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Bayesian combination of two-dimensional location estimates

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Abstract We extend a Bayesian method for combining estimates of means and variances from independent cues in a spatial cue-combination paradigm. In a typical cue-combination experiment, subjects estimate a value on a single dimension—for example, depth—on the basis of two different cues, such as retinal disparity and motion. The mathematics for this one-dimensional case is well established. When the variable to be estimated has two dimensions, such as location (which has both x and y values), then the one-dimensional method may be inappropriate due to possible correlations between x and y and the fact that the dimensions may be inseparable. A cue-combination task for location involves people or animals estimating xy locations under two single-cue conditions and in a condition in which both cues are combined. We present the mathematics for the two-dimensional case in an analogous manner to the one-dimensional case and illustrate them using a numeric example. Our example involves locations on maps, but the method illustrated is relevant for any task for which the estimated variable has two or more dimensions.

Keywords Bayesian statistics · Cue combination · Location estimation · Two-dimensional variables

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Mobile organisms need to remember places in their environment and navigate to and from them, either from memory or by using other means—for example, magnetic fields (Buehlmann, Hansson, & Knaden, 2012), celestial cues (Legge, Spetch, & Cheng, 2010), landmarks (Spetch & Kelly, 2006; Wystrach, Schwarz, Schultheiss, Beugnon, & Cheng, 2011), proximal and distal cues (Brodbeck, 1994; Spetch & Edwards, 1988), and so on. A popular hypothesis regarding how organisms combine different sources of information into a single estimate of a destination (location) is that they approximate optimal Bayesian inference. An example would be when a landmark or a proximal cue is combined with a distal cue to determine a destination. This *cue-combination* literature has been reviewed elsewhere (Cheng, Shettleworth, Huttenlocher, & Reiser, 2007) and will not be further reviewed here. Instead, our primary goal is to extend Bayesian cue combination, which typically deals with estimating one-dimensional (1-D) signals from different cues, to estimating two-dimensional (2-D) signals from different cues that themselves are 2-D signals (e.g., again, location).

In general, Bayesian cue combination allows one to infer or estimate the likelihood of values of some signal (S), given noisy measurements of S from different sources of information or cues (C) according to Bayes's rule,

$$P(S|C) = \frac{P(S) \times P(C|S)}{P(C)}, \tag{1}$$

where $P(S)$ and $P(C)$ are the prior likelihoods of the signal and cue, respectively, and $P(C|S)$ is the likelihood of the cue given the signal. The theorem thus provides a means to optimally update existing (prior) estimates, given new, additional evidence or cues. The most likely value of S as estimated by combining information from the different cues according to Bayes's rule will be the most reliable estimate of S. In the special case with uniform priors and independent Gaussians, the Bayesian method is equivalent to minimum-variance, and optimal cue combination results from a weighted linear combination rule in which the weight of each cue is inversely proportional to the variance of that cue (e.g., Colombo & Series, 2012; Ernst & Banks, 2002; Landy, Malney, Johnston, & Young, 1995; Yuille & Bulthoff, 1996). This rule has successfully modeled human performance across a variety of tasks when S is one-dimensional. Furthermore, the assumption that the prior probability for the different values of S is uniform often holds in human perception (e.g., Ernst & Banks, 2002; Landy et al., 1995).

As was noted, Bayesian cue combination has mostly been applied to properties that vary along a single dimension; for instance, depth can vary along a line of sight. The more general case occurs when a property varies along n dimensions; for instance, location on a flat surface can vary in x and y directions. Ma, Zhou, Ross, Foxe, and Parra (2009) derived a Bayesian model for n-dimensional data, which they applied to recognizing spoken words. They used Bayesian cue combination to explain how visual information improves auditory word recognition accuracy. To validate their Bayesian formulation, they experimentally varied the reliability of the auditory and visual cues by manipulating the signal-to-noise ratio of their stimuli. They then fitted the Bayesian model to the accuracy data and found a good fit between the model's predictions and human performance.

