \to \Sigma' X$ be exact triples in $\mathcal{B}$ where $I(X)$ and $I'(X)$ are injective. Then $\Sigma X$ and $\Sigma' X$ are isomorphic in the stable category $\mathcal{B}$. Moreover, there is a natural transformation $\beta : \Sigma \to \Sigma'$ such that each $\beta_X : \Sigma X \to \Sigma' X$ is an isomorphism, i.e. the functors $\Sigma$ and $\Sigma'$ are isomorphic.

**Proof.** In the ambient abelian category $\mathcal{A}$ we have short exact sequences

$$0 \to X \xrightarrow{\iota_X} I(X) \xrightarrow{\pi_X} \Sigma X \to 0 \quad \text{and} \quad 0 \to X \xrightarrow{\iota'_X} I'(X) \xrightarrow{\pi'_X} \Sigma' X \to 0.$$ 

Since $I(X)$ and $I'(X)$ are injective in $\mathcal{B}$ there are morphisms $\alpha_X : I(X) \to I'(X)$ and $\alpha'_X : I'(X) \to I(X)$ making the left hand squares in the following diagram commutative; moreover since the rows are short exact sequences these morphisms induce morphisms $\beta_X$ and $\beta'_X$ also making the right hand squares commutative.

\[
\begin{array}{ccc}
0 & \to & X \\
\downarrow{=} & & \downarrow{=} \\
0 & \to & I(X)
\end{array}
\quad \begin{array}{ccc}
X & \xrightarrow{\iota_X} & I(X) \\
\downarrow{=} & & \downarrow{=} \\
X & \xrightarrow{\iota'_X} & I'(X)
\end{array}
\quad \begin{array}{ccc}
\pi_X & \to & \Sigma X \\
\downarrow{=} & & \downarrow{=} \\
\pi'_X & \to & \Sigma' X
\end{array}
\quad \begin{array}{ccc}
0 & \to & 0 \\
\downarrow{=} & & \downarrow{=} \\
0 & \to & 0
\end{array}
\]

By commutativity it follows that

$$(\alpha'_X \alpha_X - \operatorname{id}_{I(X)}) \iota_X = \alpha'_X \iota'_X - \iota_X = \iota'_X - \iota_X = 0.$$ 

Using exactness of the top row, the injectivity of $I(X)$ implies the existence of a morphism $\sigma_X : \Sigma X \to I(X)$ such that $\sigma_X \pi_X = \alpha'_X \alpha_X - \operatorname{id}_{I(X)}$. Again using commutativity of the above diagram we then have that

$$\pi_X \sigma_X \pi_X = \pi_X \alpha'_X \alpha_X - \pi_X = (\beta'_X \beta_X - \operatorname{id}_{\Sigma X}) \pi_X.$$ 

Since $\pi_X$ is an epimorphism we deduce $\pi_X \sigma_X = \beta'_X \beta_X - \operatorname{id}_{\Sigma X}$, i.e. $\beta'_X \beta_X - \operatorname{id}_{\Sigma X}$ factors through the injective object $I(X)$ and thus $\beta'_X \beta_X = \operatorname{id}_{\Sigma X}$ in the stable category $\mathcal{B}$.

An analogous argument shows that also $\beta_X \beta'_X = \operatorname{id}_{\Sigma X}$ in $\mathcal{B}$. Thus, $\Sigma X$ and $\Sigma' X$ are isomorphic in $\mathcal{B}$, as claimed.

For naturality, consider different choices of exact triples for objects $X$ and $Y$ and a morphism $u : X \to Y$, as in the following commutative diagrams
We claim that $\beta$ induces a natural isomorphism between the functors $\Sigma$ and $\Sigma'$ resulting from different choices of exact triples. From the first part of the proof we already know that each $\beta_X$ is an isomorphism in $\mathcal{B}$. It remains to show that we indeed have a natural transformation, i.e. we have to show that $\beta_Y \Sigma(u) = \Sigma'(u)\beta_X$ in the stable category.

Note that by commutativity of the above diagrams we have

$$(\alpha_Y I(u) - I'(u)\alpha_X)\iota_X = \alpha_Y \iota_Y u - I'(u)\iota'_X = \iota'_Y u - I'(u)\iota'_X = 0.$$ 

Hence by injectivity of $I'(Y)$ there exists a morphism $\tau : \Sigma X \to I'(Y)$ such that $\tau \pi_X = \alpha_Y I(u) - I'(u)\alpha_X$. It then follows that

$$\pi'_Y \tau \pi_X = \pi'_Y (\alpha_Y I(u) - I'(u)\alpha_X) = \beta_Y \pi_Y I(u) - \Sigma'(u)\pi_X \alpha_X$$

$$= \beta_Y \Sigma(u)\pi_X - \Sigma'(u)\beta_X \pi_X = (\beta_Y \Sigma(u) - \Sigma'(u)\beta_X)\pi_X.$$