Here, we use the Bayesian method to analyze 2-D spatial data, such as location estimates, in which the dimensions of the variable are inseparable (Garner, 1974). Importantly, we show how the Bayesian method can be used straightforwardly to combine 2-D spatial locations estimated from different cues, to compute the reliabilities of the individual cues, to compute their predicted mean values under the combined condition, and to compute the relative weighting of the cues. Furthermore, this article provides a tutorial on the 2-D method using Microsoft Excel©. The supplementary material provided is an Excel© spreadsheet that performs the computations on the data for multiple subjects or items. In the Appendix, we further provide a MATLAB© script that implements the same computations. Lastly, we point out in the discussion other situations for which this method of cue combination is likely to be useful.

The Bayesian framework specifies that the optimal manner to estimate the value of a signal is to use a weighted combination of estimates from independent sources based on their relative reliability. For example, a classic 1-D cue-combination problem in vision is how observers estimate the depth of an object from different depth cues (e.g., retinal disparity and motion; Landy et al., 1995). To address this issue, observers are typically shown conditions in which depth is specified only by retinal disparity or only by motion cues (single-cue conditions) or in which depth is specified by both disparity and motion cues (combined-cue condition). Using the Bayesian method of estimating the weights, the observers' depth estimate in the combined-cue condition can then be compared with the predicted combined estimate based on their depth estimate from the single cues. Each cue signals the depth with different levels of reliability, and the noise around the signals is assumed to be independent and Gaussian. The visual system seems to take this differential noise into account through the relative weighting of the information from the different cues, thereby minimizing the variance in the combined estimate (Landy et al., 1995).

Moving from one to two dimensions

The Bayesian method for integrating two 1-D variables typically also assumes that both cues are independent. The extent to which the information from the two independent cues is used in a combined-cue condition (when they are both present) will depend on their *relative reliability*, which is determined by a weighting formula (either Eqs. 2 and 3 or Eqs. 4 and 5). The weight of each cue varies between zero and one, and the two weights sum to one. If one cue is more certain (reliable; precise) than the second, the more reliable cue will be given more weight in the combined estimate. In the case when the two cues are correlated, Oruç, Maloney, and Landy (2003) showed that a weighted linear combination rule also yields the optimal solution if the correlation between observers' internal estimates of the single cues is taken into account. It is worth noting that two inseparable dimensions within the same variable are not the same thing as two different cues that both provide information about the 2-D variable (e.g., auditory and visual cues to the location of a signal). The paradigm example is spatial location, which always has *x* and *y* values in two dimensions and *x, y,* and *z* values in three dimensions.

The formulas we present in Eqs. 2–5 illustrate the case in which Cue_1 and Cue_2 are independent conditions under which the value of a 1-D variable is estimated. These equations allow us to then combine the single estimates to predict responses in a combined-cue condition. If there is a Cue_1 condition (e.g., retinal disparity) and a Cue_2 condition (e.g., motion), then the equations to obtain the weights for predicting the optimal combination of the cues from these single-cue conditions, in a condition in which both cues are available to be combined (Cue_C), are given in Eqs. 2 and 3 below. In Eqs. 2–7, w indicates the weight, σ^2 the variance for each cue, and r indicates the reliability. The subscripts refer to the associated single-cue condition (1 or 2).

$$w_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{2}$$

$$w_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2} \tag{3}$$

Following previous work (e.g., Landy et al., 1995), we can express the weights in Eqs. 2 and 3 with respect to the reliability of the signal in that condition. Reliability (or precision) is defined as the inverse of the variance of the signal—that is, $r = \frac{1}{\sigma^2}$. We can therefore rewrite Eqs. 2 and 3 as

$$w_1 = \frac{\frac{1}{\sigma_1^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{r_1}{r_1 + r_2} \tag{4}$$

$$w_2 = \frac{\frac{1}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} = \frac{r_2}{r_1 + r_2} \tag{5}$$

That is, in Eqs. 2 and 3, the weights are parameterized with respect to variance, whereas in Eqs. 4 and 5, they are parameterized with respect to reliability. It can be shown that both parameterizations lead to the same values. Note that in Eqs. 2 and 3, the numerator of w_1 is the variance of the cue_2 condition, whereas in Eqs. 4 and 5, the numerator of w_1 is the reliability for the cue_1 condition.

Once the weights are computed, they are multiplied with the means of each single cue condition to predict the optimal value for the mean in the combined cue condition ($\overline{X}_C$), as in Eq. 6:

$$\overline{X_C} = w_1\overline{X_1} + w_2\overline{X_2} \tag{6}$$

The reliability in the combined-cue condition (r_c) is higher than the reliability of each cue alone and is equal to the sum of the reliabilities of the other two cues (i.e., $r_c = r_1 + r_2$). Because variance and reliability are inversely related, the predicted variance (σ_C^2) around $\overline{X}_C$ is the inverse of the combined reliabilities and can be computed using Eq. 7:

$$\sigma_C^2 = \frac{1}{r_C} = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2 + \sigma_2^2} \tag{7}$$

This mathematical relationship between reliability and variance provides the means to extend the Bayesian approach to the 2-D (or higher dimensional) case.

Because we want to extend the Bayesian analysis to variables with more than one dimension, such as location, we need to represent the mean location as a vector (whose elements are $\overline{X}$ and $\overline{Y}$) and the variability in the data as a matrix (the *covariance matrix,* which contains the variability for each dimension separately and jointly).

Extending Bayesian cue combination to spatial tasks is important because location estimation is used in many tasks with both humans (e.g., Friedman & Montello, 2006; Friedman, Montello, & Burte, 2012; Holden, Curby, Newcombe, & Shipley, 2010; Huttenlocher, Hedges, & Duncan, 1991) and other animals (cf. Cheng et al., 2007). An analogous situation where a 1-D versus 2-D difference in methods is required with location estimates is when estimation accuracy is determined using correlations between actual and estimated locations. For example, it is wrong to simply correlate the actual with the obtained *x* values and the actual with the obtained *y* values and conclude anything. Instead, one must use a procedure called *bidimensional regression*, first proposed by Tobler (1964) and later refined and situated within the regression context by Friedman and Kohler

(2003). There are parallels between the 1-D and 2-D cases, of course. For instance, correlating x on y does not result in the same slope as correlating y on x. Equally, the bidimensional regression procedure results in different regression coefficients using the observed values as the independent variable and the actual values as the dependent variable or vice versa.

In the case of cue combination with location estimates (i.e., in which subjects need to estimate an x and y value for each location), a given set of locations is presented in a Cue$_1$ condition for one group of subjects, a Cue$_2$ condition for a second group, and a combined-cue condition (Cue$_C$) for a third group (the three conditions can also be presented within subjects). The task for all three groups is to estimate the same locations. The issue then becomes: How does one use a Bayesian approach to (1) find the appropriate weights to predict the mean values in the combined-cue condition and (2) find the predicted covariance in the combined-cue condition? We first show the necessary 2-D computations (Ma et al., 2009) in a way that is analogous to the 1-D computations (Eqs. 2–6). With the 2-D computations, we further take into account possible correlations between subjects' internal estimates of x and y for a single location (see Oruç et al., 2003, for the case where the assumption of independent cues is violated but the noise is not assumed to be Gaussian).

Because we are using a 2-D variable, in addition to variances, we also need to use covariances, which measure the variability of both dimensions within each cue condition. In particular, we must use *covariance matrices* to find the predicted weights, variances, and covariances. For example, the covariance matrix for the Cue$_1$ condition is

$$\mathbf{cov}_1 = \begin{bmatrix} \sigma^2_{X1} & \mathrm{cov}_{XY1} \\ \mathrm{cov}_{XY1} & \sigma^2_{Y1} \end{bmatrix},$$

in which the subscripts refer to the x and y dimensions of the locations being estimated. For convenience, we abbreviate the covariance matrix for the single conditions Cue$_1$ and Cue$_2$ as $\mathbf{cov}_1$ and $\mathbf{cov}_2$, respectively, in the equations below. The covariance matrix for the combined-cue condition is abbreviated as $\mathbf{cov}_C$. To make the analogy to the 1-D case, as reliability is defined as the inverse of variance (reliability $= \frac{1}{\sigma^2}$ for 1-D), the *precision* matrix ($\mathbf{p}$) is defined as the inverse of the covariance matrix ($\mathbf{p} = \mathbf{cov}^{-1}$) in the 2-D case. For example, the estimated precision matrix for the Cue$_1$ condition is given below:

$$\mathbf{p}_1 = \begin{bmatrix} \sigma^2_{X1} & \mathrm{cov}_{XY1} \\ \mathrm{cov}_{XY1} & \sigma^2_{Y1} \end{bmatrix}^{-1} = \frac{1}{\sigma^2_{X1}\sigma^2_{Y1} - \mathrm{cov}^2_{XY1}} \times \begin{bmatrix} \sigma^2_{Y1} & -\mathrm{cov}_{XY1} \\ -\mathrm{cov}_{XY1} & \sigma^2_{X1} \end{bmatrix}$$

That is, if the 2 × 2 matrix $\mathbf{a} = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$, then $\mathbf{a}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. The MINVERSE function in Microsoft Excel© performs this computation.