Since $\pi_X$ is an epimorphism this implies that $\pi'_Y \tau = \beta_Y \Sigma(u) - \Sigma'(u)\beta_X$, i.e. $\beta_Y \Sigma(u) - \Sigma'(u)\beta_X$ factors through the injective object $I'(Y)$ and hence we have $\beta_Y \Sigma(u) = \Sigma'(u)\beta_X$ in the stable category $\mathcal{B}$. \hfill \Box

Hence we have a well-defined functor $\Sigma$ on the stable category $\mathcal{B}$. This can be shown to be an autoequivalence. Under certain assumptions it is even an automorphism; for details on this subtle issue see Happel’s book [5, Section I.2].

We now describe the construction of distinguished triangles in the stable category $\mathcal{B}$. Let $X, Y$ be objects in $\mathcal{B}$ and let $u : X \to Y$ be a morphism. For $X$ we have from the above construction an exact triple $X \xrightarrow{\iota} I(X) \xrightarrow{\pi} \Sigma X$ where $I(X)$ is injective.

In the additive category $\mathcal{B}$ there exists a coproduct $I \oplus Y$, with morphisms $\iota_I : I \to I \oplus Y$ and $\iota_Y : Y \to I \oplus Y$ satisfying the universal property given in Remark 1.2. Now we form the pushout of the morphisms $\iota : X \to I$ and $u : X \to Y$. More precisely, the pushout $M(u)$ is defined as the cokernel of the morphism $\iota_I u - \iota_Y u : X \to I \oplus Y$. By definition, this cokernel is an object $M(u)$ together with a morphism $c : I \oplus Y \to M(u)$ such that $c(\iota_I u - \iota_Y u) = 0$ and satisfying the universal property for cokernels given in Section 2. In particular, the left hand square in the following diagram is commutative

$$
\begin{array}{ccc}
X & \xrightarrow{\iota} & I(X) \\
\downarrow{u} & & \downarrow{\pi} \\
Y & \xrightarrow{\iota_Y} & M(u)
\end{array}
\quad
\begin{array}{ccc}
\Sigma X & & id \\
\downarrow{id} & & \\
\Sigma X & & \Sigma X
\end{array}
\xrightarrow{c_{\iota_I}}
\begin{array}{ccc}
\Sigma X & & \\
\downarrow{id} & & \\
\Sigma X & & \Sigma X
\end{array}
$$

We wish to complete this diagram with a morphism $w : M(u) \to \Sigma X$. To this end recall that from the properties of a coproduct there is a unique morphism $f : I \oplus Y \to \Sigma X$ such that $f \iota_I = \pi$ and $f \iota_Y = 0$. Using this morphism $f$ in the universal property for the cokernel $M(u)$ we deduce the existence of a (unique) morphism $w : M(u) \to \Sigma X$ such that $wc = f$. It follows that $wcl_I = f \iota_I = \pi$, i.e.
the following diagram of objects and morphisms in $\mathcal{B}$ is commutative

\[
\begin{array}{ccc}
X & \xrightarrow{\iota} & I(X) & \xrightarrow{\pi} & \Sigma X \\
\downarrow u & & \downarrow c_{IJ} & & \downarrow \text{id} \\
Y & \xrightarrow{\text{cl}_Y} & M(u) & \xrightarrow{w} & \Sigma X
\end{array}
\]

Note that since $\mathcal{B}$ is closed under extensions the cokernel will again be an object in $\mathcal{B}$ (and not only in the ambient abelian category $\mathcal{A}$).

The images in the stable category $\hat{\mathcal{B}}$ of any triangles of the form

\[
X \xrightarrow{u} Y \xrightarrow{\text{cl}_Y} M(u) \xrightarrow{w} \Sigma X
\]

are called standard triangles in $\hat{\mathcal{B}}$. As usual, the set of distinguished triangles in $\hat{\mathcal{B}}$ is formed by the set of all triangles in $\hat{\mathcal{B}}$ which are isomorphic (in $\hat{\mathcal{B}}$!) to a standard triangle.

The main structural result on stable categories of Frobenius categories is then the following; for a detailed proof we refer to Happel’s book [5, Chapter I.2].

**Theorem 8.9.** Let $(\mathcal{B}, \mathcal{E})$ be a Frobenius category. With the above suspension functor $\Sigma$ and the collection of distinguished triangles just defined, the stable category $\hat{\mathcal{B}}$ is a triangulated category.

**Remark 8.10.** A triangulated category is called algebraic (in the sense of B. Keller, see [8]) if it is equivalent as a triangulated category to the stable category of a Frobenius category. For more details on algebraic and non-algebraic triangulated categories we refer to S. Schwede’s article in this volume [13], see also [14]. Strikingly, there are triangulated categories which are neither algebraic nor topological [10].

**References**


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