To find the weights necessary to predict $\overline{X}$ and $\overline{Y}$, we can use Eqs. 8 and 9 or, using the precision matrix, Eqs. 10 and 11.

$$\mathbf{w}_1 = \frac{\mathbf{cov}_2}{\mathbf{cov}_1 + \mathbf{cov}_2} = \mathbf{cov}_2(\mathbf{cov}_1 + \mathbf{cov}_2)^{-1} \tag{8}$$

$$\mathbf{w}_2 = \frac{\mathbf{cov}_1}{\mathbf{cov}_1 + \mathbf{cov}_2} = \mathbf{cov}_1(\mathbf{cov}_1 + \mathbf{cov}_2)^{-1} \tag{9}$$

$$\mathbf{w}_1 = (\mathbf{p}_1 + \mathbf{p}_2)^{-1}\mathbf{p}_1 \tag{10}$$

$$\mathbf{w}_2 = (\mathbf{p}_1 + \mathbf{p}_2)^{-1}\mathbf{p}_2 \tag{11}$$

The resulting weights in Eqs. 8–11 are also 2 × 2 matrices. Note that Eqs. 8–11 are analogous to Eqs. 2–5. In addition, just as the weights in the single dimension case sum to 1, the weight matrices sum to the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. It is also important to bear in mind that matrix division is not defined. Rather, one must multiply by the inverse of the divisor matrix instead.

Once the weights have been computed, the predicted means, variances, and covariances in the combined-cue condition can be found. For example, to find the predicted means for a given combined-cue condition, we use Eq. 12:

$$\begin{bmatrix} \overline{\mathrm{X}}_C \\ \overline{\mathrm{Y}}_C \end{bmatrix} = \mathbf{w}_1 \begin{bmatrix} \overline{\mathrm{X}}_1 \\ \overline{\mathrm{Y}}_1 \end{bmatrix} + \mathbf{w}_2 \begin{bmatrix} \overline{\mathrm{X}}_2 \\ \overline{\mathrm{Y}}_2 \end{bmatrix} \tag{12}$$

Because the weights ($\mathbf{w}_1$ and $\mathbf{w}_2$) are 2 × 2 matrices and the mean location is a vector, this step requires matrix multiplication. The steps required for the multiplication will be outlined in detail in the numeric example below.

Finally, to estimate the predicted covariance matrix in the combined-cue condition on the basis of the single-cue conditions, we first calculate the precision matrix for each individual cue condition, using the inverse of the covariance matrix. Then we simply add like cells, using matrix addition, and invert the resultant matrix to get the covariance matrix:

$$\mathbf{p}_c = \mathbf{p}_1 + \mathbf{p}_2$$
$$\mathbf{cov}_c = \mathbf{p}_c^{-1}$$

Note that the predicted covariance matrix in the combined-cue condition is the same value as the first term in Eqs. 10 and 11 [i.e., $(\mathbf{p}_1 + \mathbf{p}_2)^{-1}$] for the weights. We illustrate the complete method for finding the estimated statistics for the combined-cue condition in the numeric example below.

Numeric example

We will now consider a numeric example based on a small part of a published data set (Friedman et al., 2012). Briefly, the experimental situation involved having three groups of subjects estimate identical locations in three conditions.

The single-cue conditions were called *dots-only* and *names-only*. In the dots-only condition, subjects were presented with a simplified polygon that represented the Canadian province of Alberta. On each trial, subjects were presented with one dot inside the polygonal frame. After a 4-s masking interval, a blank polygon frame appeared, and subjects had to click inside the polygon where they thought the trial's dot had been. The 26 dots used across all trials corresponded to the locations of 26 Albertan cities. This procedure resembles that used by Huttenlocher et al. (1991) and many others.

In the names-only condition, after being told that the polygon represented a map of Alberta, subjects were given a blank frame on each trial with a city's name at the top of the polygon. They were told that their task was to click on the place within the polygon where they thought the city was located. Thus, in the dots-only condition, participants could use only perceptual memory of the location of the dot, whereas in the names-only condition, they could use only their semantic knowledge of the location of the city. The key question

Table 1 Raw estimates and some descriptive statistics for 30 subjects' estimates of the location of the city of Edmonton in the three conditions in Montello, Friedman, and Burte, 2012

	Dots-Only		Names-Only		Dots-and-Names	
	EstX	EstY	Est X	Est Y	Est X	Est Y
	227	380	240	302	220	386
	236	396	226	496	244	387
	243	408	222	298	221	390
	237	418	154	348	236	381
	232	383	172	249	247	384
	215	396	215	242	202	381
	240	378	202	346	246	374
	232	417	171	346	229	340
	185	402	232	365	232	379
	241	402	195	239	221	370
	233	405	138	275	224	392
	219	377	302	464	233	409
	219	373	241	307	240	396
	251	379	183	283	220	371
	234	409	180	262	232	356
	227	390	176	250	226	372
	256	372	196	353	242	395
	254	392	185	338	226	385
	232	395	190	357	234	367
	213	378	158	241	229	379
	243	384	175	339	224	393
	218	371	164	350	218	365
	227	399	197	323	235	387
	234	415	215	336	229	390
	224	390	185	329	241	404
	233	394	153	296	219	367
	230	380	172	256	248	384
	247	372	217	251	270	365
	230	387	160	318	236	386
	234	368	142	184	225	407
Means	231.53	390.33	191.93	311.43	231.63	381.40
Variances	193.84	213.26	1232.82	4226.05	159.41	226.11
cov_{XY}	-8.36		1144.41		7.67	

$$\mathbf{cov} \begin{bmatrix} 193.84 & -8.36 \\ -8.36 & 213.26 \end{bmatrix} \begin{bmatrix} 1232.82 & 1144.41 \\ 1144.41 & 4226.05 \end{bmatrix}$$

$$\mathbf{p} = \mathbf{cov}^{-1} = \begin{bmatrix} .005168 & .000203 \\ .000203 & .004697 \end{bmatrix} \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .002316 \end{bmatrix}$$

$$\textit{Combined Precision Matrices}: \begin{bmatrix} .006251 & -9.1E-05 \\ -9.1E-05 & .005013 \end{bmatrix}$$

was how subjects estimated the 2-D location of the dot when provided with both the perceptual dot and the name label.

To address this question, in the combined *dots-and-names* condition, subjects were told that the polygon represented a map of Alberta. They were also informed that there would be a city name and a dot on the first map and that the dot would be in the correct location for the named city. After the same 4-s mask as in the dots-only condition, a blank polygon appeared, and subjects clicked in the place they thought the city/dot belonged. The procedure for this condition was thus identical to that for the dots-only condition. According to the Bayesian cue-combination method formulated here, subjects make the most reliable (i.e., least variable) estimate of the dots' locations by combining information from both short-term perceptual memory and long-term semantic memory when both cues were present in the combined-cue condition.

Table 1 details the statistics required to compute the Bayesian predictions from the 30 subjects in each group for 1 city of the 26 in the experiment—Edmonton. We chose this city to illustrate the methods because most of the subjects lived there, so that the mean familiarity ratings (taken after the location estimates were made, on a scale of 1–9, where 1 meant *no knowledge* and 9 meant *a great deal of knowledge*) were 8.6, 8.5, and 8.6 for subjects in the dots-only, names-only, and dots-and-names conditions, respectively. These familiarity ratings are of interest because, as will be seen, self-rated familiarity did not predict estimation accuracy. This example is the equivalent of an item analysis; that is, we predict the results for the combined-cue condition for one city across subjects (and could obviously do the same for each individual city). A similar analysis could be conducted separately for each subject across cities.

Given the data in Table 1, the covariance matrices for the dots-only and names-only conditions are

$$\text{Dots} - \text{only covariance matrix} = \begin{bmatrix} 193.84 & -8.36 \\ -8.36 & 213.26 \end{bmatrix}$$

$$\text{Names} - \text{only covariance matrix} = \begin{bmatrix} 1232.82 & 1144.41 \\ 1144.41 & 4226.05 \end{bmatrix}$$

To compute the precision matrices from each of these, we used the MINVERSE function in Microsoft Excel©. MINVERSE takes as input a matrix and outputs its inverse. MINVERSE is an array function, so one must (1) first select the output cells (in this case, a 2 × 2 empty set of cells), then (2) put in the MINVERSE formula in any of those cells, and (3) press the Control–Shift–Enter keys simultaneously to make the function work as an array function. For example, if the numbers from one of the covariance matrices above were in cells B2 to C3, then select the empty cells E2 to F3, put the formula = MINVERSE(B2:C3) into one of those four cells, and then press the Control–Shift–Enter keys. You should at that point see the same formula in each cell (with brace brackets around it), but the values in the spreadsheet will be the correct values for the precision matrix. By this method, the precision matrices for the dots-only and names-only conditions are

$$\text{Dots} - \text{only Precision Matrix} = \begin{bmatrix} .005168 & .000202 \\ .000202 & .004697 \end{bmatrix}$$

$$\text{Names} - \text{only Precision Matrix} = \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix}$$

The predicted precision matrix in the combined-cue condition is achieved by adding like cells:

$$\text{Dots} - \text{and} - \text{Names Precision Matrix} = \begin{bmatrix} .006251 & -9.1\text{E} - 05 \\ -9.1\text{E} - 05 & .005013 \end{bmatrix}.$$

Recall that, in the 1-D case, the inverse of the reliability in the combined-cue condition is the variance in that condition. Using a similar logic, the inverse of the predicted precision matrix, again achieved by using the Excel function MINVERSE, is the predicted covariance matrix for the combined dots-and-names condition:

$$\begin{bmatrix} 160.02 & 2.90 \\ 2.90 & 199.53 \end{bmatrix}.$$

The observed covariance matrix (in pixel units) for this condition was:

$$\begin{bmatrix} 159.4126 & 7.668966 \\ 7.668966 & 226.1103 \end{bmatrix}.$$

Note that in all of the numeric matrices we have illustrated, we have used only some of the digits in the numbers from the nonrounded matrices in Excel; the inverted covariance matrices above, for example, have many more digits than are shown.

Finally, to compute the weights for each single-cue condition, we use Eqs. 10 and 11. For the dots-only condition (Cue_1):

$$\begin{aligned}\mathbf{w}_{1(\text{Dots})} &= \left(\begin{bmatrix} .005168 & .000202 \\ .000202 & .004697 \end{bmatrix} + \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix}\right)^{-1} \begin{bmatrix} .005168 & .000202 \\ .000202 & .004697 \end{bmatrix} \\ &= \begin{bmatrix} .006251 & -9.1\text{E}-05 \\ -9.1\text{E}-05 & .005013 \end{bmatrix}^{-1} \begin{bmatrix} .005168 & .000202 \\ .000202 & .004697 \end{bmatrix} \\ &= \begin{bmatrix} 160.0155 & 2.902708 \\ 2.902708 & 199.5326 \end{bmatrix} \begin{bmatrix} .005168 & .000202 \\ .000202 & .004697 \end{bmatrix} = \begin{bmatrix} .827472 & .046034 \\ .055401 & .937783 \end{bmatrix}\end{aligned}$$

The first term in the first line expresses the addition of the two precision matrices in the single-cue conditions. The second line shows the sum. In the third line, this sum is then inverted using MINVERSE. The final result is obtained by using the MMULT function in Excel©, which is also an array function. So, if the first matrix were in columns A and B and rows 1 and 2, and the second matrix were in columns C and D and also rows 1 and 2, one would select a 2 × 2 empty matrix of cells—say, E1 to F2—put in the equation = MMULT(A1:B2,C1:D2), and push "control-shift-enter" to make the array that is the result. Note that in matrix multiplication, order is important; here, in the last step, the predicted covariance matrix is first in the formula, and the precision matrix is second in the formula for each weight. Similarly, the weight matrix for the names-only condition is:

$$\begin{aligned}\mathbf{w}_{2(\text{Names})} &= \left(\begin{bmatrix} .005168 & .000203 \\ .000203 & .004697 \end{bmatrix} + \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix}\right)^{-1} \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix} \\ &= \begin{bmatrix} .006251 & -9.1\text{E}-05 \\ -9.1\text{E}-05 & .005013 \end{bmatrix}^{-1} \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix}. \\ &= \begin{bmatrix} 160.0155 & 2.902708 \\ 2.902708 & 199.5326 \end{bmatrix} \begin{bmatrix} .001084 & -.00029 \\ -.00029 & .000316 \end{bmatrix} = \begin{bmatrix} .172527 & -.046034 \\ -.055401 & .062217 \end{bmatrix}\end{aligned}$$

We are now ready to predict the means for the combined dots-and-names condition. This calculation again involves the matrix multiplication between the weights and means for dots-only and the weights and means for names-only, as in Eq. 12.

$$\begin{aligned}&= \mathbf{w}_1 \begin{bmatrix} \overline{\text{X}}_1 \\ \overline{\text{Y}}_1 \end{bmatrix} + \mathbf{w}_2 \begin{bmatrix} \overline{\text{X}}_2 \\ \overline{\text{Y}}_2 \end{bmatrix} \\ &= \begin{bmatrix} .827472 & .046034 \\ .055401 & .937783 \end{bmatrix} \begin{bmatrix} 231.53 \\ 390.33 \end{bmatrix} + \begin{bmatrix} .1725278 & -.046034 \\ -.055401 & .062217 \end{bmatrix} \begin{bmatrix} 191.93 \\ 311.43 \end{bmatrix} \\ &= \begin{bmatrix} 209.556 \\ 378.875 \end{bmatrix} + \begin{bmatrix} 18.778 \\ 8.743 \end{bmatrix} = \begin{bmatrix} 228.333 \\ 387.618 \end{bmatrix}\end{aligned}$$

The last line in the equation above represents the results of the two matrix multiplications, and the final result is achieved by simply summing the *x*s (top row) and *y*s (bottom row) from each of the matrices. The sum of the weight matrices equals 1 on the diagonals and 0 on the off-diagonals, as it should (this is the equivalent of the weights in the 1-D case summing to 1). Thus, the predicted *X* and *Y* values in pixel units for the combined-cue condition (dots-and-names) were 228.3 and 387.6, respectively, and the obtained values were 231.6 and 381.4.

To get a sense of the relative weights of the two cue conditions, we used the Excel© function MDTERM, which gives the determinant of a matrix. To find the determinant of the weights for the dots-only condition, we would use the array formula MDTERM on the array

$$\begin{bmatrix} .827472 & .046034 \\ .055401 & .937783 \end{bmatrix}$$

to get the result .77344, and for the names-only condition, we would use the formula on the array

$$\begin{bmatrix} .172527 & -.046034 \\ -.055401 & .062217 \end{bmatrix}$$

to get the result .008184. It is important to note that these are simply the relative magnitudes of the matrices, and not the weights per se. The determinants make it easy to see that the information from dots-only (short-term perceptual memory) was given considerably more magnitude than that from names-only (long-term semantic memory) in the computation of the combined weights. Thus, we have shown how to compute the proper Bayesian predictions for 2-D measures. The end results can be used to determine whether a Bayesian model is appropriate for the psychological theory underlying the obtained data.

Extension to 3-D variables

Using the precision matrices to perform the 2-D computations lends itself easily to a 3-D Bayesian formulation. We will only give the 3-D covariance matrix here; however, once one calculates the matrix, the inverse is the 3-D precision matrix, and the computations can proceed as above.

$$3 - \text{D Covariance Matrix} = \begin{bmatrix} \sigma^2_{X1} & \text{cov}_{XY1} & \text{cov}_{XZ1} \\ \text{cov}_{XY1} & \sigma^2_{Y1} & \text{cov}_{YZ1} \\ \text{cov}_{XZ1} & \text{cov}_{YZ1} & \sigma^2_{Z1} \end{bmatrix}$$

Discussion

In the present article, we illustrated the method for the Bayesian combination of cues in the case in which the single- and combined-cue conditions are cuing specific 2-D locations. The upscaling of the method from 1-D variables to 2-D variables was straightforward. In the example, the predicted weights were higher for the dots-only condition than for the names-only condition, and indeed, the data were much less accurate for the names-only condition than for the dots-only condition (see Friedman et al., 2012). This strong reliance on perceptual memory cues in this data set is interesting because subjects were very confident in their familiarity with Edmonton (and the names-only judgments presumably were based on information in long-term memory obtained from a large variety of sources over their lifetimes).

Bayes's rule specifies how to optimally update beliefs in the face of new evidence. As applied to psychology, one core claim is that people can approximate optimal decisions given noisy data (Bowers & Davis, 2012). These beliefs can be priors, but they can also be beliefs that have already been updated by other evidence. Indeed, in their application of Bayesian inference to cue combination, Cheng et al. (2007) "divide cases of Bayesian combination of information conveniently into three different varieties . . . (a) Two current sources of information are combined in estimating a spatial value. (b) A current source of information is combined with prior information, typically an average derived from past experience. This case is closest to Bayes' original (1763) formulation. (c) A current source of information is combined with categorical information that may or may not be derived from past experience" (pp. 625–626). In all three situations, the same Bayesian formalism underlies the computation of the optimal combination, and our extension of the 1-D case to the 2-D case would apply equally well.

An interesting parallel to the 2-D method we illustrate is general recognition theory (GRT; Ashby & Townsend, 1986). GRT is a multivariate generalization of signal detection theory (Green & Swets, 1966; Tanner & Swets, 1954) and, as such, provides an important precedent to the method outlined here. According to this theory, a multidimensional perceptual space is partitioned into decision regions. A response to a stimulus is determined by deciding which region the percept elicited by the stimulus falls into—typically, using a weighted linear summation rule. Like the Bayesian method illustrated here, GRT accords with much human data, particularly in categorizing stimuli into discrete categories (Ashby & Maddox, 1990; Ashby & Perrin, 1988; Ashby & Townsend, 1986; Fific, Little, & Nosofsky, 2010). Thus, important future work would extend GRT to multidimensional response variables and to directly compare GRT with Bayesian inference.

Beyond the particular data we used as an example, the method described here should be useful for any location estimation task (e.g., one involving only perceptual variables, such as two different landmarks, which are then combined, or a landmark and a distal cue, and so on). In addition, it should be noted that if, instead of Euclidean locations, one wanted to use polar coordinates (r, θ), then the mean of these (across subjects or items) can be substituted in the equations every time $\overline{X}$ and $\overline{Y}$are used. Overall, the extension of Bayesian cue combination to the important variable of multidimensional location should be useful in a wide range of estimation, categorization, and identification tasks.

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Appendix

```
% The following lines can be executed "as is" in a Matlab command window.
% The sample data provided is from the dots-only and names-only
% conditions in the manuscript.

% cue1 sample data: row 1 = x, row 2 = y
% The matrix is transposed so the final matrix size should be 30x2
dots = [
   227 236 243 237 232 215 240 232 185 241 ...
   233 219 219 251 234 227 256 254 232 213 ...
   243 218 227 234 224 233 230 247 230 234;
   380 396 408 418 383 396 378 417 402 402 ...
   405 377 373 379 409 390 372 392 395 378 ...
   384 371 399 415 390 394 380 372 387 368
    ]';

% cue2 sample data: row 1 = x, row 2 = y
% The matrix is transposed so the final matrix size should be 30x2
names = [
   240 226 222 154 172 215 202 171 232 195 ...
   138 302 241 183 180 176 196 185 190 158 ...
   175 164 197 215 185 153 172 217 160 142;
   302 496 298 348 249 242 346 346 365 239 ...
   275 464 307 283 262 250 353 338 357 241 ...
   339 350 323 336 329 296 256 251 318 184
    ]';

% get the mean XY in the single-cue conditions
XY1 = mean(dots)';
XY2 = mean(names)';

% get the covariance matrix in the single-cue conditions
cov1 = cov(dots);
cov2 = cov(names);

% get the precision matrix in the single-cue conditions
p1 = inv(cov1);
p2 = inv(cov2);

% get weights for the single-cue conditions
w1 = inv(p1 + p2)*p1;
w2 = inv(p1 + p2)*p2;

% get determinants of weights
d1 = det(w1);
d2 = det(w2);

% get the predicted precision matrix in the combined-cue condition
pc = p1 + p2;

% get the predicted covariance matrix in the combined-cue condition
covc = inv(pc);

% get the predicted mean XY in the combined-cue condition
XYc = w1*XY1 + w2*XY2;
```

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